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ON AUSLANDER–REITEN TRANSLATES IN FUNCTORIALLY FINITE SUBCATEGORIES AND APPLICATIONS

BY

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Abstract. We consider functorially finite subcategories in module categories over Artin algebras. One main result provides a method, in the setup of bounded derived categories, to compute approximations and the end terms of relative Auslander–Reiten sequences. We also prove an Auslander–Reiten formula for the setting of functorially finite subcategories. Furthermore, we study the category of modules filtered by standard modules for certain quasi-hereditary algebras and we classify precisely when this category has finite type. The class of these algebras contains all blocks of Schur algebras $S(2, r)$.

1. Introduction. Functorially finite subcategories in module categories over Artin algebras have been introduced by Auslander and Smalø [2] to provide a convenient setting for existence of relative Auslander–Reiten sequences in subcategories.

Given a functorially finite subcategory, it is generally not well understood how to compute approximations, and the end terms of relative Auslander–Reiten sequences. In this paper, we present a general method, in the setting of bounded derived categories. This was partly inspired by [3], and it generalizes [9, Theorem 3]. Furthermore, we prove an Auslander–Reiten formula for the setting of functorially finite subcategories.

One context in which such categories occur naturally are the modules of a quasi-hereditary algebra which have a filtration by standard modules. Quasi-hereditary algebras were introduced by Cline, Parshall and Scott in the study of highest weight categories for semisimple Lie algebras and algebraic groups [4], and they were studied by Dlab and Ringel [9] and others. Let A be quasi-hereditary. Then the simple A-modules are labelled as L_i for i in some partially ordered set $(I, \leq)$. Of central importance are the standard modules, denoted by Δ_i, and the category $\mathcal{F}(\Delta)$ of modules which have a filtration where the quotients are standard modules. Then the category $\mathcal{F}(\Delta)$ is a functorially finite subcategory, and it has relative Auslander–Reiten sequences [18].

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Our original motivation was to understand this category, especially for Schur algebras. Modules for the Schur algebra $S(n,r)$ are the same as r-homogeneous polynomial modules for the general linear group $GL_n(K)$ over an infinite field K.

One would like to know when the quasi-hereditary algebra is Δ-finite, that is, when the category $\mathcal{F}(\Delta)$ has finitely many indecomposable modules, up to isomorphism. This question has also been considered in [6], [7] using techniques of vector space categories. It has also been solved for hereditary algebras (using any order on the simple modules) in [16] using quadratic forms.

By Auslander's Theorem, the algebra is Δ-finite if and only if the relative Auslander–Reiten quiver of $\mathcal{F}(\Delta)$ has a finite component, and if so this component contains all indecomposables of the subcategory. It is this approach that we pursue here.

When A is a Schur algebra, for some blocks it was shown in [12] that $\mathcal{F}(\Delta)$ has finite type. Furthermore, it was established in [5] that Schur algebras $S(2,r)$ are not Δ-finite unless r is small. In the last part of this paper we completely classify blocks of Schur algebras $S(2,r)$ which are Δ-finite.

Our result is more general; we deal with classes of quasi-hereditary algebras defined explicitly by quivers and relations (see 3.1.2), over arbitrary characteristic. We call these algebras $\Lambda_{p,i}$ where p is an odd integer > 1, and $i \leq 4$. We show that the algebra $\Lambda_{p,i}$ is Δ-finite if and only if either $i \leq 2$, or $i = 3$ and $p \leq 7$. Suppose B is a block of some Schur algebra $S(2,r)$ over a field of characteristic p (now p is a prime), and assume B has k simple modules. We show that for $p = 2$, B is Δ-finite if and only $k \leq 4$. Furthermore, for $p > 2$, the block B is Δ-finite if and only if either B has $\leq p+2$ simple modules, or B has $p+3$ simple modules *and* $p \leq 7$. We think this is surprising: the general theory and past results on Schur algebras $S(2,r)$ do not have any situation where such a condition on the characteristic occurs, for odd characteristic.

The tools used in the last part are similar to those used in [12]; unfortunately the general methods of Section 2 are not suitable for the detailed calculations required in Section 3.

2. General methods. Let A be a finite-dimensional k-algebra (where k is a field) and let T be a tilting A-module. Let $B = \operatorname{End}_A(T)^{\mathrm{op}}$. In this situation there is a well-known equivalence of triangulated categories

$$G_T = \mathbb{R}\operatorname{Hom}_A(T,-) \colon \mathcal{D}^b(A) \to \mathcal{D}^b(B)$$

between the bounded derived categories of A and B respectively. The module T is an A-B-bimodule, and a quasi-inverse of G_T is given by $F_T = T \otimes^{\mathbb{L}}_B - \colon \mathcal{D}^b(B) \to \mathcal{D}^b(A)$. The B-module $T^* = \operatorname{Hom}_k(T,k)$ is a cotilting B-module.

The equivalence above restricts to an equivalence between certain module categories. Define

$$\mathcal{Y} = {}_AT^{\perp} = \{N \in A\text{-mod} \mid \mathrm{Ext}^i_A(T,N) = 0,\ i > 0\},$$
$$\mathcal{Z} = {}^{\perp}_{B}T^* = \{M \in B\text{-mod} \mid \mathrm{Ext}^i_B(M,T^*) = 0,\ i > 0\}.$$

We get an equivalence $\mathrm{Hom}_A(T,-)\colon \mathcal{Y} \to \mathcal{Z}$ with quasi-inverse $T \otimes_B -$.

$$\begin{array}{ccc} \mathcal{D}^b(A) & \xrightarrow{\mathbb{R}\,\mathrm{Hom}_A(T,-)} & \mathcal{D}^b(B) \\ \uparrow & & \uparrow \\ \mathcal{Y} & \xrightarrow{\mathrm{Hom}_A(T,-)} & \mathcal{Z} \end{array}$$

In this paper our interest will lie in the following special case. Suppose A is a quasi-hereditary algebra and T the canonical tilting module. Let $B = \mathrm{End}_A(T)^{\mathrm{op}}$, a quasi-hereditary algebra that is called the *Ringel dual* of A. Then $\mathcal{Y} = \mathcal{F}_A(\nabla)$ and $\mathcal{Z} = \mathcal{F}_B(\Delta)$, where $\mathcal{F}_A(\nabla)$ and $\mathcal{F}_B(\Delta)$ denote the categories of modules with a filtration in costandard A-modules and standard B-modules respectively (see [18] or Section 3.1.1).

2.1. Approximations. Due to a theorem by Kleiner and Perez [14], relative almost split sequences can be found by approximating absolute ones. Approximations of modules are defined as follows.

Definition 1. Suppose $\mathcal{X}$ is a subcategory of a category $\mathcal{C}$, and let M be an object of $\mathcal{C}$. A morphism $f\colon X^M \to M$ with $X^M \in \mathcal{X}$ is called a *right $\mathcal{X}$-approximation* of M (or an *$\mathcal{X}$-precover*) if for any morphism $g\colon X \to M$ with $X \in \mathcal{X}$ there is a morphism $h\colon X \to X^M$ such that $g = fh$.

$$\begin{array}{ccc} & & X \\ & \overset{h}{\swarrow} & \downarrow g \\ X^M & \xrightarrow{f} & M \end{array}$$

It is called a *minimal right $\mathcal{X}$-approximation* (or an *$\mathcal{X}$-cover*) if for any morphism $g\colon X^M \to X^M$ satisfying $fg = f$,

$$\begin{array}{ccc} X^M & & \\ {\scriptstyle g}\downarrow & \overset{f}{\searrow} & \\ & & M \\ & \overset{f}{\nearrow} & \\ X^M & & \end{array}$$

the morphism g is an isomorphism. The subcategory $\mathcal{X}$ is called *contravariantly finite* if every object in $\mathcal{C}$ has a right $\mathcal{X}$-approximation. *Left $\mathcal{X}$-approximations* (also called *$\mathcal{X}$-preenvelopes*), *minimal left $\mathcal{X}$-approximations* (also called *$\mathcal{X}$-envelopes*) and *covariantly finite subcategories* are defined dually.

The subcategory $\mathcal{X}$ is called *functorially finite* if it is both contravariantly finite and covariantly finite.

According to [2], a functorially finite and extension closed subcategory has relative Auslander–Reiten sequences. The extension closed subcategories $\mathcal{Y}$ and $\mathcal{Z}$ above are functorially finite according to [1], and hence they have relative Auslander–Reiten sequences. We next state the theorem by Kleiner and Perez.

