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ON COMMUTATIVITY AND OVALS FOR A PAIR OF SYMMETRIES OF A RIEMANN SURFACE

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Abstract. We study the upper bounds for the total number of ovals of two symmetries of a Riemann surface of genus g, whose product has order n. We show that the natural bound coming from Bujalance, Costa, Singerman and Natanzon's original results is attained for arbitrary even n, and in case of n odd, there is a sharper bound, which is attained. We also prove that two $(M-q)$- and $(M-q')$-symmetries of a Riemann surface X of genus g commute for $g \geq q+q'+1$ (by $(M-q)$-symmetry we understand a symmetry having $g+1-q$ ovals) and we show that actually, with just one exception for any $g>2$, $q+q'+1$ is the minimal lower bound for g which guarantees the commutativity of two such symmetries.

1. Introduction. Let X be a compact Riemann surface of genus $g > 1$. By a *symmetry* of X we mean an antiholomorphic involution a of X which has fixed points. By the classical result of Harnack the set of fixed points of a consists of at most $g+1$ disjoint simple closed curves, which are called *ovals*. If a has $g+1-q$ ovals then we shall call it an $(M-q)$*-symmetry*.

In [4] we observed (see also Corollary 3 in [1]) that for $g \geq q+q' + 1$, arbitrary $(M-q)$- and $(M-q')$-symmetries of a Riemann surface X commute. Here, using a method developed in [2], we show that with just one exception for any $g>2$, $q+q'+1$ is the minimal lower bound for g which guarantees the commutativity of arbitrary $(M-q)$- and $(M-q')$-symmetries. We show (Theorems 4.1 and 4.2) that for $2 \leq g \leq q+q'$ there exists a configuration of two non-commuting $(M-q)$- and $(M-q')$-symmetries, unless $g>2$ and $\{q,q'\} = \{1,g\}$, as in that case such symmetries always commute. It is worth recalling here that in [6] Natanzon gives a topological classification of pairs of commuting symmetries.

In [1] and [5] it was shown that two symmetries of a Riemann surface of genus g, whose product has order n, have at most $4g/n+2$ or $2(g-1)/n+4$ ovals in total for n even and odd respectively. Also it was shown that these

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bounds are attained for arbitrary n such that n divides $4g$ or $g-1$, depending on the parity of n. We recall Bujalance, Costa and Singerman's result from [1] and we study natural bounds following from it, i.e. $[4g/n]+2$ for n even and $[2(g-1)/n]+4$ for n odd. We show (Theorem 3.3) that for n odd this new bound is not attained for n not dividing $g-1$, we find a sharper bound and show its attainment for given n for infinitely many values of g. In contrast, for n even, the bound $[4g/n]+2$ is attained for a wider range of g and n than in [1], as we show in Theorem 3.4. Similar problems, concerning the numbers of ovals of two symmetries, were also studied in [3].

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2. Preliminaries. We shall prove our results using the theory of non-euclidean crystallographic groups (*NEC groups* for short), by which we mean discrete and cocompact subgroups of the group $\mathcal{G}$ of all isometries of the hyperbolic plane $\mathcal{H}$. The algebraic structure of such a group Λ is determined by its *signature*

$$(1)\quad s(\Lambda)=(g;\pm;[m_1,\dots,m_r];\{(n_{11},\dots,n_{1s_1}),\dots,(n_{k1},\dots,n_{ks_k})\}),$$

where the brackets $(n_{i1},\dots,n_{is_i})$ are called the *period cycles*, the integers n_{ij} are the *link periods*, m_i the *proper periods* and finally g the *orbit genus* of Λ.

A group Λ with signature (1) has the presentation with the following generators, called *canonical generators*:

$$\begin{array}{ll} x_1,\dots,x_r,e_i,c_{ij}, & 1\le i\le k,\ 0\le j\le s_i,\\ a_1,b_1,\dots,a_g,b_g & \text{if the sign is } +,\\ d_1,\dots,d_g & \text{otherwise,} \end{array}$$

and relators

$$\begin{array}{l} x_i^{m_i},\quad i=1,\dots,r,\\ c_{i,j-1}^2,c_{ij}^2,(c_{i,j-1}c_{ij})^{n_{ij}},c_{i0}e_i^{-1}c_{is_i}e_i,\quad i=1,\dots,k,\ j=1,\dots,s_i, \end{array}$$

