COLLOQUIUM MATHEMATICUM

VOL. 109 2007 NO. 1

RIESZ POTENTIALS DERIVED BY ONE-MODE INTERACTING FOCK SPACE APPROACH

BY

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Abstract. The main aim of this short paper is to study Riesz potentials on one-mode interacting Fock spaces equipped with deformed annihilation, creation, and neutral operators with constants $c_{0,0}, c_{1,1} \in \mathbb{R}$ and $c_{0,1} > 0$, $c_{1,2} \geq 0$ as in equations (1.4)–(1.6). First, to emphasize the importance of these constants, we summarize our previous results on the Hilbert space of analytic L^2 functions with respect to a probability measure on $\mathbb{C}$. Then we consider the Riesz kernels of order 2α, $\alpha = c_{0,1}/c_{1,2}$, on $\mathbb{C}$ if $0 < c_{0,1} < c_{1,2}$, which can be derived from the Bessel kernels of order 2α, $\gamma_{\alpha,c_{1,2}}$, on $\mathbb{C}$. Moreover, we prove that if $c_{1,2}/2 < c_{0,1} < c_{1,2}$, then the Riesz potentials are continuous linear operators on the Hilbert space of analytic L^2 functions with respect to $\gamma_{\alpha,c_{1,2}}$.

1. Preliminaries. Let μ be a probability measure on $I \subset \mathbb{R}$ with finite moments of all orders such that the linear span of the monomials x^n, $n \geq 0$, is dense in $L^2(I,\mu)$. Then it is known [8] that there exist a complete orthogonal system $\{P_n(x)\}_{n=0}^\infty$ of polynomials with leading coefficient 1 for $L^2(I,\mu)$ with $P_0 = 1$, a sequence $\{\omega_n\}_{n=0}^\infty$ of nonnegative real numbers, and a sequence $\{\alpha_n\}_{n=0}^\infty$ of real numbers such that the following recurrence formula holds:

$$(x-\alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \qquad n \geq 0,$$

where $\omega_0 = 1$ and $P_{-1} = 0$ by convention. The numbers ω_n, α_n are called the *Jacobi–Szegö parameters* of μ. In this paper, it will be enough to consider probability measures having the Jacobi–Szegö parameters of the form

$$\omega_0 = 1, \qquad \omega_n = n(c_{0,1} + c_{1,2}(n-1)), \qquad n \geq 1, \tag{1.1}$$

$$\alpha_n = c_{0,0} + c_{1,1}n, \qquad n \geq 0, \tag{1.2}$$

where $c_{0,1} > 0$ and $c_{1,2} \geq 0$ and $c_{0,0}, c_{1,1} \in \mathbb{R}$.

For $f \in L^2(I,\mu)$, the author [3] introduced the S_μ-*transform* given by

$$(S_\mu f)(z) = \langle E_\lambda(\cdot,\overline{z}), f\rangle_{L^2(\mu)} = \int_I E_\lambda(x,z)f(x)\,d\mu(x), \qquad z \in \Omega_\lambda, \tag{1.3}$$

2000 *Mathematics Subject Classification*: 46N30, 33D45, 60J45.

Key words and phrases: one-mode interacting Fock space, Jacobi–Szegö parameters, Segal–Bargmann transform, Bessel kernel measures, Riesz potentials, Hilbert space of analytic L^2 functions.

where

$$E_\lambda(x,z) = \sum_{n=0}^{\infty} \frac{P_n(x)}{\lambda_n} z^n, \qquad \lambda_n = \omega_0\omega_1\cdots\omega_n,$$

and Ω_λ is the set of all z in $\mathbb{C}$ such that $\|E_\lambda(\cdot,z)\|_{L^2(\mu)} < \infty$. The set $\{E_\lambda(\cdot,z) : z \in \Omega_\lambda\}$ is linearly independent and spans a dense subspace of $L^2(I,\mu)$. The S_μ-transform in (1.3) is a non-Gaussian analogue of the well-known Segal–Bargmann transform. See [6], [10] for μ being the Gaussian measure. The S_μ-transform maps $L^2(I,\mu)$ isomorphically onto the Hilbert space $\mathcal{H}_\lambda$ of all analytic functions $F(z) = \sum_{n=0}^{\infty} a_n z^n$ on Ω_λ with the norm

$$\|F\|_{\mathcal{H}_\lambda} := \Big(\sum_{n=0}^{\infty} \lambda_n |a_n|^2\Big)^{1/2} < \infty.$$