THEOREM 2 ([14, 5.4]). *Let $\mathcal{X}$ be a contravariant and extension closed subcategory of A-mod. Let $Z \in \mathcal{X}$ be indecomposable and not Ext-projective and let $\eta\colon 0 \to \tau Z \to E \to Z \to 0$ be an Auslander–Reiten sequence in A-mod. Then there exists a unique (up to isomorphism) exact commutative diagram*

$$\begin{array}{ccccccccc} \epsilon\colon 0 & \longrightarrow & X^{\tau Z} & \longrightarrow & X^{E} & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \| & & \\ \eta\colon 0 & \longrightarrow & \tau Z & \longrightarrow & E & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

where $u\colon X^{\tau Z} \to \tau Z$ and $v\colon X^{E} \to E$ are minimal right $\mathcal{X}$-approximations of τZ and E respectively.

The exact sequence ϵ is a direct sum of a relative almost split sequence $0 \to \tau_{\mathcal{X}} Z \to E' \to Z \to 0$ and a split sequence $0 \to Y = Y \to 0 \to 0$ with Y Ext-injective.

Despite the name *functorially finite subcategory*, approximations are usually not functorial. What is lacking is *uniqueness*: a minimal right $\mathcal{X}$-approximation $X^M \to M$ is unique up to (a non-unique!) isomorphism, but the map $X \to X^M$ in the definition is not necessarily unique. For instance, the subcategory $\mathcal{Z}$ is coreflective, that is, the inclusion functor $\mathrm{inc}\colon \mathcal{Z} \to B$-mod has a right adjoint, only in the trivial case when $\mathcal{Z} = B$-mod, in other words only when T^* is injective. Nevertheless, it is always possible to find right $\mathcal{Z}$-approximations by using the following procedure.

Computing $\mathcal{Z}$-approximations. Suppose M is an object in B-mod. The following steps will produce a right $\mathcal{Z}$-approximation of M.

1. Find a projective B-resolution of M, call it $\mathbf{P}_M$.
2. Put $\mathbf{T}_M = \mathbf{T} \otimes_B \mathbf{P}_M$, a complex of tilting A-modules. Note that $\mathbf{T}_M \simeq F(M)$.
3. Make an injective A-resolution of the complex $\mathbf{T}_M$, call it $\mathbf{I}^\bullet(\mathbf{T}_M)$.
4. Let $N = Z^0(\mathbf{I}^\bullet(\mathbf{T}_M))$ and let f be the map $f\colon N \to \mathbf{I}^\bullet(\mathbf{T}_M)$. We prove below that $N \in \mathcal{Y}$.
5. Let $g\colon \mathrm{Hom}_A(T, N) \to M$ be the composition

$$\mathrm{Hom}_A(T,N) \xrightarrow{\sim} G_T(N) \xrightarrow{G_T f} G_T(\mathbf{I}^\bullet(\mathbf{T}_M)) \xrightarrow{\sim} G_T F_T(M) \xrightarrow{\sim} M.$$

THEOREM 3.

(a) *The module* $N = Z^0(\mathbf{I}^\bullet(\mathbf{T}_M))$ *is an object in* $\mathcal{Y}$.

(b) *The map* $g\colon \operatorname{Hom}_A(T, N) \to M$ *is a right* $\mathcal{Z}$*-approximation.*

Proof. (a) Suppose

$$\mathbf{I}^\bullet(\mathbf{T}_M) = \cdots \to I^{-2} \to I^{-1} \to I^0 \to I^1 \to \cdots$$

and let

$$K = \cdots \to I^{-2} \to I^{-1} \to 0 \to 0 \to \cdots,$$
$$L = \cdots \to 0 \to 0 \to I^0 \to I^1 \to \cdots.$$

Then there is a triangle

$$K[-1] \to L \to \mathbf{I}^\bullet(\mathbf{T}_M) \to K.$$

We have $\operatorname{Hom}_{\mathcal{D}^b(A)}(T, K[i]) = 0$ for $i \geq 0$ and

$$\operatorname{Hom}_{\mathcal{D}^b(A)}(T, \mathbf{I}^\bullet(\mathbf{T}_M)[i]) \simeq \operatorname{Hom}_{\mathcal{D}^b(B)}(B, M[i]) \simeq \operatorname{Ext}^i_B(B, M) = 0$$

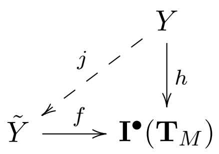
for $i > 0$, so $\operatorname{Hom}_{\mathcal{D}^b(A)}(T, L[i]) = 0$ for $i > 0$. But $\mathbf{I}^\bullet(\mathbf{T}_M)$ has homology only in non-positive degrees, so L is quasi-isomorphic to $N = Z^0(\mathbf{I}^\bullet(\mathbf{T}_M))$. Therefore

$$\operatorname{Ext}^i_A(T, N) \simeq \operatorname{Hom}_{\mathcal{D}^b(A)}(T, L[i]) = 0$$

for $i > 0$, and it follows that $N \in \mathcal{Y}$.

(b) Since B-mod is a full subcategory of $\mathcal{D}^b(B)$, a map $g\colon X^M \to M$ is a right $\mathcal{Z}$-approximation of M considered as an object in B-mod if and only if it is a right $\mathcal{Z}$-approximation of M considered as an object in $\mathcal{D}^b(B)$, and since F_T is an equivalence, this is true if and only if $F_T g\colon F_T(X^M) \to F_T(M) \simeq \mathbf{I}^\bullet(\mathbf{T}_M)$ is a right $\mathcal{Y}$-approximation.

A morphism $f\colon \tilde{Y} \to \mathbf{I}^\bullet(\mathbf{T}_M)$ with $\tilde{Y} \in \mathcal{Y}$ is a right $\mathcal{Y}$-approximation if and only if for each morphism $h\colon Y \to \mathbf{I}^\bullet(\mathbf{T}_M)$ with $Y \in \mathcal{Y}$ there is a morphism $j\colon Y \to \tilde{Y}$ such that $h = fj$.

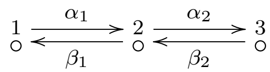


As observed in [13, Lemma 4.5], the map $f\colon Z^0(\mathbf{I}^\bullet(\mathbf{T}_M)) \to \mathbf{I}^\bullet(\mathbf{T}_M)$ has the factorisation property. From (a) we also have $Z^0(\mathbf{I}^\bullet(\mathbf{T}_M)) \in \mathcal{Y}$, so $f\colon N \to \mathbf{I}^\bullet(\mathbf{T}_M)$ is a right $\mathcal{Y}$-approximation. Since G_T is an equivalence, we deduce that $G_T f\colon G_T(N) \to G_T(\mathbf{I}^\bullet(\mathbf{T}_M))$ is a right $\mathcal{Z}$-approximation. The proposition follows. ∎

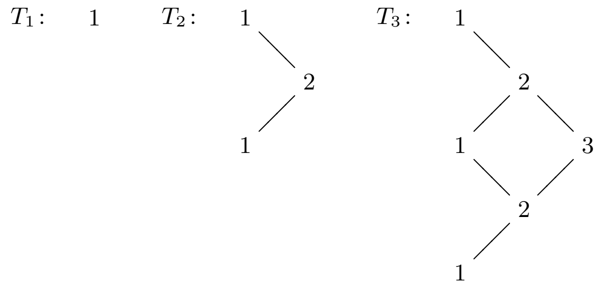
The non-functorial part of the above method is Step 4. The space of 0-cycles depends on which particular injective resolution was chosen in Step 3.

Extra injective summands might be produced, which end up as extra Ext-injective summands in the approximation. To get a minimal approximation, unneeded Ext-injective summands must be removed.

Example 4. We illustrate the method on a small example (see again Section 3.1.1 for the standard notation concerning quasi-hereditary algebras). Let A be the algebra $A = kQ/I$, where Q is the quiver

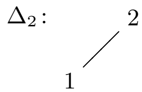


and I is the ideal generated by $\{\alpha_1\beta_1 + \beta_2\alpha_2, \alpha_2\beta_2\}$. This algebra is quasi-hereditary. The indecomposable summands of the canonical tilting module are as follows:



The algebra A is Ringel self-dual in such a way that $F_T(P_1) = T_3$, $F_T(P_2) = T_2$ and $F_T(P_3) = T_1$.

