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MINIMALITY OF THE SYSTEM OF ROOT FUNCTIONS OF STURM–LIOUVILLE PROBLEMS WITH DECREASING AFFINE BOUNDARY CONDITIONS

BY

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Abstract. We consider Sturm–Liouville problems with a boundary condition linearly dependent on the eigenparameter. We study the case of decreasing dependence where non-real and multiple eigenvalues are possible. By determining the explicit form of a biorthogonal system, we prove that the system of root (i.e. eigen and associated) functions, with an arbitrary element removed, is a minimal system in $L_2(0,1)$, except for some cases where this system is neither complete nor minimal.

Introduction. Consider the following spectral problem:

$$-y'' + q(x)y = \lambda y, \quad 0 < x < 1, \tag{0.1}$$

$$y'(0)\sin\beta = y(0)\cos\beta, \tag{0.2}$$

$$y'(1) = (a\lambda + b)y(1), \tag{0.3}$$

where a, b, β are real constants, $0 \le \beta < \pi$, $a < 0$, λ is a spectral parameter and $q(x)$ is a real-valued and continuous function over the interval $[0,1]$.

It was proved in [2] (see also [1]) that the eigenvalues of the boundary value problem (0.1)–(0.3) form an infinite sequence accumulating only at ∞ and only the following cases are possible: (a) all eigenvalues are real and simple; (b) all eigenvalues are real and all, except one double, are simple; (c) all eigenvalues are real and all, except one triple, are simple; (d) all eigenvalues are simple and all, except a conjugate pair of non-real ones, are real.

Let $\{v_n\}_{n=1}^{\infty}$ be a sequence of elements from $L_2(0,1)$ and V_k the closure (in the norm of $L_2(0,1)$) of the linear span of $\{v_n\}_{n=1,\, n\neq k}^{\infty}$. The system $\{v_n\}_{n=1}^{\infty}$ is called *minimal* in $L_2(0,1)$ if $v_k \notin V_k$ for all $k = 1, 2, \ldots$ (see [9, Ch. I, §2]).

The present article concerns the minimality in $L_2(0,1)$ of the system of root functions of the boundary value problem (0.1)–(0.3). In cases (a) and (d), we complete the results of [2] by showing that the system of eigenfunctions of (0.1)–(0.3), with an arbitrary element removed, is minimal in

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$L_2(0,1)$. In cases (b) and (c) we discuss all the choices of the removed element and find necessary and sufficient conditions for the system of root functions, with one element removed, to be minimal in $L_2(0,1)$. Using the method of [10–12] one can show that such a minimal system is a basis in $L_p(0,1)$ $(1 < p < \infty)$. The precise statements and proofs of our results are contained in Section 4.

The eigenvalues λ_n $(n \geq 0)$ will be listed according to their non-decreasing real part and repeated according to their algebraic multiplicity. The asymptotics of eigenvalues and oscillations of eigenfunctions of the boundary value problem (0.1)–(0.3), with the linear function in the boundary condition replaced by a general rational function, were studied in a recent paper [3]. For an affine (linear) decreasing function this asymptotics is as follows [2]:

$$\lambda_n = \begin{cases} (n-1/2)^2\pi^2 + O(1) & \text{if } \beta \neq 0, \\ n^2\pi^2 + O(1) & \text{if } \beta = 0. \end{cases} \tag{0.4}$$

This asymptotic formula plays an important role in the passage from minimality theorems to basis properties in $L_2(0,1)$ (cf. [10–12]).

The case $a > 0$ of our problem is considerably simpler and can be found as a special case in [10, 11]. In [13] the following boundary value problem was considered:

$$-y'' = \lambda y, \qquad 0 < x < 1, \tag{0.5}$$

$$y'(0) = 0, \qquad y'(1) = a\lambda y(1), \qquad a \neq 0. \tag{0.6}$$

For this problem only cases (a) and (b) are possible, and in [13] a complete solution of the problem of the basis properties in $L_p(0,1)$ $(1 < p < \infty)$ of the system of root functions was given. We shall discuss this problem further in the last section. The situation for (0.1)–(0.3) is much more complicated, with the possibility of non-real eigenvalues and of an eigenvalue with algebraic multiplicity 3.

There is a vast literature on the boundary value problems with a spectral parameter in the boundary conditions (see e.g. [4, 7, 15] and a recent contribution [5]).

1. Inner products and norms of eigenfunctions. Let y_n be an eigenfunction corresponding to an eigenvalue λ_n. By (0.1)–(0.3) we have

$$\begin{aligned} -y_n'' + q(x)y_n &= \lambda_n y_n, \\ y_n'(0)\sin\beta &= y_n(0)\cos\beta, \\ y_n'(1) &= (a\lambda_n + b)y_n(1). \end{aligned}$$

Let $y(x,\lambda)$ be a non-zero solution of (0.1)–(0.2), and consider the characteristic function

$$\varpi(\lambda) = y'(1,\lambda) - (a\lambda + b)y(1,\lambda). \tag{1.1}$$

By (0.3), λ_n is an eigenvalue of (0.1)–(0.3) if $\varpi(\lambda_n) = 0$. It is a simple eigenvalue if $\varpi(\lambda_n) = 0 \neq \varpi'(\lambda_n)$, a double eigenvalue if

$$\varpi(\lambda_k) = \varpi'(\lambda_k) = 0 \neq \varpi''(\lambda_k), \tag{1.2}$$

and a triple eigenvalue if

$$\varpi(\lambda_k) = \varpi'(\lambda_k) = \varpi''(\lambda_k) = 0 \neq \varpi'''(\lambda_k). \tag{1.3}$$

We also note that $y(x,\lambda) \to y(x,\lambda_n)$ uniformly as $\lambda \to \lambda_n$, because $y(x,\lambda)$ is an entire function of λ (see [6, Sect. 10.72]).

Throughout this paper we denote by $(\cdot,\cdot)$ the scalar product in $L_2(0,1)$.