Let b and b^* be the Bosonic annihilation and creation operators, respectively, defined by

$$b\cdot 1 = 0, \quad bz^n = nz^{n-1}, \qquad n \ge 1,$$

and

$$b^* z^n = z^{n+1}, \qquad n \ge 0.$$

Moreover, introduce the operators

$$B^- = c_{0,1}b + c_{1,2}b^*b^2, \tag{1.4}$$

$$B^+ = b^*, \tag{1.5}$$

$$B^\circ = c_{0,0}I + c_{1,1}b^*b. \tag{1.6}$$

Then the Hilbert space $\mathcal{H}_\lambda$ equipped with $\{B^-, B^+, B^\circ\}$ becomes the *one-mode interacting Fock space* discussed in [1], [4]. We call B^-, B^+ and B° the *deformed annihilation operator*, *deformed creation operator*, and *neutral* (*preservation*) *operator*, respectively. The constants $c_{0,0}$ and $c_{0,1}$ correspond to the mean and variance of a classical random variable x, respectively. The roles of $c_{1,1}$, $c_{1,2}$ and $c_{0,1}/c_{1,2}$ will be seen later on.

In our previous papers [4], [5] we have managed to realize the operators B^-, B^+, B° on $\mathcal{H}L^2(\mathbb{C},\gamma)$, a certain Hilbert space of analytic L^2 functions with respect to a probability measure γ on $\mathbb{C}$. To construct such a measure, it is quite important to see whether or not the structure constant $c_{1,2}$ is zero.

2. Hilbert spaces of analytic functions associated with Gaussian and Bessel kernel measures. In this section, we summarize the key results from [5] for the case of $c_{1,2} = 0$ and [4] for $c_{1,2} \neq 0$. Then the readers can recognize that the constant $c_{1,2}$ plays an important role in our analysis.

First, let us state the following theorem for the case of $c_{1,2} = 0$:

THEOREM 2.1 ([5]). *Suppose that the Jacobi–Szegö parameters have the form* (1.1), (1.2) *with* $c_{1,2} = 0$. *Then*:

(1) *There exists a unique probability measure* $h_{c_{0,1}}$ *on* $\mathbb{C}$ *satisfying*

$$\mathcal{H}_\lambda = \mathcal{H}L^2(\mathbb{C}, h_{c_{0,1}}).$$

In fact, $h_{c_{0,1}}$ *is the Gaussian measure on* $\mathbb{C}$ *of the form*

$$dh_{c_{0,1}}(z) := h(z, c_{0,1})dz$$

where

$$h(z, c_{0,1}) = \frac{1}{\pi c_{0,1}} \exp\left(-\frac{|z|^2}{\pi c_{0,1}}\right).$$

(2) *The Segal–Bargmann transform* S_μ *is a unitary operator from* $L^2(I, \mu)$ *onto* $\mathcal{H}L^2(\mathbb{C}, h_{c_{0,1}})$ *satisfying*

$$S_\mu^{-1}(c_{0,0} + c_{0,1}b + b^* + c_{1,1}b^*b)S_\mu = Q_x$$

where Q_x *is the multiplication operator by* x *on* $L^2(I, \mu)$.

EXAMPLE 2.2. The Gaussian measure on $\mathbb{R}$ and the Poisson measure on $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ have $c_{1,2} = 0$. Note that $c_{1,1} = 0$ for the Gaussian measure. See [5] for the details.

Secondly, the case of $c_{1,2} \neq 0$ is as follows:

THEOREM 2.3 ([4]). *Assume that the Jacobi–Szegö parameters have the form* (1.1), (1.2) *with* $c_{1,2} \neq 0$. *Then*:

(1) *There exists a unique probability measure* $\gamma_{\alpha,c_{1,2}}$ *on* $\mathbb{C}$ *satisfying*

$$\mathcal{H}_\lambda = \mathcal{H}L^2(\mathbb{C}, \gamma_{\alpha,c_{1,2}}).$$

In fact, $\gamma_{\alpha,c_{1,2}}$ *is the Bessel kernel measure on* $\mathbb{C}$ *of the form*

$$d\gamma_{\alpha,c_{1,2}}(z) := \frac{2c_{1,2}^{-(1+\alpha)/2}}{\pi\Gamma(\alpha)} |z|^{\alpha-1} K_{1-\alpha}(2c_{1,2}^{-1/2}|z|)\, dz, \qquad \alpha = c_{0,1}/c_{1,2}.$$