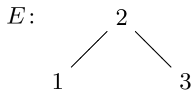
Consider the standard module



The Auslander–Reiten sequence in A-mod ending in Δ_2 is

$$0 \to S_3 \xrightarrow{j} E \xrightarrow{p} \Delta_2 \to 0,$$

where E is the module



We now want to find the relative Auslander–Reiten sequence ending in Δ_2. Using Theorem 2, it suffices to find a minimal right $\mathcal{Z}$-approximation of E. We apply our method. The module E has projective resolution

$$0 \to P_3 \to P_2 \to P_2 \to 0.$$

Tensoring with T we get the complex

$$0 \to T_1 \to T_2 \to T_2 \to 0.$$

This complex is quasi-isomorphic to

$$0 \to 0 \to \nabla_2 \to T_2 \to 0.$$

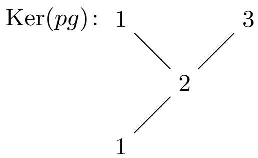
Taking a pushout with the map $i\colon \nabla_2 \to I_2$, we get another quasi-isomorphic complex

$$0 \to 0 \to I_2 \to T_2 \oplus I_3 \to 0,$$

so $N = T_2 \oplus I_3$. Since $G_T(N) = P_2 \oplus T_1$, we get a right $\mathcal{Z}$-approximation $g\colon P_2 \oplus T_1 \to E$. This approximation is minimal. From Theorem 2 we get the commutative diagram

$$\begin{array}{ccccccccc}
\epsilon\colon 0 & \longrightarrow & \mathrm{Ker}(pg) & \longrightarrow & P_2 \oplus T_1 & \longrightarrow & \Delta_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow g & & \| & & \\
\eta\colon 0 & \longrightarrow & S_3 & \xrightarrow{\;j\;} & E & \xrightarrow{\;p\;} & \Delta_2 & \longrightarrow & 0
\end{array}$$

The module $\mathrm{Ker}(pg)$ is given by



Since $\mathrm{Ker}(pg)$ is indecomposable, the sequence ϵ is a relative Auslander–Reiten sequence.

This general method can in principle always be used, but, as we will see in the next section, in concrete examples there are often more convenient ways of finding relative Auslander–Reiten sequences.

2.2. Auslander–Reiten formula in functorially finite subcategories coming from a cotilting module. As in the previous section, let A be a finite-dimensional algebra and $\mathcal{Z} := {}^{\perp}T$ be a functorially finite subcategory of A-mod where ${}_AT$ is a cotilting module. It is known ([1]) that in the case of algebras of finite global dimension, all functorially finite resolving subcategories have such an Ext-injective generator T, that is, all functorially finite subcategories which are closed under extensions and kernels of surjections and which contain all projective A-modules. We denote by $\tau_{\mathcal{Z}}M$ the relative Auslander–Reiten translate for $M \in \mathcal{Z}$, by τM the normal Auslander–Reiten translate in A-mod and by $\tau_{D^b(A)}M$ the Auslander–Reiten translate in the bounded derived category $D^b(A)$. For $M, N, K \in A$-mod, we let $\underline{\mathrm{Hom}}_K(M, N)$ denote the quotient of $\mathrm{Hom}_A(M, N)$ by the ideal $\mathrm{KHom}_A(M, N)$ of those morphisms which factor through $\mathrm{add}\, K$.

We first need the following technical lemma.

Lemma 5. *Let* $D \rightarrowtail B \overset{b}{\twoheadrightarrow} C$ *be a short exact sequence in* A-mod*, and* $K, N \in A$-mod *such that* $\mathrm{Hom}(N,B) \overset{b_*}{\to} \mathrm{Hom}(N,C)$ *and* $\mathrm{Hom}(K,B) \overset{b_*}{\to} \mathrm{Hom}(K,C)$ *are epimorphisms. Then the sequence*

$$\underset{K}{\underline{\mathrm{Hom}}}(N,D) \rightarrowtail \underset{K}{\underline{\mathrm{Hom}}}(N,B) \twoheadrightarrow \underset{K}{\underline{\mathrm{Hom}}}(N,C)$$

is also exact.

Proof. We start with the diagram

$$\begin{array}{ccccc}
 & & \mathrm{KHom}(N,B) & & \\
 & & \downarrow & & \\
\mathrm{Hom}(N,D) & \rightarrowtail & \mathrm{Hom}(N,B) & \overset{b_*}{\twoheadrightarrow} & \mathrm{Hom}(N,C) \\
 & & \downarrow & & \\
 & & \underset{K}{\underline{\mathrm{Hom}}}(N,B) & &
\end{array}$$

Since a morphism $N \to B$ which factors over K will still factor over K after composition with b, we obtain induced maps b_*^K and $\underline{b_*}$ from $\mathrm{KHom}(N,B)$ to $\mathrm{KHom}(N,C)$ and $\underset{K}{\underline{\mathrm{Hom}}}(N,B)$ to $\underset{K}{\underline{\mathrm{Hom}}}(N,C)$ respectively. We claim that b_*^K is onto. Let $f \in \mathrm{KHom}(N,C)$. Then f can be decomposed as $f : N \overset{u}{\to} K \overset{v}{\to} C$. But since $\mathrm{Hom}(K,B) \overset{b_*}{\twoheadrightarrow} \mathrm{Hom}(K,C)$, there exists a map $w \in \mathrm{Hom}(K,B)$ such that $v = b \circ w$. Hence $f = v \circ u = b \circ w \circ u$ and b_*^K is surjective. Now we have the diagram

$$\begin{array}{ccccc}
\mathrm{KHom}(N,D) & \rightarrowtail & \mathrm{KHom}(N,B) & \overset{b_*^K}{\twoheadrightarrow} & \mathrm{KHom}(N,C) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(N,D) & \rightarrowtail & \mathrm{Hom}(N,B) & \overset{b_*}{\twoheadrightarrow} & \mathrm{Hom}(N,C) \\
 & & \downarrow & & \downarrow \\
 & & \underset{K}{\underline{\mathrm{Hom}}}(N,B) & \overset{\underline{b_*}}{\twoheadrightarrow} & \underset{K}{\underline{\mathrm{Hom}}}(N,C)
\end{array}$$

where the monomorphisms in the upper left corner and the epimorphisms in the lower right corner follow trivially. The snake lemma then implies that $\mathrm{Ker}\,\underline{b_*}$ is indeed $\underset{K}{\underline{\mathrm{Hom}}}(N,D)$ as desired. ∎

Theorem 6. *The Auslander–Reiten formula holds in* $\mathcal{Z}$*, i.e. for any* $M, N \in \mathcal{Z}$*,*

$$\mathrm{Ext}^1(M,N)^* \cong \underset{T}{\underline{\mathrm{Hom}}}(N, \tau_{\mathcal{Z}} M).$$

Proof. Let $N \hookrightarrow T(N) \twoheadrightarrow N'$ be the first step in an $\operatorname{add} T$-resolution of $N \in \mathcal{Z}$ and let $M \in \mathcal{Z}$. Then we obtain an exact sequence

$$\begin{aligned}\operatorname{Hom}(N', \tau M) \hookrightarrow \operatorname{Hom}(T(N), \tau M) \to \operatorname{Hom}(N, \tau M) \xrightarrow{\omega} \operatorname{Hom}(M, N')^* \\ \to \operatorname{Hom}(M, T(N))^* \twoheadrightarrow \operatorname{Hom}(M, N)^*\end{aligned}$$

where the image of ω is $\underset{T}{\underline{\operatorname{Hom}}}(N, \tau M)$. Since $\operatorname{Ext}^1(M, T(N)) = 0$, the kernel of the sequence

$$\operatorname{Hom}(M, N')^* \to \operatorname{Hom}(M, T(N))^* \twoheadrightarrow \operatorname{Hom}(M, N)^*$$

is $\operatorname{Ext}^1(M, N)^*$, hence

$$\operatorname{Ext}^1(M, N)^* \cong \underset{T}{\underline{\operatorname{Hom}}}(N, \tau M). \tag{1}$$

We now wish to show that $\underset{T}{\underline{\operatorname{Hom}}}(N, \tau_{\mathcal{Z}} M) \cong \underset{T}{\underline{\operatorname{Hom}}}(N, \tau M)$ for $M \in \mathcal{Z}$. We proceed in two steps:

Claim 1. *$\underset{T}{\underline{\operatorname{Hom}}}(N, Z^{\tau M}) \cong \underset{T}{\underline{\operatorname{Hom}}}(N, \tau M)$ for $M \in \mathcal{Z}$, where $r : Z^{\tau M} \to \tau M$ is a right $\mathcal{Z}$-approximation of τM.*

Claim 2. $\underset{T}{\underline{\operatorname{Hom}}}(N, \tau_{\mathcal{Z}} M) \twoheadrightarrow \underset{T}{\underline{\operatorname{Hom}}}(N, \tau M)$.