LEMMA 1.1. *Let y_n, y_m be eigenfunctions corresponding to the eigenvalues λ_n, λ_m ($\lambda_n \neq \overline{\lambda}_m$). Then*

$$(y_n, y_m) = -a y_n(1)\overline{y_m(1)}. \tag{1.4}$$

Proof. To begin we note that

$$\frac{d}{dx}\left[y(x,\lambda)\overline{y'(x,\mu)} - y'(x,\lambda)\overline{y(x,\mu)}\right] = (\lambda - \overline{\mu})y(x,\lambda)\overline{y(x,\mu)}.$$

By integrating this identity from 0 to 1, we obtain

$$(\lambda - \overline{\mu})(y(\cdot,\lambda), y(\cdot,\mu)) = (y(x,\lambda)\overline{y'(x,\mu)} - y'(x,\lambda)\overline{y(x,\mu)})\big|_0^1. \tag{1.5}$$

From (0.2), we obtain

$$y(0,\lambda)\overline{y'(0,\mu)} - y'(0,\lambda)\overline{y(0,\mu)} = 0. \tag{1.6}$$

By (1.1),

$$\begin{aligned} y(1,\lambda)\overline{y'(1,\mu)} - y'(1,\lambda)\overline{y(1,\mu)} = & -a(\lambda - \overline{\mu})y(1,\lambda)\overline{y(1,\mu)} \\ & + y(1,\lambda)\overline{\varpi(\mu)} - \overline{y(1,\mu)}\varpi(\lambda). \end{aligned} \tag{1.7}$$

From (1.5)–(1.7), it follows that for $\lambda \neq \overline{\mu}$,

$$(y(\cdot,\lambda), y(\cdot,\mu)) = -a y(1,\lambda)\overline{y(1,\mu)} + y(1,\lambda)\frac{\overline{\varpi(\mu)}}{\lambda - \overline{\mu}} - \overline{y(1,\mu)}\frac{\varpi(\lambda)}{\lambda - \overline{\mu}}, \tag{1.8}$$

which is a generalization of an analogous formula in [6, Sect. 10.72]. Since λ_n, λ_m are eigenvalues of (0.1)–(0.3), we have $\varpi(\lambda_n) = \varpi(\lambda_m) = 0$, hence by letting $\lambda \to \lambda_n$ ($\overline{\mu} \neq \lambda_n$) and then $\mu \to \lambda_m$ we obtain (1.4). ∎

Now we collect some easy facts about inner products of eigenfunctions.

LEMMA 1.2. *If λ_n is a real eigenvalue then*

$$\|y_n\|_2^2 = -a y_n(1)^2 - y_n(1)\varpi'(\lambda_n). \tag{1.9}$$

Proof. Since $\varpi(\lambda_n) = 0$, we have $\varpi(\lambda)/(\lambda - \lambda_n) \to \varpi'(\lambda_n)$ as $\lambda \to \lambda_n$. Therefore, by letting $\mu \to \lambda_n$ ($\lambda \neq \lambda_n$) and then $\lambda \to \lambda_n$ in (1.8) we obtain (1.9). ∎

COROLLARY 1.1. *If λ_k is a multiple eigenvalue then*

$$\|y_k\|_2^2 = -a y_k(1)^2. \tag{1.10}$$

An immediate corollary of (1.4) is the following

COROLLARY 1.2. *If λ_r is a non-real eigenvalue then*

$$\|y_r\|_2^2 = -a|y_r(1)|^2. \tag{1.11}$$

Proof. Since $\lambda_r \neq \overline{\lambda}_r$, (1.11) follows at once from (1.4) by replacing λ_n, λ_m by λ_r. ∎

For the eigenfunction y_n define

$$B_n = \|y_n\|_2^2 + a|y_n(1)|^2. \tag{1.12}$$

The following corollary of (1.9) and (1.11) will be useful (cf. [1, Theorem 4.3]).

COROLLARY 1.3. *$B_n \neq 0$ if and only if the corresponding eigenvalue λ_n is real and simple.*

If λ_k is a multiple (double or triple) eigenvalue ($\lambda_k = \lambda_{k+1}$) then $B_k = -y_k(1)\omega'(\lambda_k) = 0$ and B_{k+1} is not defined, so we set $B_{k+1} = -y_k(1)\omega''(\lambda_k)/2$. If λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$) then $B_{k+1} = 0$ and B_{k+2} is not defined, so we set $B_{k+2} = -y_k(1)\omega'''(\lambda_k)/6$.

We conclude this section with the following

LEMMA 1.3. *If λ_r and $\lambda_s = \overline{\lambda}_r$ are a conjugate pair of non-real eigenvalues then*

$$(y_r, y_s) = -ay_r(1)^2 - y_r(1)\varpi'(\lambda_r). \tag{1.13}$$

The proof is similar to the proof of (1.9). We also note that $\varpi'(\lambda_r) \neq 0$ in (1.13) since all non-real eigenvalues of (0.1)–(0.3) are simple.

2. Inner products and norms of associated functions. We shall need the results of this and subsequent sections only for real eigenvalues, so throughout these sections we assume that all the eigenvalues are real.

If λ_k is a double eigenvalue ($\lambda_k = \lambda_{k+1}$) then for the associated function y_{k+1} corresponding to the eigenfunction y_k, the following relations hold:

$$\begin{aligned} &-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k, \\ &y'_{k+1}(0)\sin\beta = y_{k+1}(0)\cos\beta, \\ &y'_{k+1}(1) = (a\lambda_k + b)y_{k+1}(1) + ay_k(1). \end{aligned}$$

If λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$) then together with the associated function y_{k+1} there exists a second associated function y_{k+2} for which

$$\begin{aligned} &-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1}, \\ &y'_{k+2}(0)\sin\beta = y_{k+2}(0)\cos\beta, \\ &y'_{k+2}(1) = (a\lambda_k + b)y_{k+2}(1) + ay_{k+1}(1). \end{aligned}$$

The following well known properties of associated functions play an important role in our investigation. The functions $y_{k+1} + cy_k$ and $y_{k+2} + dy_k$, where c and d are arbitrary constants, are also associated functions of the first and second order respectively. Next we observe that if we replace the associated function y_{k+1} by $y_{k+1} + cy_k$, then the associated function y_{k+2} changes to $y_{k+2} + cy_{k+1}$. For a fuller discussion of the theory of associated functions see [14, Ch. I, §2].