and

$$x_1\cdots x_re_1\cdots e_ka_1b_1a_1^{-1}b_1^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}\quad\text{or}\quad x_1\cdots x_re_1\cdots e_kd_1^2\cdots d_g^2$$

according as the sign is $+$ or $-$. The elements x_i are elliptic transformations, a_i,b_i hyperbolic translations, d_i glide reflections and c_{ij} hyperbolic reflections. The reflections $c_{i,j-1}$, c_{ij} are said to be *consecutive*. Every element of finite order in Λ is conjugate to a canonical reflection, a power of some canonical elliptic element, or a power of the product of two consecutive canonical reflections.

Now an abstract group with the above presentation can be realized as an NEC group Λ if and only if the value

$$2\pi\left(\varepsilon g+k-2+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1-\frac{1}{n_{ij}}\right)\right)$$

is positive where $\varepsilon=2$ or 1 according as the sign is $+$ or $-$. This value turns out to be the hyperbolic area $\mu(\Lambda)$ of an arbitrary fundamental region for the group, and we have the *Hurwitz–Riemann formula*

$$[\Lambda:\Lambda']=\mu(\Lambda')/\mu(\Lambda)$$

for any subgroup Λ' of finite index in an NEC group Λ.

Now NEC groups having no orientation-reversing elements are classical Fuchsian groups. They have signatures $(g;+;[m_1,\ldots,m_r];\{-\})$, which will be abbreviated as $(g;m_1,\ldots,m_r)$. Given an NEC group Λ, the subgroup Λ^+ of Λ consisting of the orientation-preserving elements is called the *canonical Fuchsian subgroup* of Λ and for a group with signature (1) it has, by [7], the signature

$$(\varepsilon g+k-1;m_1,m_1,\ldots,m_r,m_r,n_{11},\ldots,n_{ks_k}). \tag{2}$$

A torsion free Fuchsian group Γ is called a *surface group* and it has signature $(g;-)$. In that case $\mathcal{H}/\Gamma$ is a compact Riemann surface of genus g, and conversely, each compact Riemann surface can be represented as such an orbit space for some Γ. Furthermore, given a Riemann surface so represented, a finite group G is a group of automorphisms of X if and only if $G=\Lambda/\Gamma$ for some NEC group Λ. The following result from [2] is crucial for the paper.

PROPOSITION 2.1. *Let $X=\mathcal{H}/\Gamma$ be a Riemann surface and G the group of all automorphisms of X. Let $G=\Lambda/\Gamma$ for some NEC group Λ and let $\theta:\Lambda\to G$ be the canonical epimorphism. Then the number of ovals of a symmetry a of X equals*

$$\sum[C(G,\theta(c_i)):\theta(C(\Lambda,c_i))],$$

where the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to a.

For a symmetry a we shall denote by $\|a\|$ the number of its ovals. The index $w_i=[C(G,\theta(c_i)):\theta(C(\Lambda,c_i))]$ will be called the *contribution* of c_i to $\|a\|$.

LEMMA 2.2 (see also Theorem 2 in [1]). *Let $\mathrm{D}_n=\Lambda/\Gamma$ be the group of automorphisms of a Riemann surface $X=\mathcal{H}/\Gamma$ generated by two non-central symmetries a and b and let $C=(n_1,\ldots,n_s)$ be a period cycle of Λ. If n is odd then the reflections corresponding to C contribute to $\|a\|$ and $\|b\|$*

at most two ovals in total. If n is even then the reflections corresponding to C contribute to $\|a\|$ and $\|b\|$ at most t ovals in total, where t is the number of even link periods if $s \geq 1$ and some n_i is even, and at most two ovals in total in the remaining cases.

Proof. Let $\theta : \Lambda \to \mathrm{D}_n$ be the canonical epimorphism. The case of n odd is trivial; here all canonical reflections belonging to C are conjugate, $C(\mathrm{D}_n, \theta(c))$ has order 2 and $c \in C(\Lambda, c)$.