Assuming both claims, the statement follows since

$$\underset{T}{\underline{\operatorname{Hom}}}(N, Z^{\tau M}) \cong \underset{T}{\underline{\operatorname{Hom}}}(N, \tau_{\mathcal{Z}} M) \oplus \underset{T}{\underline{\operatorname{Hom}}}(N, C)$$

for some $C \in \mathcal{Z}$ with $\operatorname{Ext}^1(M, C) = 0$. If this direct sum is isomorphic to $\underset{T}{\underline{\operatorname{Hom}}}(N, \tau_{\mathcal{Z}} M)$ and one summand already surjects onto it, then the other must be zero and the surjection in Claim 2 is an isomorphism.

Proof of Claim 1. From the definition of right approximation, we obtain a short exact sequence

$$\operatorname{Hom}(N, \operatorname{Ker} r) \hookrightarrow \operatorname{Hom}(N, Z^{\tau M}) \overset{r_*}{\twoheadrightarrow} \operatorname{Hom}(N, \tau M).$$

As $T \in \mathcal{Z}$, the same holds for T instead of N. Hence we can apply Lemma 5 to obtain an exact sequence

$$\underset{T}{\underline{\operatorname{Hom}}}(N, \operatorname{Ker} r) \hookrightarrow \underset{T}{\underline{\operatorname{Hom}}}(N, Z^{\tau M}) \overset{r_*}{\twoheadrightarrow} \underset{T}{\underline{\operatorname{Hom}}}(N, \tau M).$$

But r being a right approximation implies $\underset{T}{\underline{\operatorname{Hom}}}(N, \operatorname{Ker} r) = 0$, whence Claim 1 holds.

Proof of Claim 2. By Jørgensen [13], $\tau_{\mathcal{Z}} M$ is the right $\mathcal{Z}$-approximation of $\tau_{D^b(A)} M$, hence we get an epimorphism

$$\operatorname{Hom}(N, \tau_{\mathcal{Z}} M) \twoheadrightarrow \operatorname{Hom}_{D^b(A)}(N, \tau_{D^b(A)} M)$$

for every $N \in \mathcal{Z}$. But $\tau_{D^b(A)}M$ is $\nu P^\bullet[1]$ where $P^\bullet = \cdots \xrightarrow{f_2} P^2 \xrightarrow{f_1} P^1 \xrightarrow{f_0} P^0$ is a minimal projective resolution of M and ν is the Nakayama functor. Since $\mathcal{Z}$ is considered to be concentrated in degree zero, we have

$$\mathrm{Hom}_{D^b(A)}(N, \tau_{D^b(A)}M) \cong \mathrm{Hom}_A(N, \mathrm{Ker}\, \nu f_0)/J$$

where J is the ideal generated by those morphisms factoring over νf_1, hence over the injective νP^2. On the other hand, $\mathrm{Ker}\, \nu f_0 \cong \tau M$, so that $\mathrm{Hom}_A(N, \tau M)/J$ surjects onto $\underline{\mathrm{Hom}}_{A^*}(N, \tau M)$. Composing epimorphisms, we obtain an epimorphism

$$\mathrm{Hom}(N, \tau_{\mathcal{Z}} M) \twoheadrightarrow \underline{\mathrm{Hom}}_{A^*}(N, \tau M).$$

From the comparison of the usual Auslander–Reiten formula and (1), we see that for $N \in \mathcal{Z}$ we have

$$\underline{\mathrm{Hom}}_{A^*}(N, \tau M) \cong \underline{\mathrm{Hom}}_{T}(N, \tau M),$$

therefore Claim 2 holds and the result follows. ■

3. Δ-finiteness for $S(2, r)$

Preliminaries for $\mathcal{F}(\Delta)$

3.1.1. *Background.* Assume A is a finite-dimensional basic algebra over some field K, with simple modules L_i labelled by a set $I \subset \mathbb{Z}$, and let $\leq$ be the natural order on I. For $i \in I$ let P_i be the indecomposable projective module with simple quotient L_i. Recall that the standard module Δ_i is the largest quotient of P_i with composition factors L_j for $j \leq i$. Dually, the costandard module ∇_i is the largest submodule of the injective hull I_i with composition factors L_j for $j \leq i$. We assume A is quasi-hereditary with respect to the natural order on I. That is, for all $i \in I$ the simple module L_i occurs only once as a composition factor of Δ_i, and furthermore P_i has a Δ-filtration where Δ_i occurs once, and if Δ_j occurs then $j \geq i$.

Recall that indecomposable modules which have both a Δ-filtration and a ∇-filtration are labelled as T_i where i is maximal with $\Delta_i \subset T_i$. These are canonical tilting modules, which will be referred to simply as *tilting modules* in the following. These are the Ext-injective objects in $\mathcal{F}(\Delta)$, i.e. $\mathrm{Ext}^1(M, T_i) = 0$ for any M in $\mathcal{F}(\Delta)$.

Fix a set $\{e_i\}$ of orthogonal primitive idempotents such that $\sum e_i = 1_A$, and where $Ae_i = P_i$. For $i \in I$, let $f_i := \sum_{j>i} e_i$, and let

$$\bar{A}\ (= \bar{A}_{>i}) = A/Af_iA.$$

Then $\bar{A}$ is quasi-hereditary with respect to $\{j \in I : j \leq i\}$, with standard modules Δ_j for $j \leq i$. Furthermore, the algebra fAf is quasi-hereditary with respect to $\{j : j > i\}$, with standard modules $f\Delta_j$. We call $\bar{A}$ a *good* *quotient* of A, and fAf a *good subalgebra* of A. Clearly the category $\mathcal{F}(\Delta_{\bar{A}})$ of $\bar{A}$-modules with Δ-filtration is a subcategory of $\mathcal{F}(\Delta_A)$, the A-modules with Δ-filtration. Furthermore, the category $\mathcal{F}(\Delta_{fAf})$ is equivalent to a full subcategory of $\mathcal{F}(\Delta_A)$ (see [12, 2.4]). This implies that if A is Δ-finite then so are $\bar{A}$ and fAf.

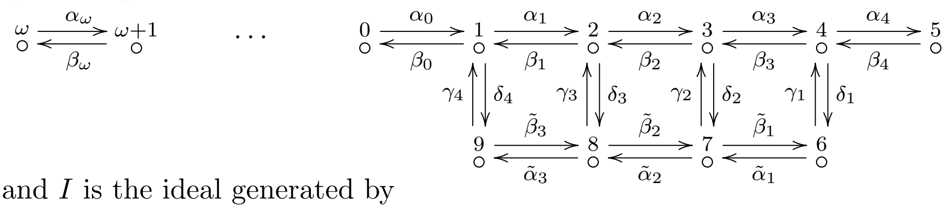
Remark 7. One general principle we will often use is the following. Suppose

$$0 \to \tau_\Delta(X) \to E \to X \to 0$$

is a relative Auslander–Reiten sequence of $\bar{A}$-modules. In general, if the relative Auslander–Reiten sequence of $\tau_\Delta(X)$ as a module for A ends in Y then $X = Y/Af_iX$. If then $\mathrm{Ext}^1(X, \Delta_j) = 0$ for $j > i$ then $X \cong Y$, so that the relative AR-sequence for $\bar{A}$ remains a relative Auslander–Reiten sequence for A (see [12, 3.1]). We will use this frequently, when i is the largest element in I.

3.1.2. *Algebras.* We will consider classes of quasi-hereditary algebras defined by quivers and relations. Take an odd integer p. Then the algebras will be $\Lambda_{p,i}$ for $i \le 4$. The presentation for $\Lambda_{p,4}$ and $p \ge 5$ is given in detail below. To obtain $\Lambda_{p,3}$ one factors out the ideal generated by the idempotent f_9 (with the notation of the previous section). Similarly to obtain $\Lambda_{p,2}$ one factors out the ideal generated by f_8, and the algebra $\Lambda_{p,1}$ is obtained by factoring out the ideal generated by f_7. This algebra $\Lambda_{p,1}$ appears in [12] and [10] (it is called $\mathcal{D}_{p+1}$); and $\Lambda_{p,0}$ is obtained by factoring out the ideal generated by f_6 (this is called $\mathcal{A}_p$ in [12] and [10]). We also define $\Lambda_{3,i} := f_2\Lambda_{5,i}f_2$ for $i = 1, 2, 3$, and similarly $\Lambda_{2,2} = f_3\Lambda_{5,2}f_3$.