From (0.1), (0.2) and (1.1) we obtain

$$\begin{aligned}
&-y''_\lambda + q(x)y_\lambda = \lambda y_\lambda + y,\\
&y'_\lambda(0)\sin\beta = y_\lambda(0)\cos\beta,\\
&\varpi'(\lambda) = y'_\lambda(1) - (a\lambda + b)y_\lambda(1) - ay(1),
\end{aligned}$$

where the subscript denotes differentiation with respect to λ.

Let λ_k be a multiple (double or triple) eigenvalue of (0.1)–(0.3). Since $\varpi(\lambda_k) = \varpi'(\lambda_k) = 0$ it follows that $y(x,\lambda) \to y_k$ and $y_\lambda(x,\lambda) \to \widetilde{y}_{k+1}$ as $\lambda \to \lambda_k$, where $\widetilde{y}_{k+1} = y_{k+1} + \widetilde{c}y_k$ is an associated function of the first order, and $\widetilde{c} = (\widetilde{y}_{k+1}(1) - y_{k+1}(1))/y_k(1)$.

Similarly, we may write

$$\begin{aligned}
&-y''_{\lambda\lambda} + q(x)y_{\lambda\lambda} = \lambda y_{\lambda\lambda} + 2y_\lambda,\\
&y'_{\lambda\lambda}(0)\sin\beta = y_{\lambda\lambda}(0)\cos\beta,\\
&\varpi''(\lambda) = y'_{\lambda\lambda}(1) - (a\lambda + b)y_{\lambda\lambda}(1) - 2ay_\lambda(1).
\end{aligned}$$

We note again that if λ_k is a triple eigenvalue of (0.1)–(0.3) then $\varpi''(\lambda_k) = 0$, hence $y_{\lambda\lambda} \to 2\widetilde{y}_{k+2}$ as $\lambda \to \lambda_k$, where $\widetilde{y}_{k+2} = y_{k+2} + \widetilde{c}y_{k+1} + \widetilde{d}y_k$ is an associated function of the second order corresponding to the first associated function $\widetilde{y}_{k+1}$, and $\widetilde{d} = (\widetilde{y}_{k+2}(1) - y_{k+2}(1) - \widetilde{c}y_{k+1}(1))/y_k(1)$. We shall use the fact that the functions $y(x,\lambda)$, $y_\lambda(x,\lambda)$, $y_{\lambda\lambda}(x,\lambda)$ are continuous in both x and λ (see [8, Ch. 3, §4]). So, differentiation and subsequent limit passages in the integrals below are meaningful.

Lemma 2.1. *If λ_k is a multiple eigenvalue and $\lambda_n \neq \lambda_k$ then*

$$(y_{k+1}, y_n) = -ay_{k+1}(1)y_n(1). \tag{2.1}$$

Proof. Differentiating (1.8) with respect to λ we obtain

$$\begin{aligned}
(y_\lambda(\cdot,\lambda), y(\cdot,\mu)) = -ay_\lambda(1,\lambda)y(1,\mu) + y_\lambda(1,\lambda)\,\frac{\varpi(\mu)}{\lambda-\mu}\\
- y(1,\lambda)\,\frac{\varpi(\mu)}{(\lambda-\mu)^2} - y(1,\mu)\,\frac{\varpi'(\lambda)}{\lambda-\mu} + y(1,\mu)\,\frac{\varpi(\lambda)}{(\lambda-\mu)^2}.
\end{aligned} \tag{2.2}$$

Letting $\mu \to \lambda_n$ ($\lambda \neq \lambda_n$) and then $\lambda \to \lambda_k$ in (2.2) we obtain $(\widetilde{y}_{k+1}, y_n) = -a\widetilde{y}_{k+1}(1)y_n(1)$. We note that $\widetilde{y}_{k+1} = y_{k+1} + \widetilde{c}y_k$. Therefore,

$$(y_{k+1}, y_n) + \widetilde{c}(y_k, y_n) = -ay_{k+1}(1)y_n(1) - a\widetilde{c}y_k(1)y_n(1).$$

Combining this with $(y_k, y_n) = -ay_k(1)y_n(1)$ we obtain (2.1). ∎

Lemma 2.2. *If λ_k is a multiple eigenvalue then*

$$(y_{k+1}, y_k) = -a y_{k+1}(1) y_k(1) - y_k(1) \frac{\varpi''(\lambda_k)}{2}. \tag{2.3}$$

Proof. Letting $\mu \to \lambda_k$ ($\lambda \neq \lambda_k$) and then $\lambda \to \lambda_k$ in (2.2) we obtain

$$(\widetilde{y}_{k+1}, y_k) = -a \widetilde{y}_{k+1}(1) y_k(1) - y_k(1) \frac{\varpi''(\lambda_k)}{2}.$$

In analogy with the previous lemma, using (1.10), we obtain (2.3). ∎

Lemma 2.3. *If λ_k is a multiple eigenvalue then*

$$\begin{aligned} \|y_{k+1}\|_2^2 &= (y_{k+1}, y_{k+1}) \\ &= -a y_{k+1}(1)^2 - \widehat{y}_{k+1}(1) \frac{\varpi''(\lambda_k)}{2} - y_k(1) \frac{\varpi'''(\lambda_k)}{6}, \end{aligned} \tag{2.4}$$

where $\widehat{y}_{k+1} = y_{k+1} - \widetilde{c} y_k$.