Now for n even the centralizer of any non-central element of D_n has order 4. Since $c_i \in C(\Lambda, c_i)$, we have $w_i \leq 2$, and since a and b are not conjugate, we can assume that either $s \geq 2$, or $s = 1$ and n_1 is even. If c belongs to two odd link periods then we can assume that c contributes to neither $\|a\|$ nor $\|b\|$, while if c belongs to an even link period n' and cc' has order n' then $(cc')^{n'/2} \in C(\Lambda, c)$. Now $\theta((cc')^{n'/2}c) \neq 1$ since $\ker\theta$ is a Fuchsian group and therefore we see that $\theta(C(\Lambda, c))$ has order 4. ∎

3. Bounds for the total number of ovals of two symmetries of a Riemann surface. The starting point for this paper is the result of Bujalance, Costa and Singerman from [1] (see also Natanzon [5]), which we recall below. In this work we show that the natural bound for n not satisfying the divisibility conditions from [1] is attained for arbitrary even n. In contrast, for odd n there is a sharper bound, which is attained for arbitrary n not dividing $g-1$ for infinitely many values of g.

THEOREM 3.1 (Bujalance, Costa, Singerman, Natanzon). *Let a and b be two symmetries of a Riemann surface X of genus g, whose product has order n. Then a and b have at most $2(g-1)/n + 4$ and $4g/n + 2$ ovals in total for n odd and even respectively.*

COROLLARY 3.2. *Any $(M-q)$- and $(M-q')$-symmetries of a Riemann surface of genus g commute for $g \geq q + q' + 1$.*

Proof. Observe that for the total number t of ovals of both symmetries, $t = 2g + 2 - q - q' \geq g + 3$. Let n denote the order of the product of our symmetries and assume to the contrary that $n \neq 2$. By Theorem 3.1 for n even we get $g + 3 \leq 4g/n + 2 \leq g + 2$, a contradiction. For n odd, $g+3 \leq 2(g-1)/n+4 \leq 2(g-1)/3+4$ and so $g \leq 1$, which is not the case. ∎

The bounds given in the previous theorem were shown in [1] to be attained for arbitrary n and g for which n divides $g-1$ and $4g$ respectively. Theorem 3.1 gives in particular the bounds $[2(g-1)/n] + 4$ and $[4g/n] + 2$ (where $[\cdot]$ denotes the integer part), which we shall study now. In particular, the first bound turns out to be attained only for n dividing $g-1$.

THEOREM 3.3. *Let a and b be two symmetries of a Riemann surface X of genus g, whose product has order n. If n is odd and n does not divide*

$g-1$, *then* a *and* b *have at most* $[2(g-1)/n]+3$ *ovals in total, and this bound is attained for arbitrary* n *for infinitely many values of* g.

Proof. Let t denote the total number of ovals of a and b, and let $G = \langle a,b\rangle = \mathrm{D}_n$. Now $G = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature

$$(h;\pm;[m_1,\ldots,m_r];\{C_1,\ldots,C_k,(n_1),\ldots,(n_l),(-),\overset{m}{\ldots},(-)\}),$$

where $C_i = (n_{i1},\ldots,n_{is_i})$ with $s_i \geq 2$. Now as $\mu(\Lambda) = 2\pi(g-1)/n$ and n does not divide $g-1$, we see that the signature of Λ has link periods or proper periods. If there is a proper period or at least two link periods, then

$$\begin{aligned}2\pi(g-1)/n = \mu(\Lambda) &> 2\pi(k+l+m-2+1/2)\\ &\geq \pi(2(k+l+m)-3) \geq \pi(t-3)\end{aligned}$$

and so $t \leq [2(g-1)/n]+3$ as t is an integer. Obviously the number of link periods cannot be 1 if $r=0$ as otherwise $\Lambda^+ = (h';n_0)$ by (2) for the unique link period n_0 in the signature of Λ. As $\Lambda^+/\Gamma = \mathrm{Z}_n$, the relation $x_1'[a_1',b_1']\ldots[a_{h'}',b_{h'}'] = 1$ in Λ^+ would give $\theta(x_1') = 1$ for the canonical epimorphism $\theta:\Lambda\to G$, which is impossible.