Note that K_ν *is the so-called modified Bessel function given by*

$$K_\nu(x) = \frac{\pi}{2\sin(\nu\pi)} (I_{-\nu}(x) - I_\nu(x))$$

where

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!\Gamma(n+\nu+1)}.$$

(2) *The Segal–Bargmann transform* S_μ *is a unitary operator from* $L^2(I, \mu)$ *onto* $\mathcal{H}L^2(\mathbb{C}, \gamma_{\alpha,c_{1,2}})$ *satisfying*

$$S_\mu^{-1}(c_{0,0} + c_{0,1}b + b^* + c_{1,1}b^*b + c_{1,2}b^*b^2)S_\mu = Q_x$$

where $\alpha = c_{0,1}/c_{1,2}$ *and* Q_x *is the multiplication operator by* x *on* $L^2(I,\mu)$.

(3) *The measure* $\gamma_{\alpha,c_{1,2}}$ *has the following integral representation*:

$$d\gamma_{\alpha,c_{1,2}}(z) = \frac{1}{\Gamma(\alpha)}\Big(\int_0^\infty h(z,c_{1,2}t)e^{-t}t^{\alpha-1}\,dt\Big)\,dz$$

where $\alpha = c_{0,1}/c_{1,2}$.

EXAMPLE 2.4. For $c_{1,2} \neq 0$, we have three examples classified by the sign of $c_{1,1}^2 - 4c_{1,2}$:

(1) If μ is the Gamma distribution on $\mathbb{R}_+$, then $c_{1,1} \neq 0$ and $c_{1,1}^2 = 4c_{1,2}$.
(2) If μ is the negative binomial distribution on $\mathbb{N}_0$, then $c_{1,1}^2 > 4c_{1,2}$.
(3) If μ is the Meixner distribution on $\mathbb{R}$, then $c_{1,1}^2 < 4c_{1,2}$.

The reader can refer to Appendix of [4] for the details.

So, if $c_{1,2} \neq 0$, a classical random variable x in $L^2(I,\mu)$ is realized in a Hilbert space of analytic L^2 functions with respect to $\gamma_{\alpha,c_{1,2}}$, different from $\mathcal{H}L^2(\mathbb{C}, h_{c_{0,1}})$ in Theorem 2.1. On the other hand, the constants $c_{0,0}, c_{1,1}$ do not contribute anything to the construction of $h_{c_{0,1}}$ and $\gamma_{\alpha,c_{1,2}}$. This is because these two measures on $\mathbb{C}$ are derived from the complex moment problem for the sequence $\{\lambda_n\}$.

3. Riesz potentials. There are some natural relationships between $\mathcal{H}L^2(\mathbb{C}, \gamma_{\alpha,c_{1,2}})$ in Theorem 2.3 and the Riesz potentials on it. To see them, let us discuss the case $c_{1,2} \neq 0$ as $|z| \to 0$, which was not considered in our previous papers [4], [5].

It is known [2], [11] that the asymptotic behavior of the Bessel kernels $\gamma_{\alpha,c_{1,2}}$ as $|z| \to 0$ is given by

$$\gamma_{\alpha,c_{1,2}}(z) \sim \frac{\Gamma(1-\alpha)}{c_{1,2}^{\alpha}\pi\Gamma(\alpha)}\,\frac{1}{|z|^{2(1-\alpha)}} =: R_{\alpha,c_{1,2}}(z) \qquad \alpha = c_{0,1}/c_{1,2}, \tag{3.1}$$

if $0 < c_{0,1} < c_{1,2}$. In this paper, the right hand side of (3.1) is called the *Riesz kernel* of order 2α.

Note that the order 2α of the kernel depends on two constants $c_{0,1}$ and $c_{1,2}$. To see the roles of these constants in our analysis, let us consider the Laplace operator $\Delta_c = 4\partial^2/\partial z\partial\bar{z}$ and its fractional power

$$\Big(-\frac{c_{1,2}}{4}\,\Delta_c\Big)^{-\alpha}, \qquad \alpha = c_{0,1}/c_{1,2}.$$

This is the so-called *Riesz potential*. By using the Gamma function, one can formally give the integral representation

$$\left(-\frac{c_{1,2}}{4}\,\Delta_c\right)^{-\alpha} F = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}_+} (e^{(c_{1,2}/4)t\Delta_c} F) t^{\alpha-1}\, dt \tag{3.2}$$

for $F \in \mathcal{H}L^2(\mathbb{C}, \gamma_{\alpha,c_{1,2}})$. We shall prove that the Riesz potentials as defined by (3.2) make sense as continuous linear operators on $\mathcal{H}L^2(\mathbb{C}, \gamma_{\alpha,c_{1,2}})$ due to the following.