Proof. Differentiating (2.2) with respect to μ we obtain

$$\begin{aligned} (y_\lambda(\cdot, \lambda), y_\mu(\cdot, \mu)) &= -a y_\lambda(1, \lambda) y_\mu(1, \mu) + y_\lambda(1, \lambda) \frac{\varpi'(\mu)}{\lambda - \mu} \\ &+ y_\lambda(1, \lambda) \frac{\varpi(\mu)}{(\lambda - \mu)^2} - y(1, \lambda) \frac{\varpi'(\mu)}{(\lambda - \mu)^2} - y(1, \lambda) \frac{2\varpi(\mu)}{(\lambda - \mu)^3} - y_\mu(1, \mu) \frac{\varpi'(\lambda)}{\lambda - \mu} \\ &- y(1, \mu) \frac{\varpi'(\lambda)}{(\lambda - \mu)^2} + y_\mu(1, \mu) \frac{\varpi(\lambda)}{(\lambda - \mu)^2} + y(1, \mu) \frac{2\varpi(\lambda)}{(\lambda - \mu)^3}. \end{aligned} \tag{2.5}$$

Letting $\mu \to \lambda_k$ ($\lambda \neq \lambda_k$) and then $\lambda \to \lambda_k$ we obtain

$$(\widetilde{y}_{k+1}, \widetilde{y}_{k+1}) = -a \widetilde{y}_{k+1}(1)^2 - \widetilde{y}_{k+1}(1) \frac{\varpi''(\lambda_k)}{2} - y_k(1) \frac{\varpi'''(\lambda_k)}{6}.$$

As in the previous lemmas, substituting $\widetilde{y}_{k+1} = y_{k+1} + \widetilde{c} y_k$, after some computations we get (2.4). ∎

Lemma 2.4. *If λ_k is a triple eigenvalue and $\lambda_n \neq \lambda_k$ then*

$$(y_{k+2}, y_n) = -a y_{k+2}(1) y_n(1). \tag{2.6}$$

Proof. Differentiating (2.2) with respect to λ we obtain

$$\begin{aligned} (y_{\lambda\lambda}(\cdot, \lambda), y(\cdot, \mu)) &= -a y_{\lambda\lambda}(1, \lambda) y(1, \mu) + y_{\lambda\lambda}(1, \lambda) \frac{\varpi(\mu)}{\lambda - \mu} - y_\lambda(1, \lambda) \frac{2\varpi(\mu)}{(\lambda - \mu)^2} \\ &+ y(1, \lambda) \frac{2\varpi(\mu)}{(\lambda - \mu)^3} - y(1, \mu) \frac{\varpi''(\lambda)}{\lambda - \mu} + y(1, \mu) \frac{2\varpi'(\lambda)}{(\lambda - \mu)^2} - y(1, \mu) \frac{2\varpi(\lambda)}{(\lambda - \mu)^3}. \end{aligned}$$

Letting $\lambda \to \lambda_k$ ($\mu \neq \lambda_k$) we obtain

$$\begin{aligned} (\widetilde{y}_{k+2}, y(\cdot, \mu)) = &- a \widetilde{y}_{k+2}(1) y(1, \mu) + \widetilde{y}_{k+2}(1) \frac{\varpi(\mu)}{\lambda_k - \mu} \\ &- \widetilde{y}_{k+1}(1) \frac{\varpi(\mu)}{(\lambda_k - \mu)^2} + y_k(1) \frac{\varpi(\mu)}{(\lambda_k - \mu)^3}. \end{aligned} \tag{2.7}$$

Letting $\mu \to \lambda_n$ gives $(\widetilde{y}_{k+2}, y_n) = -a\widetilde{y}_{k+2}(1)y_n(1)$, from which applying $\widetilde{y}_{k+2} = y_{k+2} + \widetilde{c}y_{k+1} + \widetilde{d}y_k$, $(y_k, y_n) = -ay_k(1)y_n(1)$ and (2.1) we obtain (2.6). ∎

LEMMA 2.5. *If λ_k is a triple eigenvalue then*

$$(y_{k+2}, y_k) = -ay_{k+2}(1)y_k(1) - y_k(1)\,\frac{\varpi'''(\lambda_k)}{6}. \tag{2.8}$$

Proof. Letting $\mu \to \lambda_k$ in (2.7) and applying (1.3) we obtain

$$(\widetilde{y}_{k+2}, y_k) = -a\widetilde{y}_{k+2}(1)y_k(1) - y_k(1)\,\frac{\varpi'''(\lambda_k)}{6}.$$

Similar to the previous lemma, using (2.3) and (1.3) yields (2.8). ∎

LEMMA 2.6. *If λ_k is a triple eigenvalue then*

$$(y_{k+2}, y_{k+1}) = -ay_{k+2}(1)y_{k+1}(1) - \widehat{y}_{k+1}(1)\,\frac{\varpi'''(\lambda_k)}{6} - y_k(1)\,\frac{\varpi^{IV}(\lambda_k)}{24}. \tag{2.9}$$

Proof. Differentiating (2.7) with respect to μ we obtain

$$\begin{aligned}(\widetilde{y}_{k+2}, y_\mu(\cdot,\mu)) = &- a\widetilde{y}_{k+2}(1)y_\mu(1,\mu)\\ &+ \widetilde{y}_{k+2}(1)\,\frac{\varpi'(\mu)}{\lambda_k-\mu} + \widetilde{y}_{k+2}(1)\,\frac{\varpi(\mu)}{(\lambda_k-\mu)^2}\\ &- \widetilde{y}_{k+1}(1)\,\frac{\varpi'(\mu)}{(\lambda_k-\mu)^2} - \widetilde{y}_{k+1}(1)\,\frac{2\varpi(\mu)}{(\lambda_k-\mu)^3}\\ &+ y_k(1)\,\frac{\varpi'(\mu)}{(\lambda_k-\mu)^3} + y_k(1)\,\frac{3\varpi(\mu)}{(\lambda_k-\mu)^4}.\end{aligned} \tag{2.10}$$

Letting $\mu \to \lambda_k$, after simplifications we obtain (2.9). ∎

LEMMA 2.7. *If λ_k is a triple eigenvalue then*

$$\begin{aligned}\|y_{k+2}\|_2^2 = &- ay_{k+2}(1)^2\\ &- \widehat{y}_{k+2}(1)\,\frac{\varpi'''(\lambda_k)}{6} - \widehat{y}_{k+1}(1)\,\frac{\varpi^{IV}(\lambda_k)}{24} - y_k(1)\,\frac{\varpi^{V}(\lambda_k)}{120},\end{aligned} \tag{2.11}$$

where $\widehat{y}_{k+2} = y_{k+2} - \widetilde{c}\,\widehat{y}_{k+1} - \widetilde{d}y_k$.