We now show that for arbitrary m there exist two symmetries a and b on a Riemann surface X of genus $g = n(m+1)$, whose product has order n and which have $[2(g-1)/n]+3$ ovals in common. Indeed, consider an NEC group with signature

$$(0;+;[-];\{(-),\overset{m+1}{\ldots},(-),(n,n)\})$$

and let $\theta:\Lambda\to\mathrm{D}_n$ be an epimorphism defined by $\theta(e_i) = 1$ for $i = 1,\ldots,m+2$, $\theta(c_{i0}) = a$ for $i = 1,\ldots,m+1$ and $\theta(c_{m+2,0}) = \theta(c_{m+2,2}) = a$, $\theta(c_{m+2,1}) = b$. Then by the Hurwitz–Riemann formula for $\Gamma = \ker\theta$, $X = \mathcal{H}/\Gamma$ is a Riemann surface of genus g, and by Proposition 2.1 each of the symmetries a and b has $m+2$ ovals. ∎

In contrast to the previous theorem, the bound $[4g/n]+2$ for n,g not satisfying the divisibility conditions from [1] cannot be improved for n even.

THEOREM 3.4. *For arbitrary even* $n>4$ *there are infinitely many values of* g *for which* n *does not divide* $4g$ *and there exists a Riemann surface of genus* g *having two symmetries whose product has order* n, *with* $[4g/n]+2$ *ovals in total.*

Proof. Let Λ be an NEC group with signature

$$(0;+;[-];\{(-),(2,\overset{2m}{\ldots},2)\})$$

and consider an epimorphism $\theta:\Lambda\to\mathrm{D}_n = \langle a,b \mid a^2, b^2, (ab)^n\rangle$ defined by $\theta(e_1) = \theta(e_2) = 1$, $\theta(c_{10}) = a$ and which sends the reflections corresponding to the unique non-empty period cycle alternately to b and $(ab)^{n/2-1}a$. As before θ defines the configuration of two symmetries of a Riemann surface of genus $g = mn/2+1$, which have, by Proposition 2.1, $2m+2$ ovals in total. ■

4. Commutativity of a pair of $(M-q)$- and $(M-q')$-symmetries. By Corollary 3.2, a pair of $(M-q)$- and $(M-q')$-symmetries of a Riemann surface X of genus g commutes for $g \geq q+q'+1$. Now, using the method introduced in Proposition 2.1, we shall show that $q+q'+1$ is in fact the minimal lower bound for g which guarantees commutativity of a pair of $(M-q)$- and $(M-q')$-symmetries of a Riemann surface X of genus g. The only exception is the case of $(M-1)$- and $(M-g)$-symmetries for $g > 2$. Recall that we only consider symmetries with fixed points.

THEOREM 4.1. *For $2 \leq g \leq q+q'$ but $g > 2$ and $\{q,q'\} = \{1,g\}$, there exists a Riemann surface of genus g, having a pair of non-commuting $(M-q)$- and $(M-q')$-symmetries.*

Proof. Let $q \leq q'$ and observe that $g \geq q'$ as both symmetries have ovals.

For $q+q'-g \equiv 0 \bmod 4$ consider an NEC group Λ with signature

$$(h; -; [-]; \{(2, \overset{s}{\ldots}, 2, 4, 2, \overset{s'}{\ldots}, 2, 4)\}),$$

where $h = (q+q'-g)/4$, $s = g-q$, $s' = g-q'$, and an epimorphism $\theta : \Lambda \to G = \mathrm{D}_4$ for which $\theta(e) = 1$, $\theta(d_i) = a$ and the consecutive canonical reflections corresponding to the non-empty period cycle are mapped to

$$\underbrace{a\ bab\ a\ bab\ \ldots\ a(ab)^{2s}}_{s+1}\ \underbrace{b\ aba\ b\ aba\ \ldots\ b(ab)^{2s'}}_{s'+1}\ a.$$

Then by the Hurwitz–Riemann formula for $\Gamma = \ker\theta$, $X = \mathcal{H}/\Gamma$ has genus g, and by Proposition 2.1 the symmetries a and b have $g+1-q$ and $g+1-q'$ ovals respectively.

For $q'+q-g \equiv 2 \bmod 4$ consider an NEC group with signature

$$(h; -; [2]; \{(2, \overset{s}{\ldots}, 2, 4, 2, \overset{s'}{\ldots}, 2, 4)\}),$$

where $h = (q'+q-2-g)/4$, s, s' are as above, and the epimorphism defined as in the previous case with $\theta(x) = \theta(e) = (ab)^2$. As before θ defines a desired configuration of non-commuting $(M-q)$- and $(M-q')$-symmetries of a Riemann surface of genus g.