Definition 8. The algebra $\Lambda_{p,4}$ is defined to be the algebra KQ/I where Q is the quiver

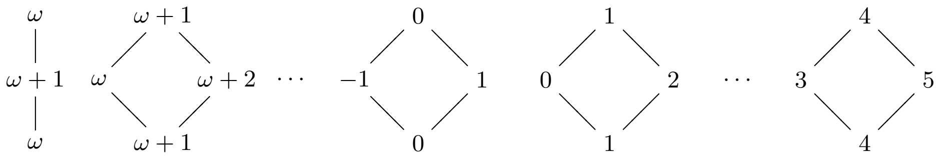


and I is the ideal generated by

$$\alpha_i\beta_i - \beta_{i+1}\alpha_{i+1}, \quad \tilde{\alpha}_i\tilde{\beta}_i - \tilde{\beta}_{i+1}\tilde{\alpha}_{i+1},$$
$$\alpha_i\alpha_{i-1}, \quad \beta_i\beta_{i+1}, \quad \tilde{\alpha}_{i+1}\tilde{\alpha}_i, \quad \tilde{\beta}_i\tilde{\beta}_{i+1},$$
$$\alpha_4\beta_4, \quad \tilde{\alpha}_3\tilde{\beta}_3,$$
$$\delta_i\gamma_i$$

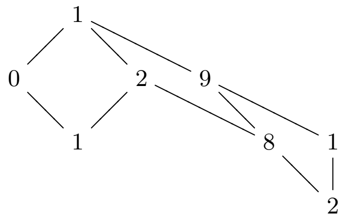
for all i where these expressions make sense, and all possible $\rho - \mu$ where ρ and μ are paths of length 2 around a square starting and ending in the same vertex. The quiver has $p + 4 = 5 + 2k + 4$ vertices, we label them from $\omega := -2k + 1, -2k + 2, \ldots, 7, 8, 9$.

We now describe the submodule structures of the projective modules and the tilting modules. The tilting module T_ω is just the simple module corresponding to ω. The tilting modules $T_{\omega+1}, \ldots, T_5$ are given by

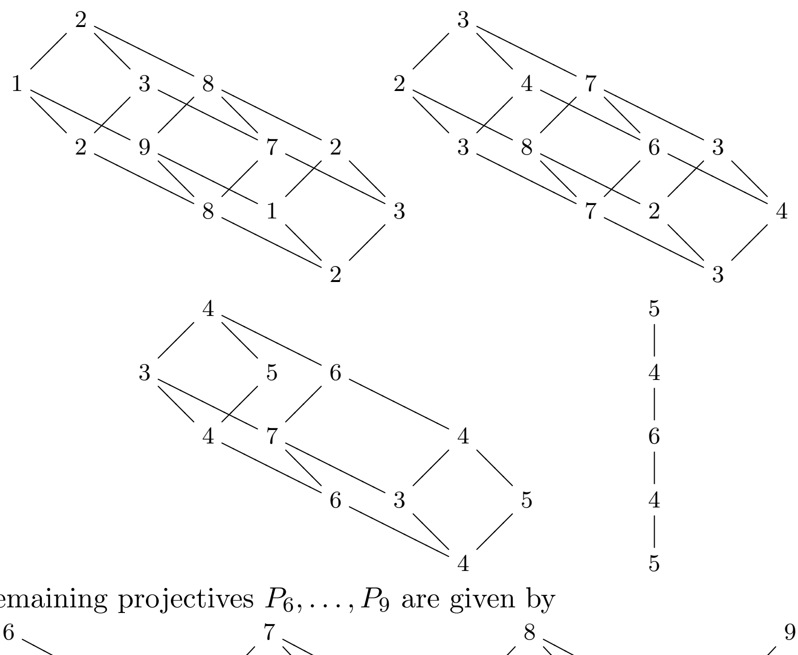


Up to the module with top 0 these are also the projectives $P_\omega, \ldots, P_0$.

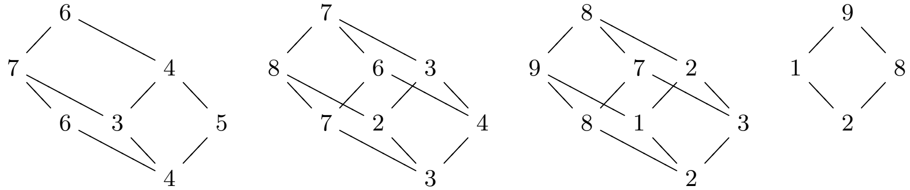
The projective P_1 has composition structure



The projectives $P_2 = T_9$, $P_3 = T_8$, $P_4 = T_7$, $P_5 = T_6$ have respective composition structures



The remaining projectives $P_6, \ldots, P_9$ are given by



3.1.3. *Blocks of Schur algebras.* Suppose B is a block of a Schur algebra $S(2,r)$ over a field F of characteristic p, with k simple modules. The Morita equivalence class of B is completely determined by the number k (see [11]). In fact, one can deduce from [11, 1.2] and [10] the following. If B, B' are blocks of Schur algebras $S(2,r_1)$ and $S(2,r_2)$, with k and k' simple modules respectively and $k < k'$, then B is Morita equivalent to a good quotient of B'.

Furthermore, a presentation of the basic algebra of B by a quiver and relations has been determined in [17], using work of [15] and of [8], and if B has finite type, already in [10].

For $k \leq p$ the algebra has finite type and is Morita equivalent to an algebra denoted as $\mathcal{A}_k$ in [10]. If $k = p+i$ for $1 \leq i \leq 4$ and $p \geq 5$ then B is Morita equivalent to $\Lambda_{p,i}$ (over characteristic p). When $k > p+4$ and $p \geq 5$ then B always has a good quotient which is Morita equivalent to $\Lambda_{p,4}$. If $p = 3$ and $k = 3+i \leq 6$ then B is Morita equivalent to $\Lambda_{3,i}$. If $p = 2$ and $k = 4$ then B is Morita equivalent to $\Lambda_{2,2}$.

3.1.4. *Main results.* We will classify Δ-finiteness for the algebras $\Lambda_{p,i}$. Our results are as follows.

THEOREM 9. *Assume p is an odd integer.*

(a) *The algebras $\Lambda_{p,i}$ for $i = 1,2$ are Δ-finite.*
(b) *The algebras $\Lambda_{p,3}$ are Δ-finite if and only if $p \leq 7$.*
(c) *The algebras $\Lambda_{p,4}$ are Δ-infinite for $p \geq 5$.*

COROLLARY 10. *Assume B is a block of $S(2,r)$ over characteristic p with k simple modules. Then B is Δ-finite if and only if one of the following holds:*

(i) *$p = 2$ and $k \leq 4$.*
(ii) *$p \leq 7$ and p odd, then $k \leq p+3$,*
(iii) *$p \geq 11$, then $k \leq p+2$.*

Proof of the Corollary. Most of this follows directly from the theorem and the description of the Morita equivalence classes of blocks in 3.1.3. We only need to establish Δ-infiniteness for $p = 2$ and $p = 3$. This follows from [5, 6.11]: there it is shown that $S(n,r)$ is Δ-infinite if either $p > 2$ and $r \geq 2p^2 + p - 2$, or if $p = 2$ and $d \geq 8$ (if d is even), $d \geq 17$ (if d is odd). The proof shows that, in each case, the block containing the simple module $L(r)$ is Δ-infinite. If $p = 2$, then $S(2,8)$ is one block, with five simple modules. If $p = 3$, the block of $S(2,19)$ which contains $L(19)$ has seven simple modules. ■

The proof of part (b) in Theorem 9 entails most of the work. As for Schur algebras, we were surprised to find this dependence on p.

3.2. The algebras $\mathcal{A}_p$ and $\Lambda_{p,1}$. Throughout, p is an odd integer. The characteristic of the field is arbitrary. We will describe the relative AR-quiver of $\mathcal{A}_p$ and of $\Lambda_{p,1}$; this will be the starting point for dealing with $\Lambda_{p,2}$.

3.2.1. *Algebras $\mathcal{A}_p$ for $p \geq 5$ odd.* These are of finite type, and the relative AR-quiver can be found in [19]. We write ω for the smallest weight, that is, $\omega = -(2k-1)$ if $p > 5$, and $\omega = 1$ for $p = 5$. Then the relative AR-quiver has an odd wing and an even wing. The odd wing is spanned by the Δ_r with r odd, and we have $\tau_\Delta(\Delta_r) = \Delta_{r+2}$ for $\omega \leq r < 5$.

The even wing is spanned by the Δ_r with r even, and $\tau_\Delta(\Delta_r) = \Delta_{r+2}$ for $\omega + 1 \leq r < 4$.

For each $\omega \leq r \leq s \leq 5$ of the same parity, there is a unique indecomposable module in the wing of this parity with Δ-quotients $\Delta_r, \Delta_{r+2}, \ldots, \Delta_s$. We denote this module by $W_{r,s}$. [If $r = s$ then $W_{r,s} = \Delta_r$.]