Proof. Differentiating (2.10) with respect to μ we obtain

$$\begin{aligned}(\widetilde{y}_{k+2}, y_{\mu\mu}(\cdot,\mu)) = &-a\widetilde{y}_{k+2}(1)y_{\mu\mu}(1,\mu)\\ &+ \widetilde{y}_{k+2}(1)\,\frac{\varpi''(\mu)}{\lambda_k-\mu} + \widetilde{y}_{k+2}(1)\,\frac{2\varpi'(\mu)}{(\lambda_k-\mu)^2} + \widetilde{y}_{k+2}(1)\,\frac{2\varpi(\mu)}{(\lambda_k-\mu)^3}\\ &- \widetilde{y}_{k+1}(1)\,\frac{\varpi''(\mu)}{(\lambda_k-\mu)^2} - \widetilde{y}_{k+1}(1)\,\frac{4\varpi'(\mu)}{(\lambda_k-\mu)^3} - \widetilde{y}_{k+1}(1)\,\frac{6\varpi(\mu)}{(\lambda_k-\mu)^4}\\ &+ y_k(1)\,\frac{\varpi''(\mu)}{(\lambda_k-\mu)^3} + y_k(1)\,\frac{6\varpi'(\mu)}{(\lambda_k-\mu)^4} + y_k(1)\,\frac{12\varpi(\mu)}{(\lambda_k-\mu)^5}.\end{aligned}$$

Letting $\mu \to \lambda_k$, after elementary but lengthy computations, we obtain (2.11). ∎

3. Existence of auxiliary associated functions. In this section, we shall prove the existence of some associated functions which have the properties of an eigenfunction in inner products with original associated functions. In the proof of these results, we shall require some facts about the inner products of root functions, which have been gathered in Sections 1 and 2.

LEMMA 3.1. *If λ_k is a double eigenvalue then there exists an associated function of the form $y^*_{k+1} = y_{k+1} + c_1 y_k$, where c_1 is a constant, such that*

$$(y^*_{k+1}, y_{k+1}) = -a y^*_{k+1}(1) y_{k+1}(1). \tag{3.1}$$

Proof. Adding (2.4) to (2.3) multiplied by c_1 we obtain

$$\begin{aligned}(y_{k+1} + c_1 y_k, y_{k+1}) = &- a(y_{k+1}(1) + c_1 y_k(1)) y_{k+1}(1) \\ &- (\widehat{y}_{k+1}(1) + c_1 y_k(1)) \frac{\varpi''(\lambda_k)}{2} - y_k(1) \frac{\varpi'''(\lambda_k)}{6}.\end{aligned}$$

The equality (3.1) holds true if we take

$$c_1 = -\frac{y_k(1)\varpi'''(\lambda_k) + 3\widehat{y}_{k+1}(1)\varpi''(\lambda_k)}{3y_k(1)\varpi''(\lambda_k)}. \ ∎$$

Here, it should be pointed out that $y^*_{k+1}(1) = 0$ if and only if $\varpi'''(\lambda_k) = 3\widetilde{c}\varpi''(\lambda_k)$. We shall not need y^*_{k+1} in the triple eigenvalue case, but it is worthwhile to note that nothing of the kind exists if λ_k is a triple eigenvalue. Before proceeding, we also note that for $\lambda_n \neq \lambda_k$,

$$(y^*_{k+1}, y_n) = -a y^*_{k+1}(1) y_n(1), \tag{3.2}$$

$$(y^*_{k+1}, y_k) = -a y^*_{k+1}(1) y_k(1) - y_k(1) \frac{\varpi''(\lambda_k)}{2}. \tag{3.3}$$

We shall now concentrate on the triple eigenvalue case.

LEMMA 3.2. *If λ_k is a triple eigenvalue then there exist associated functions of the form $y^{**}_{k+1} = y_{k+1} + c_2 y_k$, $y^{**}_{k+2} = y_{k+2} + c_2 y_{k+1}$, where c_2 is a constant, such that*

$$(y^{**}_{k+1}, y_{k+2}) = -a y^{**}_{k+1}(1) y_{k+2}(1), \tag{3.4}$$

$$(y^{**}_{k+2}, y_{k+1}) = -a y^{**}_{k+2}(1) y_{k+1}(1). \tag{3.5}$$

Proof. The reasoning is very similar to that in the proof of Lemma 3.1, so we only sketch it. Adding (2.9) to (2.8) multiplied by c_2, and (2.9) to (2.4) multiplied by c_2, where

$$c_2 = -\frac{y_k(1)\varpi^{IV}(\lambda_k) + 4\widehat{y}_{k+1}(1)\varpi'''(\lambda_k)}{4y_k(1)\varpi'''(\lambda_k)},$$

we obtain (3.4) and (3.5), respectively. ∎

We now indicate some relations between y^{**}_{k+1}, y^{**}_{k+2} and other root functions:

$$(3.6)\qquad (y^{**}_{k+1}, y_n) = -ay^{**}_{k+1}(1)y_n(1) \quad (n \neq k+1, k+2),$$

$$(3.7)\qquad (y^{**}_{k+1}, y_{k+1}) = -ay^{**}_{k+1}(1)y_{k+1}(1) - y_k(1)\,\frac{\varpi'''(\lambda_k)}{6},$$

$$(3.8)\qquad (y^{**}_{k+2}, y_n) = -ay^{**}_{k+2}(1)y_n(1) \quad (n \neq k, k+1, k+2),$$

$$(3.9)\qquad (y^{**}_{k+2}, y_k) = -ay^{**}_{k+2}(1)y_k(1) - y_k(1)\,\frac{\varpi'''(\lambda_k)}{6}.$$

Since y^{**}_{k+1} and y^{**}_{k+2} are associated functions, the equalities (3.6) and (3.8) are obvious from (2.1) ((2.3) if $n = k$) and (2.6), respectively. By adding (2.4) and (2.8) to (2.3) multiplied by c_2, and applying (1.3), we obtain (3.7) and (3.9), respectively.