Now let $q'+q-g \equiv 3 \bmod 4$. Consider an NEC group with signature

$$(h; -; [4]; \{(2, \overset{s}{\ldots}, 2, 4, 2, \overset{s'}{\ldots}, 2, 4)\}),$$

where $h = (q'+q-3-g)/4$, s, s' are as above, and an epimorphism defined as follows for the consecutive canonical reflections corresponding to the non-empty period cycle:

$$\underbrace{a\ bab\ a\ bab\ \ldots\ a(ab)^{2s}}_{s+1}\ \underbrace{b\ aba\ b\ aba\ \ldots\ b(ab)^{2s'}}_{s'+1}\ bab$$

and $\theta(x) = ab$, $\theta(e) = ba$. Also here θ gives rise to the configuration of symmetries we looked for.

Now if $q + q' - g \equiv 1 \bmod 4$ and $g < q + q' - 1$ consider an NEC group with signature

$$(h; -; [2, 4]; \{(2, \overset{s}{\ldots}, 2, 4, 2, \overset{s'}{\ldots}, 2, 4)\}),$$

where $h = (q'+q-5-g)/4$, s, s' are as above, and an epimorphism defined for the consecutive canonical reflections corresponding to the non-empty period cycle as follows:

$$\underbrace{a\ bab\ a\ bab\ \ldots\ a(ab)^{2s}}_{s+1}\ \underbrace{b\ aba\ b\ aba\ \ldots\ b(ab)^{2s'}}_{s'+1}\ bab$$

and $\theta(x_1) = (ab)^2$, $\theta(x_2) = \theta(e) = ab$. As before for $\Gamma = \ker\theta$, $X = \mathcal{H}/\Gamma$ is a Riemann surface of genus g having two non-commuting $(M - q)$- and $(M - q')$-symmetries.

Finally, for $g = q + q' - 1$ assume first that $q \geq 2$ and let Λ be an NEC group with signature

$$(0; \pm; [-]; \{(2, \overset{q-2}{\ldots}, 2, 4, 2, \overset{q'-2}{\ldots}, 2, 4, 4, 4)\})$$

and an epimorphism $\theta : \Lambda \to G = \mathrm{D}_4$ for which $\theta(e) = 1$ and the reflections corresponding to the non-empty period cycle are mapped onto

$$\underbrace{a\ bab\ a\ bab\ \ldots\ a(ab)^{2(q-1)}}_{q-1}\ \underbrace{b\ aba\ b\ aba\ \ldots\ b(ab)^{2(q'-1)}}_{q'-1}\ a\ b\ a.$$

Here again we get a configuration of two non-commuting symmetries a and b, which have q and q' ovals respectively. For $g = 2$, $\{q, q'\} = \{1, 2\}$, we can take $n = 8$; in this case the bound $4g/n + 2$ is attained by Theorem 4 in [1], and one of our symmetries has two ovals and the other has one oval by Theorem 6 from [1]. ∎

Theorem 4.2. *For $g > 2$ any $(M - 1)$- and $(M - g)$-symmetries of a Riemann surface of genus g commute.*

Proof. Assume to the contrary that there exists pair a, b of non-commuting $(M - 1)$- and $(M - g)$-symmetries, and let $n > 2$ denote the order of their product. Observe that the total number t of ovals of both symmetries is $g + 1$.

Obviously n cannot be odd, as in this case the symmetries would be conjugate and so they would have the same number of ovals, which is clearly not the case. So let n be even. By Theorem 3.1 we see that in this case the two symmetries have at most $4g/n+2$ ovals in total. In particular for $n \geq 8$, $4g/n + 2 \leq g/2 + 2$ and so $g + 1 \leq g/2 + 2$ would be necessary for such symmetries to exist. But then we have $g \leq 2$, which is not the case again.

Assume now that such a pair of symmetries a, b exists for $n = 4$, and let a and b have g ovals and 1 oval respectively. Let Λ be an NEC group with signature

$$(h; \pm; [m_1, \ldots, m_r]; \{C_1, \ldots, C_k, (-), \overset{m}{\ldots}, (-)\}),$$

where $C_i = (n_{i1}, \ldots, n_{is_i})$, and set $s = s_1 + \cdots + s_k$. Observe now that if $k = 0$, then either $m \geq 3$, or $m = 2$ and $h + r \geq 1$. In addition, $2m \geq t + 1$ by Lemma 2.2, as the symmetry b has exactly one oval. So we have $\pi(g-1)/2 = \mu(\Lambda) \geq 2\pi(m - 2 + h + r/2) \geq 2\pi(m/2 + (h + m + r)/2 - 2) \geq \pi(-1 + t)/2$ and hence $t \leq g$, a contradiction.