The two wings are connected by relative AR-sequences in which all projective-injective modules occur:

(1) The sequences which start at the odd wing and end at the even wing are

$$0 \to W_{\omega,5} \to W_{\omega,3} \oplus P_4 \to W_{4,6} \to 0,$$

and for $\omega + 2 \leq r \leq 3$ and r odd,

$$0 \to W_{\omega,r} \to W_{r+1,6} \oplus P_{r-1} \oplus W_{\omega,r-2} \to W_{r-1,6} \to 0.$$

(2) The sequences which start at the even wing and end at the odd wing are

$$0 \to W_{\omega+1,4} \to W_{\omega+1,2} \oplus P_3 \oplus \Delta_5 \to W_{3,5} \to 0,$$

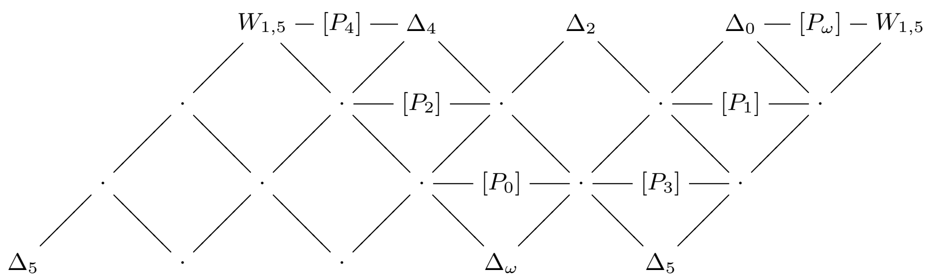
and for $\omega + 3 \leq r \leq 4$ and r even,

$$0 \to W_{\omega+1,r} \to W_{\omega+1,r-2} \oplus P_{r-1} \oplus W_{r+1,5} \to W_{r-1,5} \to 0.$$

The last one is

$$0 \to \Delta_{\omega+1} \to P_\omega \oplus W_{\omega+2,5} \to W_{\omega,5} \to 0.$$

For $p = 7$, we have $\omega = -1$, and the relative Auslander–Reiten quiver is given by



3.2.2. *The algebras $\Lambda_{p,1}$ for odd p.* These algebras are not of finite type but they are Δ-finite. The relative AR-quiver was determined in [12]. The general description is, roughly speaking, that one ray is inserted into the relative Auslander–Reiten quiver for $\mathcal{A}_p$, after the even wing and before the odd wing.

For r even and $\omega + 1 \leq r \leq 4$ we have a unique indecomposable module with Δ-quotients $\Delta_r, \Delta_{r+2}, \ldots, \Delta_6$, which we call $W_{r,6}$. The modules $W_{r,6}$, together with P_4, P_5 and Δ_6, are the only indecomposable modules with Δ-filtration which have Δ_6 as a quotient. Almost all of the relative Auslander–Reiten quiver of $\Lambda_{p,1}$ remains. The change is that the sequences for $\mathcal{A}_p$ described in (1) are replaced by the relative Auslander–Reiten sequences in which the $W_{r,6}$ occur. These are

$$0 \to \Delta_6 \to P_4 \to T_5 \to 0,$$
$$0 \to P_4 \to T_5 \oplus W_{4,6} \to \Delta_4 \to 0,$$
$$0 \to W_{\omega,5} \to P_4 \oplus W_{\omega,3} \to W_{4,6} \to 0,$$
$$0 \to W_{4,6} \to \Delta_4 \oplus W_{2,6} \to W_{2,4} \to 0,$$

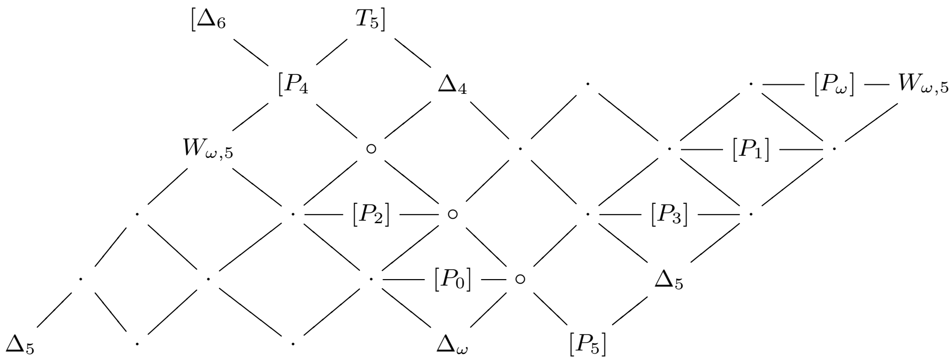
and for $\omega + 1 < r \leq 3$ and r even,

$$0 \to W_{\omega,r+1} \to W_{r+2,6} \oplus P_r \oplus W_{\omega,r-1} \to W_{r,6} \to 0,$$
$$0 \to W_{r,6} \to W_{r,4} \oplus W_{r-2,6} \to W_{r-2,4} \to 0.$$

Finally,

$$0 \to W_{\omega,\omega+2} \to W_{\omega+3,6} \oplus P_{\omega+1} \oplus \Delta_\omega \to W_{\omega+1,6} \to 0,$$
$$0 \to W_{\omega+1,6} \to W_{\omega+1,4} \oplus P_5 \to \Delta_5 \to 0.$$

Again for $p = 7$, the relative Auslander–Reiten quiver is given by

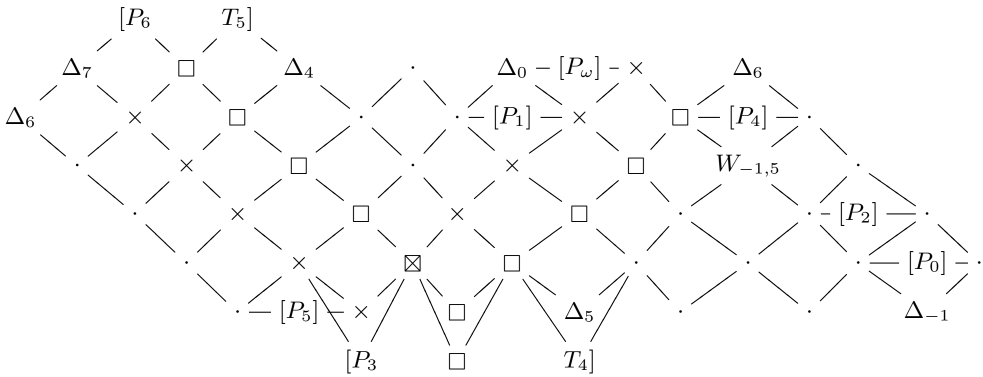


where the circles denote modules of the form $W_{r,6}$.

In the following we will use the term "extended odd wing" for the full subquiver consisting of the odd wing together with all relative Auslander–Reiten sequences ending in $W_{r,6}$.

3.3. Algebras $\Lambda_{p,2}$ for p odd. Let $\Gamma_{p,2}$ be the relative Auslander–Reiten quiver of $\Lambda_{p,2}$. We will prove the following.

THEOREM 11. *The quiver $\Gamma_{p,2}$ can be obtained from the relative Auslander–Reiten quiver of $\Lambda_{p,1}$ by inserting two rays connecting the even wing and the extended odd wing. The shape of the quiver $\Gamma_{p,2}$ is (in the example where $p = 7$)*



where crosses signify modules which are extensions of modules for $\Lambda_{p,1}$ with Δ_7, and boxes denote modules which are extensions of modules for $\Lambda_{p,1}$ with P_6.

Whenever we have a one-dimensional Ext-space between two modules M and N, we denote their extension by $M * N$. In particular, we will generally write modules for $\Lambda_{p,j}$ as $M * N$ where M factors over the quotient algebra $\mathcal{A}_p$ and all constituents in a Δ-filtration of N are of the form Δ_j for $j \geq 6$. In this notation, the modules $W_{r,6}$ from the previous section would be called $W_{r,4} * \Delta_6$.

Proof of Theorem 11. We apply Remark 7, using that $\Lambda_{p,1} \cong \Lambda_{p,2}/I$ where I is the ideal generated by f_7.

(1) We first observe that $\mathrm{Ext}^1(\Delta_j, \Delta_7) = 0$ for $j \leq 3$. This implies, by Remark 7 that the extended odd wing is a full subquiver of $\Gamma_{p,2}$. Similarly the even wing is a full subquiver of $\Gamma_{p,2}$. In particular, we have located the projectives P_r for $r \leq 2$ and r even.