It is worthwhile to note that $y^{**}_{k+1}(1) = 0$ if and only if $\varpi^{IV}(\lambda_k) = 4\widetilde{c}\varpi'''(\lambda_k)$.

LEMMA 3.3. *If λ_k is a triple eigenvalue then there exists an associated function of the form $y^{\#}_{k+2} = y_{k+2} + d_1 y_k$, where d_1 is a constant, such that*

$$(3.10)\qquad (y^{\#}_{k+2}, y_{k+2}) = -ay^{\#}_{k+2}(1)y_{k+2}(1).$$

Proof. Adding (2.11) to (2.8) multiplied by d_1, where

$$d_1 = -\frac{y_k(1)\varpi^{V}(\lambda_k) + 5\widehat{y}_{k+1}(1)\varpi^{IV}(\lambda_k) + 20\widehat{y}_{k+2}(1)\varpi'''(\lambda_k)}{20y_k(1)\varpi'''(\lambda_k)},$$

we obtain (3.10). ∎

With the above notations, we also have

$$(3.11)\qquad (y^{\#}_{k+2}, y_n) = -ay^{\#}_{k+2}(1)y_n(1) \quad (n \neq k, k+1, k+2),$$

$$(3.12)\qquad (y^{\#}_{k+2}, y_k) = -ay^{\#}_{k+2}(1)y_k(1) - y_k(1)\,\frac{\varpi'''(\lambda_k)}{6}.$$

Indeed, by adding (2.6), the equality $(y_k, y_n) = -ay_k(1)y_n(1)$ multiplied by d_1, and (2.8) to (1.10) multiplied by d_1, we obtain (3.11) and (3.12), respectively.

LEMMA 3.4. *If λ_k is a triple eigenvalue then there exists an associated function of the form $y^{\#\#}_{k+2} = y^{**}_{k+2} + d_2 y_k$, where d_2 is a constant, such that*

$$(3.13)\qquad (y^{\#\#}_{k+2}, y_{k+1}) = -ay^{\#\#}_{k+2}(1)y_{k+1}(1),$$

$$(3.14)\qquad (y^{\#\#}_{k+2}, y_{k+2}) = -ay^{\#\#}_{k+2}(1)y_{k+2}(1).$$

Proof. By adding (3.5) to (2.3) multiplied by d_2, and applying (1.3), we obtain (3.13). Note that for (3.13) the value of d_2 is not important.

By adding (2.11) to (2.9) multiplied by c_2, we obtain

$$(y^{**}_{k+2}, y_{k+2}) = -a y^{**}_{k+2}(1) y_{k+2}(1) - Q_k,$$

where

$$Q_k = \widehat{y}_{k+2}(1)\,\frac{\varpi'''(\lambda_k)}{6} + \widehat{y}_{k+1}(1)\,\frac{\varpi^{IV}(\lambda_k)}{24} + y_k(1)\,\frac{\varpi^{V}(\lambda_k)}{120} + c_2\left(\widehat{y}_{k+1}(1)\,\frac{\varpi'''(\lambda_k)}{6} + y_k(1)\,\frac{\varpi^{IV}(\lambda_k)}{24}\right).$$

By adding this equality to (2.8) multiplied by d_2, where

$$d_2 = -\frac{6Q_k}{y_k(1)\varpi'''(\lambda_k)},$$

we obtain (3.14). ∎

Note also that, for $y^{\#\#}_{k+2}$, the counterparts of (3.11), (3.12) are true:

(3.15) $$(y^{\#\#}_{k+2}, y_n) = -a y^{\#\#}_{k+2}(1) y_n(1) \quad (n \neq k, k+1, k+2),$$

(3.16) $$(y^{\#\#}_{k+2}, y_k) = -a y^{\#\#}_{k+2}(1) y_k(1) - y_k(1)\,\frac{\varpi'''(\lambda_k)}{6}.$$

These follow from (3.8) and (3.9), respectively.

We remark that $y^{\#\#}_{k+2}(1) = 0$ if and only if

$$5\varpi^{IV}(\lambda_k)(\varpi^{IV}(\lambda_k) - 4\widetilde{c}\varpi'''(\lambda_k)) = 4\varpi'''(\lambda_k)(\varpi^{V}(\lambda_k) - 20\widetilde{d}\varpi'''(\lambda_k)).$$

4. Minimality of the system of root functions. We discuss various cases. In each case we determine the explicit form of a biorthogonal system.

Case (a).

Theorem 4.1. *If all the eigenvalues of* (0.1)–(0.3) *are real and simple then the system*

(4.1) $$\{y_n\} \quad (n = 0, 1, \ldots; n \neq l),$$

where l is any non-negative integer, *is minimal in* $L_2(0,1)$.

Proof. It suffices to show the existence of a system (see Theorem 2 in [9, Ch. I, §2])

(4.2) $$\{u_n\} \quad (n = 0, 1, \ldots; n \neq l),$$

biorthogonal to (4.1). We define

(4.3) $$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y_l(1)} y_l(x)}{B_n}.$$

It remains to note that, by (1.4), (1.9) and (1.12),

(4.4) $$(u_n, y_m) = \delta_{nm},$$

where δ_{nm} $(n, m = 0, 1, \ldots; n, m \neq l)$ is Kronecker's symbol. ∎

CASE (b).