For $k \geq 2$ we have $\pi(g-1)/2 = \mu(\Lambda) \geq 2\pi(m + s/4) \geq 2\pi(m/2 + s/4)$ and as $t \leq s + 2m$, by Lemma 2.2, we get $t \leq g - 1$. So we can assume that $k = 1$. If $m \geq 2$ then $\pi(g-1)/2 = \mu(\Lambda) \geq 2\pi(-2 + k + m + s/4) \geq 2\pi(m/2 + s/4)$ and as before we have $t \leq g - 1$, which is not the case.

Let now $k = m = 1$. We can assume $h = r = 0$ as otherwise $\pi(g-1)/2 = \mu(\Lambda) \geq 2\pi(1/2 + s/4) = 2\pi(m/2 + s/4)$ and we would have $t \leq g - 1$ as above. Observe now that $s \geq 2$, since otherwise $\Lambda^+ = (h'; n_0)$ by (2) for the unique link period n_0 in the signature of Λ. As $\Lambda^+/\Gamma = \mathrm{Z}_4$, the relation $x_1'[a_1', b_1'] \cdots [a_{h'}', b_{h'}'] = 1$ in Λ^+ would give $\theta(x_1') = 1$ for the canonical epimorphism $\theta : \Lambda \to G$, which is impossible. Now if all link periods are equal to 2 then, by Proposition 2.1, the non-empty period cycle contributes ovals only to the symmetry a as $s \geq 2$ and the order of the product of an element conjugate to a and an element conjugate to b is 4. So by Lemma 2.2 we have $s + 2 \geq t + 1$, which gives $\pi(g-1)/2 = \mu(\Lambda) \geq \pi s/2 \geq \pi(t-1)/2$ and so $t \leq g$, which is not the case. Observe now that if there is a link period 4, then there has to be another link period 4. Indeed, the conjugates of a have product of order 2 and so $\theta(c_i)$ is conjugate to b for the unique i in the range $0 \leq i \leq s - 1$. But then for $i \neq 0$, $\theta(c_{i-1})$, $\theta(c_{i+1})$ are conjugates of a and so $n_i = n_{i+1} = 4$. For $i = 0$, $\theta(c_s)$ is conjugate to b, while $\theta(c_1)$ and $\theta(c_{s-1})$ are conjugate to a, so $n_1 = n_s = 4$. In both cases all other link periods are equal to 2. Thus $\pi(g-1)/2 = \mu(\Lambda) \geq 2\pi((s-2)/4 + 3/4) = \pi(s+1)/2 \geq \pi(t-1)/2$ since $s + 2 \geq t$ by Lemma 2.2 and so $t \leq g$, which is not the case.

So we can assume that Λ has signature of the form

$$(h; \pm; [m_1, \ldots, m_r]; \{(n_1, \ldots, n_s)\})$$

and by Proposition 2.1 and Lemma 2.2 we see that $t = s = g + 1$. Since both a and b have ovals, it follows, as shown above, that $n_j = n_{j+1} = 4$ for a unique integer j with $1 \leq j \leq s$ and all other n_i are equal to 2.

Observe first that $h = 0$ as otherwise $\pi(g-1)/2 = \mu(\Lambda) \geq 2\pi((g-1)/4 + 3/4)$ and so $g + 2 \leq g - 1$, a contradiction. Now if $r > 0$ then we have $\pi(g-1)/2 = \mu(\Lambda) \geq 2\pi(-1 + (g-1)/4 + 3/4 + 1/2) = \pi g/2$ and we get $g \leq g - 1$, a contradiction again. So finally let $r = 0$. Then $\pi(g-1)/2 = \mu(\Lambda) = \pi(g-2)/2$, and also in this case we get a contradiction.

Observe now that for $n = 6$, $g + 1 \leq 2g/3 + 2$ by Theorem 3.1 and so $g \leq 3$. Now for $g = 3$, $4g/n = 2$ is an integer, $4g/n + 2 = g + 1$ and by Theorems 4 and 6 from [1] each of our symmetries has two ovals, which is not the case. ∎

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