(2) Some translates are easily found. We have

$$\tau_\Delta(T_5) = P_6, \quad \tau_\Delta(\Delta_4) = T_5 * P_6 \quad \tau_\Delta(T_5 * P_6) = \Delta_7.$$

Note that $T_5 * P_6$ is the middle term of the relative AR-sequence starting in P_6, so we have a part of the relative AR-quiver which contains one tilting module and one projective.

(3) We extend the even wing to the right, and this will also give the positions of the projective-injectives P_r with $r \leq 1$ odd.

LEMMA 12. *There are relative AR-sequences in* $\Lambda_{p,2}$

$$0 \to \Delta_{\omega+1} \to P_\omega \oplus (W_{\omega+2,5} * \Delta_7) \to W_{\omega,5} * \Delta_7 \to 0,$$

$$0 \to W_{\omega+1,k} \to W_{\omega+1,k-2} \oplus P_{k-1} \oplus W_{k-3,5} * \Delta_7 \to W_{k-1,5} * \Delta_7 \to 0$$

for $k = \omega + 3, \omega + 5, \ldots, 2$.

Proof. We use Remark 7, using the fact that $\Lambda_{p,1} \cong \Lambda_{p,2}/I$ where I is the ideal generated by f_7.

(a) Let $0 \to \Delta_{\omega+1} \to X \to Y \to 0$ be the relative AR-sequence in $\Lambda_{p,2}$. By Remark 7, Y is an extension of $W_{\omega,5}$ by a direct sum of copies of Δ_7 (or just $W_{\omega,5}$). One finds that $\mathrm{Ext}^1(W_{\omega,5}, \Delta_7) \cong K$, furthermore by inspection there is an indecomposable module of the form $W_{\omega,5} * \Delta_7$. So either Y is this module, or $Y = W_{\omega,5}$.

We must show that $Y = W_{\omega,5} * \Delta_7$. Assume (for a contradiction) that $Y = W_{\omega,5}$. Then the relative AR-sequence is the same as in $\Lambda_{p,1}$,

$$(\xi) \qquad 0 \to \Delta_{\omega+1} \to P_\omega \oplus W_{\omega+2,5} \xrightarrow{\phi} W_{\omega,5} \to 0.$$

There is an indecomposable module U which is an extension of $W_{\omega,5}$ by P_6, and an epimorphism $\pi : U \to W_{\omega,5}$. This must factor through ϕ, say $\pi = \phi \circ \gamma$. The simple module L_6 is not a composition factor of the middle in ξ. This implies that γ maps the submodule P_6 of U to zero. Since π is onto, it follows that the image of γ is isomorphic to $W_{\omega,5}$. But this means that the sequence ξ is split, which is a contradiction since it is a relative AR-sequence.

Now we have identified the end term of the sequence starting in $\Delta_{\omega+1}$. It follows that the middle is an extension of $P_\omega \oplus W_{\omega+2,5}$ by Δ_7. The module P_ω is projective and injective, so it is a direct summand. We have already found the position of Δ_7 and it is easy to deduce that the middle of the relative AR-sequence ending in $\Delta_{\omega+1}$ is $P_\omega \oplus Z$ where $Z = W_{\omega+2,5} * \Delta_7$.

(b) It follows now that for $k = \omega + 3$ we have $\tau_\Delta^{-1}(W_{k,\omega+1}) = Z$. The argument as in part (a) shows that the middle of the relative AR-sequence ending in $W_{k,\omega+1}$ is $\Delta_{\omega+1} \oplus P_{\omega+2} \oplus W_{\omega+4,5} * \Delta_7$. Similar arguments prove the rest of the lemma. ■

We now continue the proof of Theorem 11.

(4) The same reasoning shows that there is a relative AR-sequence

$$0 \to W_{\omega+1,6} \to P_5 \oplus W_{\omega+1,4} * P_6 \to \Delta_5 * \Delta_7 \to 0.$$

Note that $W_{\omega+1,6}$ is located at the right side of the extended odd wing. Note also that P_5 is projective-injective.

(5) We locate the remaining projectives and tilting modules. As for the position of P_4 ($= T_7$), one shows that there is a relative AR-sequence

$$0 \to W_{\omega,5} * P_6 \to \Delta_6 \oplus P_4 \oplus W_{\omega,5} \to T_5 * \Delta_6 \to 0.$$

Note that $W_{\omega,5}$ is at the top of the odd wing, and $T_5 * \Delta_6$ is at the top of the extended odd wing. [For $\Lambda_{p,1}$, the module $T_5 * \Delta_6$ is the projective P_4.]

Furthermore, it is easy to see that $\tau_\Delta(T_4) = \Delta_5 * P_6$ and $\tau_\Delta(\Delta_5 * P_6) = P_3$. The following shows that T_4 is linked to the odd wing.

LEMMA 13. *There is an indecomposable module* $M = (T_4 \oplus \Delta_5) * P_6$. *Furthermore, we have relative AR-sequences*

$$0 \to \Delta_5 * P_6 \to M \to T_4 \to 0, \tag{1}$$

$$0 \to T_4 * P_6 \to M \to \Delta_5 \to 0 \tag{2}$$

and $\mathrm{Ext}^1(\Delta_5, T_4 * P_6) = K$.

Proof. The module M is constructed by factoring out Δ_6 from the diagonal in $T_4 * P_6 \oplus P_5$. We can write down a non-split exact sequence as in (1). Furthermore, one checks that $\mathrm{Ext}^1(T_4, \Delta_5 * P_6) = K$. Since $\tau_\Delta(T_4) = \Delta_5 * P_6$ the sequence (1) must be the relative AR-sequence. Exactly the same reasoning shows that the relative AR-sequence ending in Δ_5 must be as given in (2). ∎

We have now located all projectives and all tilting modules. Therefore, anywhere else, the translation acts regularly, and this allows us to see easily how the parts found so far are connected.

(6) It is easy to see that $\tau_\Delta(\Delta_6) = W_{\omega,5} * \Delta_7$, which is in the extended even wing, has translate $\Delta_{\omega+1}$.

(7) We have $\tau_\Delta(T_5 * \Delta_7) = \Delta_6$, which is easy to see. Now, $T_5 * \Delta_7$ occurs in the middle of the relative AR-sequence starting with Δ_7, so we get a connection from the left side of the even wing to the right side of the odd wing.

The rest is now obtained by knitting.

3.4. Algebras $\Lambda_{p,3}$. Let $\Gamma_{p,3}$ be the relative Auslander–Reiten quiver of $\Lambda_{p,3}$. We have the following result.

THEOREM 14. *The algebra $\Lambda_{p,3}$ is Δ-finite if and only if $p \le 7$. For $p = 7$, the relative Auslander–Reiten quiver $\Gamma_{7,3}$ is given by Figure* 1.

Proof. The strategy is as follows. For $q < p$ and q odd, $\Lambda_{q,3}$ is a good subalgebra of $\Lambda_{p,3}$, and $\mathcal{F}_{\Lambda_{q,3}}(\Delta)$ is equivalent to a subcategory of $\mathcal{F}_{\Lambda_{p,3}}(\Delta)$ (see Remark 7). Hence if $\Lambda_{p,3}$ is Δ-finite then so is $\Lambda_{q,3}$; and if $\Lambda_{q,3}$ is Δ-infinite then so is $\Lambda_{p,3}$.

One shows that $\Lambda_{7,3}$ is Δ-finite by finding the complete relative Auslander–Reiten quiver. Furthermore, one shows that $\Lambda_{9,3}$ is Δ-infinite. To start, we assume $p \ge 5$, and one determines parts of $\Gamma_{p,3}$ in general, by the following steps.