THEOREM 4.2. *If λ_k is a double eigenvalue then the system*

$$\{y_n\} \quad (n = 0, 1, \ldots; n \neq k + 1)$$

is minimal in $L_2(0, 1)$.

Proof. In this case, the biorthogonal system is defined by $(n \neq k, k + 1)$

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y_k(1)} y_k(x)}{B_n}, \tag{4.5}$$

$$u_k(x) = \frac{y_{k+1}(x) - \frac{y_{k+1}(1)}{y_k(1)} y_k(x)}{B_{k+1}}.$$

Using (1.4), (1.9), (1.10), (1.12), (2.1), (2.3) one can easily verify (4.4) for $n, m = 0, 1, \ldots$ $(n, m \neq k + 1)$. ■

THEOREM 4.3. *If λ_k is a double eigenvalue, and if $y^*_{k+1}(1) \neq 0$, then the system*

$$\{y_n\} \quad (n = 0, 1, \ldots; n \neq k) \tag{4.6}$$

is minimal in $L_2(0, 1)$.

Proof. The elements of the biorthogonal system are defined as follows $(n \neq k, k + 1)$:

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y^*_{k+1}(1)} y^*_{k+1}(x)}{B_n}, \quad u_{k+1}(x) = \frac{y_k(x) - \frac{y_k(1)}{y^*_{k+1}(1)} y^*_{k+1}(x)}{B_{k+1}}.$$

The relation (4.4) for $n, m \neq k$ follows from (1.4), (1.9), (1.12), (2.1), (2.3), (3.1), (3.2). ■

REMARK 4.3. Before proceeding we comment on the condition $y^*_{k+1}(1) \neq 0$ above. Let $y^*_{k+1}(1) = 0$. Then by (3.1), (3.2) the function y^*_{k+1} is orthogonal to all the elements of the system (4.6). Therefore this system is not complete (cf. [13, Theorem 3]) in $L_2(0.1)$. It is not minimal either. Indeed, otherwise using the method of [10–12] and the asymptotic formula (0.4), we could prove that (4.6) is a basis in $L_2(0, 1)$, which contradicts its incompleteness.

THEOREM 4.4. *If λ_k is a double eigenvalue then the system*

$$\{y_n\} \quad (n = 0, 1, \ldots; n \neq l),$$

where $l \neq k, k + 1$ is a non-negative integer, is minimal in $L_2(0, 1)$.

Proof. The biorthogonal system is given by (4.3) for $n \neq k, k + 1$, and

$$u_{k+1}(x) = \frac{y_k(x) - \frac{y_k(1)}{y_l(1)} y_l(x)}{B_{k+1}}, \quad u_k(x) = \frac{y^*_{k+1}(x) - \frac{y^*_{k+1}(1)}{y_l(1)} y_l(x)}{B_{k+1}}.$$

The relation (4.4) for $n, m \neq l$ follows from (1.4), (1.9), (1.10), (1.12), (2.1), (2.3), (3.1)–(3.3). ■

Case (c).

Theorem 4.5. *If λ_k is a triple eigenvalue then the system*

$$\{y_n\} \quad (n = 0, 1, \ldots; n \neq k+2)$$

is minimal in $L_2(0,1)$.

Proof. The biorthogonal system is given by (4.5) for $n \neq k, k+1, k+2$, and

$$u_{k+1}(x) = \frac{y_{k+1}(x) - \frac{y_{k+1}(1)}{y_k(1)} y_k(x)}{B_{k+2}}, \qquad u_k(x) = \frac{y^{**}_{k+2}(x) - \frac{y^{**}_{k+2}(1)}{y_k(1)} y_k(x)}{B_{k+2}}.$$

The relation (4.4) for $n, m \neq k+2$ follows from the above mentioned results of Sections 1 and 2, and formulas (3.5), (3.8), (3.9). ■

Theorem 4.6. *If λ_k is a triple eigenvalue, and if $y^{**}_{k+1}(1) \neq 0$, then the system*

$$\{y_n\} \quad (n = 0, 1, \ldots; n \neq k+1) \tag{4.7}$$

is minimal in $L_2(0,1)$.

Proof. In this case, the elements of the biorthogonal system are ($n \neq k, k+1, k+2$)

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y^{**}_{k+1}(1)} y^{**}_{k+1}(x)}{B_n},$$

$$u_{k+2}(x) = \frac{y_k(x) - \frac{y_k(1)}{y^{**}_{k+1}(1)} y^{**}_{k+1}(x)}{B_{k+2}},$$

$$u_k(x) = \frac{y^{\#}_{k+2}(x) - \frac{y^{\#}_{k+2}(1)}{y^{**}_{k+1}(1)} y^{**}_{k+1}(x)}{B_{k+2}}. \tag{4.8}$$

The relation (4.4) for $n, m \neq k+1$ can be verified using the above mentioned results of Sections 1 and 2, and formulas (3.4), (3.6), (3.10)–(3.12). ■

Using the reasoning of Remark 4.3, we can show that if $y^{**}_{k+1}(1) = 0$ then y^{**}_{k+1} is orthogonal to all elements of (4.7); hence the system (4.7) is neither complete nor minimal.

Theorem 4.7. *If λ_k is a triple eigenvalue, and if $y^{\#\#}_{k+2}(1) \neq 0$, then the system*

$$\{y_n\} \quad (n = 0, 1, \ldots; n \neq k) \tag{4.9}$$

is minimal in $L_2(0,1)$.

Proof. We define, for $n \neq k, k+1, k+2$,

$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y^{\#\#}_{k+2}(1)} y^{\#\#}_{k+2}(x)}{B_n},$$

$$u_{k+2}(x) = \frac{y_k(x) - \frac{y_k(1)}{y^{\#\#}_{k+2}(1)} y^{\#\#}_{k+2}(x)}{B_{k+2}}, \quad u_{k+1}(x) = \frac{y_{k+1}(x) - \frac{y_{k+1}(1)}{y^{\#\#}_{k+2}(1)} y^{\#\#}_{k+2}(x)}{B_{k+2}}.$$

The relation (4.4) for $n, m \neq k$ follows from the results of Sections 1 and 2, and formulas (3.13)–(3.15). ■

Note that, again, for $y^{\#\#}_{k+2}(1) = 0$, the system (4.9) is neither complete nor minimal.