THEOREM 3.1. *Let $F \in \mathcal{H}L^2(\mathbb{C}, \gamma_{\alpha,c_{1,2}})$ and $c_{1,2}/2 < c_{0,1} < c_{1,2}$. Then*

$$\left\| \left(-\frac{c_{1,2}}{4}\,\Delta_c\right)^{-\alpha} F \right\|_{\mathcal{H}L^2} \le C\|F\|_{\mathcal{H}L^2}, \qquad \alpha = c_{0,1}/c_{1,2},$$

for some $C > 0$.

Proof. It is easy to see that

$$\begin{aligned}\left(-\frac{c_{1,2}}{4}\,\Delta_c\right)^{-\alpha} F(z) &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}_+} \Big\{\int_{\mathbb{C}} h(z-w, c_{1,2}t) F(w)\, dw\Big\} t^{\alpha-1}\, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_{\mathbb{C}} \bigg\{\int_{\mathbb{R}_+} \frac{se^{-s}}{\pi|z-w|^2} \left(\frac{|z-w|^2}{c_{1,2}s}\right)^{\alpha-1} \left(\frac{|z-w|^2}{c_{1,2}s^2}\right) ds\bigg\} F(w)\, dw \\ &= \frac{1}{c_{1,2}^{\alpha}\pi\Gamma(\alpha)} \Big(\int_{\mathbb{R}_+} e^{-s}s^{-\alpha}\, ds\Big)\Big(\int_{\mathbb{C}} |z-w|^{2(\alpha-1)} F(w)\, dw\Big) \\ &= \int_{\mathbb{C}} R_{\alpha,c_{1,2}}(z-w) F(w)\, dw =: R_{\alpha,c_{1,2}} * F(z).\end{aligned} \tag{3.3}$$

By using Young's inequality for convolution, we get

$$\|R_{\alpha,c_{1,2}} * F\|_{\mathcal{H}L^2} \le \|R_{\alpha,c_{1,2}}\|_{\mathcal{H}L^1} \|F\|_{\mathcal{H}L^2}.$$

With the help of Hölder's inequality, we obtain

$$\begin{aligned}\|R_{\alpha,c_{1,2}}\|_{\mathcal{H}L^1} &= \int_{\mathbb{C}} R_{\alpha,c_{1,2}}(z)\gamma_{\alpha,c_{1,2}}(z)\, dz \\ &= \int_{|z|<1} R_{\alpha,c_{1,2}}(z)\gamma_{\alpha,c_{1,2}}(z)\, dz + \int_{|z|\ge 1} R_{\alpha,c_{1,2}}(z)\gamma_{\alpha,c_{1,2}}(z)\, dz \\ &\le \Big(\int_{|z|<1} R_{\alpha,c_{1,2}}(z)^2\, dz\Big)^{1/2} \Big(\int_{|z|<1} \gamma_{\alpha,c_{1,2}}(z)^2\, dz\Big)^{1/2} + \frac{\Gamma(1-\alpha)}{c_{1,2}^{\alpha}\pi\Gamma(\alpha)} \\ &< \infty\end{aligned}$$

due to $c_{1,2}/2 < c_{0,1} < c_{1,2}$. Therefore,

$$\|R_{\alpha,c_{1,2}} * F\|_{\mathcal{H}L^2} \le C\|F\|_{\mathcal{H}L^2} \quad \text{for some } C > 0. \ ■$$

Since the Riesz potentials are closely related to fractional calculus, our approach from the point of view of deformed creation and annihilation operators on one-mode interacting Fock spaces and the Hilbert space of analytic L^2 functions could be useful to study (complex) fractional Brownian motions (fBm's) and fractional white noises. Our parameter $\alpha = c_{0,1}/c_{1,2}$ is related to the Hurst parameter H in $(0, 1)$. See [7], [9] and papers cited therein for fBm's and related applications.

Acknowledgments. The author thanks the referees for pointing out several misprints and making useful comments to improve this paper.

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Received 12 April 2006;
revised 21 December 2006 (4770)