THEOREM 4.8. *If λ_k is a triple eigenvalue then the system*

$$\{y_n\} \quad (n = 0, 1, \dots; n \neq l),$$

where $l \neq k, k+1, k+2$ is a non-negative integer, is minimal in $L_2(0,1)$.

Proof. The elements of the biorthogonal system can be represented by (4.3) for $n \neq k, k+1, k+2,\ l$, and by

$$u_{k+2}(x) = \frac{y_k(x) - \frac{y_k(1)}{y_l(1)} y_l(x)}{B_{k+2}},$$

$$u_{k+1}(x) = \frac{y^{**}_{k+1}(x) - \frac{y^{**}_{k+1}(1)}{y_l(1)} y_l(x)}{B_{k+2}}, \quad u_k(x) = \frac{y^{\#\#}_{k+2}(x) - \frac{y^{\#\#}_{k+2}(1)}{y_l(1)} y_l(x)}{B_{k+2}}.$$

The relation (4.4) for $n, m \neq l$ follows from the results of Sections 1 and 2 and formulas (3.4), (3.6), (3.7), (3.13)–(3.16). ■

CASE (d).

THEOREM 4.9. *If λ_r and $\lambda_s = \overline{\lambda}_r$ are a conjugate pair of non-real eigenvalues then each of the systems*

(4.10) $$\{y_n\} \quad (n = 0, 1, \dots; n \neq r),$$

(4.11) $$\{y_n\} \quad (n = 0, 1, \dots; n \neq l),$$

where $l \neq r, s$ is a non-negative integer, is minimal in $L_2(0,1)$.

Proof. The biorthogonal system for (4.10) is as follows ($n \neq r, s$):

(4.12)
$$u_n(x) = \frac{y_n(x) - \frac{y_n(1)}{y_s(1)} y_s(x)}{B_n},$$

$$u_s(x) = \frac{y_r(x) - \frac{y_r(1)}{y_s(1)} y_s(x)}{-y_r(1)\varpi'(\lambda_r)}.$$

The equality (4.4) for $n, m \neq r$ can be verified using (1.4), (1.9), (1.11)–(1.13).

The biorthogonal system for (4.11) is defined by (4.3) for $n \neq r, s$, by (4.12), and

$$u_r(x) = \frac{y_s(x) - \frac{y_s(1)}{y_r(1)} y_r(x)}{-y_s(1)\varpi'(\lambda_s)}. \ \blacksquare$$

In conclusion, we note that in some cases it is possible to define the elements of the biorthogonal system in a different way. For example the element (4.8) of the biorthogonal system of (4.7) can be replaced by

$$u_k(x) = \frac{y_{k+2}^{\#\#}(x) - \frac{y_{k+2}^{\#\#}(1)}{y_{k+1}^{**}(1)} y_{k+1}^{**}(x)}{B_{k+2}}.$$

But using the equality $d_2 = d_1 + c_2^2$, which is easily verified, we can show that this representation coincides with (4.8). This observation agrees with the well known fact that the biorthogonal system of a basis is unique.

5. Example. Let us illustrate the above theory by a particular result for the problem (0.5), (0.6). It was noted in [13] that if $a = -1$ then $\lambda_0 = \lambda_1 = 0$ is a double eigenvalue and the eigenvalues $0 < \lambda_2 < \lambda_3 < \cdots$ are solutions of the equation $\tan\sqrt{\lambda} = \sqrt{\lambda}$. Eigenfunctions are $y_0 = 1$, $y_n = \cos\sqrt{\lambda_n}x$ ($n \geq 2$) and an associated function corresponding to y_0 is $y_1 = -\frac{1}{2}x^2 + c$, where c is an arbitrary constant. We look for an auxiliary associated function in the form $y_1^* = -\frac{1}{2}x^2 + c'$. That is, $c_1 = c' - c$. By (3.1),

$$\int_0^1 \left(-\frac{1}{2}x^2 + c\right)\left(-\frac{1}{2}x^2 + c'\right) dx = \left(-\frac{1}{2} + c\right)\left(-\frac{1}{2} + c'\right).$$

From this equality we obtain $c' = -c + \frac{3}{5}$, so $y_1^*(1) = c - \frac{1}{10}$. Therefore the above condition $y_1^*(1) = 0$ in Theorem 4.3 is equivalent to $c = \frac{1}{10}$. This result coincides with [13, Theorem 3] if we note that the definition of the first associated function in [13] differs from ours in sign.

We shall now indicate another approach to this problem. Note that $y(x, \lambda) = \cos\sqrt{\lambda}x$ is a solution of (0.5), satisfying the first boundary condition in (0.6), hence $y_\lambda(x, \lambda) = -\frac{x\sin\sqrt{\lambda}x}{2\sqrt{\lambda}}$. In particular, $\widetilde{y}_1 = \lim_{\lambda\to 0} y_\lambda(x, \lambda)$ $= -x^2/2$. Let $y_1 = -\frac{1}{2}x^2 + c$. Then $\widetilde{c} = -c$. Note also that $\varpi(\lambda) = \lambda\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda}$, and consequently

$$\varpi''(0) = \lim_{\lambda\to 0} \varpi''(\lambda) = -2/3, \qquad \varpi'''(0) = \lim_{\lambda\to 0} \varpi'''(\lambda) = 1/5.$$

As was pointed out in the comments following the proof of Lemma 3.1, the condition $y_1^*(1) = 0$ is equivalent to $\varpi'''(\lambda_k) = 3\widetilde{c}\varpi''(\lambda_k)$, from which we obtain, once again, $c = \frac{1}{10}$. These calculations are in perfect agreement with our result stated in Theorem 4.3.

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