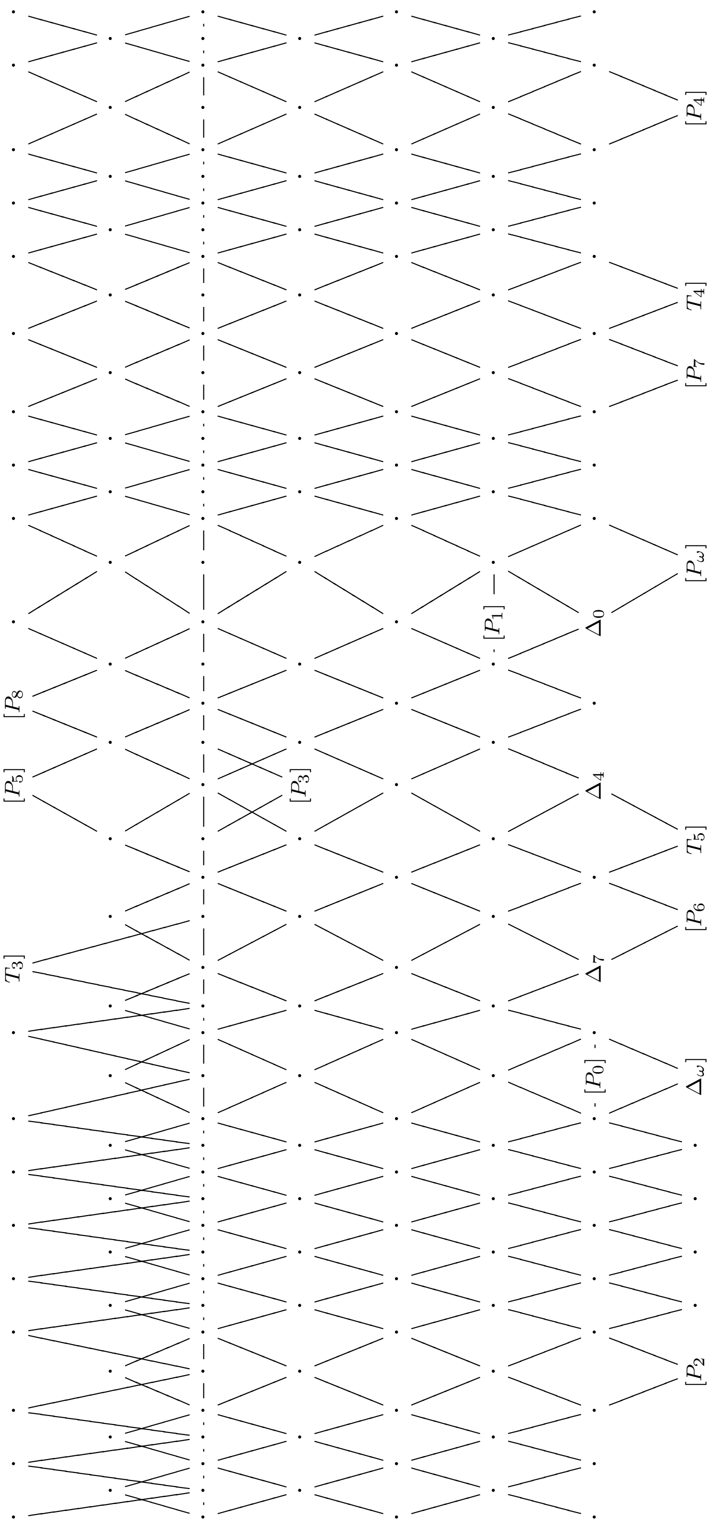


Fig. 1

(I) Extend $\Gamma_{p,2}$, in particular locate all P_r for $r \leq 1$.
(II) Locate the remaining projectives and tilting modules.
(III) Find a full sectional path.

When $p = 7$, we can then knit to deduce the graph structure of the complete $\Gamma_{p,3}$ and see that it is finite. On the other hand, when $p = 9$, we use knitting to show that it is infinite.

The proofs for parts (I) to (III) use methods similar to those in the previous section, and we will only give details where the arguments are different.

(I) Parts of the $\Gamma_{p,3}$ which come from $\Lambda_{p,2}$. One uses Remark 7. One checks extensions between standard modules Δ_j and Δ_8 for all j and obtains

$$\mathrm{Ext}^1(\Delta_j, \Delta_8) = \begin{cases} K, & j = 3, 6, 7, \\ 0, & \text{otherwise.} \end{cases}$$

(Ia) The even wing. In addition to the previous calculations of Exts between some Δ_j and Δ_8, we have $\mathrm{Ext}^1(W_{k,5} * \Delta_7, \Delta_8) = 0$ for $k \leq 5$ and k odd. These conditions imply:

Corollary 15. *The extended even wing in $\Gamma_{p,2}$ (from step (3) in the proof of Theorem 11) is a full subquiver of $\Gamma_{p,3}$. In particular, this part contains the projective-injectives P_k for k odd and $k \leq 1$. Furthermore, $\Gamma_{p,3}$ also contains the next path to the left, from P_6 to $W_{\omega+1,4} * P_6$.*

(Ib) Some sectional paths. The relative AR-sequence ending in Δ_4 from $\Lambda_{p,2}$ survives. The translate of Δ_4 is $T_5 * P_6$ and one checks that this satisfies $\mathrm{Ext}^1(T_5 * \Delta_6, \Delta_8) = 0$. So the relative AR-sequence ending in $T_5 * P_6$ also survives; this starts in Δ_7. The same argument shows that the two sectional paths in $\Gamma_{p,2}$ which are next to each other,

$$P_6 \to T_5 * P_6 \to \Delta_4 * P_6 \to W_{2,4} * P_6 \to \cdots \to W_{\omega+1,4} * P_6,$$
$$\Delta_7 \to T_5 * \Delta_7 \to \Delta_4 * \Delta_7 \to W_{2,4} * \Delta_7 \to \cdots \to W_{\omega+1,4} * \Delta_7,$$

both survive. This will be important later.

(Ic) The odd wing. By the Ext-calculations above most of the odd wing is unchanged, and is part of $\Gamma_{p,3}$. The exception is the left edge, that is, the modules $W_{k,5}$ for k odd, including Δ_5. We can however extend the wing spanned by Δ_r for $\omega \leq r \leq 3$ and r odd to the left as follows. It is easy to see that

$$\tau_\Delta(\Delta_3) = T_5 * P_7, \quad \tau_\Delta(T_5 * P_7) = \Delta_6, \quad \tau_\Delta(\Delta_6) = P_2.$$

If $0 \to P_2 \to M \to \Delta_6 \to 0$ is the relative AR-sequence then M is indecomposable and $\tau_\Delta(M) = W_{\omega,3} * \Delta_8$.

Furthermore, the relative AR-sequence which has T_6 ($= P_5$) as middle term, is as for $\Gamma_{p,2}$, namely

$$0 \to W_{\omega+1,6} \to T_6 \oplus Z \to \Delta_5 * \Delta_7 \to 0.$$

To see this, one computes $\mathrm{Ext}^1(\Delta_5 * \Delta_7, \Delta_8) = 0$.

We extend the odd wing to the right, and also locate the P_r even for $r \leq 0$. We have extensions $W_{r,6}$ and Δ_8 which we call $W_{r,4} * W_{6,8}$.

Lemma 16. *The relative AR-sequence starting with* $W_{\omega,k}$ *for* $\omega < k < 3$ *and* k *odd is of the form*

$$0 \to W_{\omega,k} \to W_{\omega,k-2} \oplus P_{k-1} \oplus W_{k+1,4} * W_{6,8} \to W_{k-1,4} * W_{6,8} \to 0.$$

Moreover, $\tau_\Delta(\Delta_7) = W_{\omega+1,4} * W_{6,8}$ *and hence* Δ_7 *is located to the bottom right of the extended odd wing.*

We can continue one more step, and we obtain

Lemma 17. *The relative AR-sequence starting with* $W_{\omega,3}$ *is of the form*

$$0 \to W_{\omega,3} \to W_{\omega,1} \oplus Y \to W_{2,4} * W_{6,8} \to 0$$

where $Y = (\Delta_4 \oplus T_3) * W_{6,8}$*, and* Y *is indecomposable.*

(II) The position of projectives and of tilting modules. We have already located all P_r for $r \leq 2$ and also P_5 ($= T_6$) and T_4, T_5 and P_6, Δ_ω. It remains to locate P_3, P_4, T_4, P_7, T_3 and Δ_8.

(IIa) The position of P_4 ($= T_7$)

Lemma 18. *There is a relative AR-sequence*

$$0 \to W_{\omega,5} * P_6 \to T_7 \oplus W_{\omega,5} * W_{6,8} \to T_5 * W_{6,8} \to 0.$$

Proof. From $\Lambda_{p,2}$ we have the relative AR-sequence

$$0 \to M = W_{\omega,5} * P_6 \to T_7 \oplus \Delta_6 \oplus W_{\omega,5} \to T_5 * \Delta_6 \to 0.$$

By Remark 7 we can deduce that $\tau_\Delta^{-1}(M)$ is an extension of $T_5 * \Delta_6$ by copies of Δ_8, or just $T_5 * \Delta_6$. Similarly to previous situations, one shows that the sequence is as stated. ∎

We also need to know the location in relation to the two wings when $p = 7$. First, there is an irreducible map from $W_{\omega,3} * \Delta_8$ to P_2, from the extension of the odd wing to the left. Next, we calculate translates of $W_{\omega,3}$: they are

$$W_{\omega+2,5}, \qquad \Delta_3 * P_7, \qquad T_5 * W_{6,8},$$

and hence we have reached the sequence in which T_7 occurs.

(IIb) The position of P_3 ($= T_8$)

Lemma 19. *There is a relative AR-sequence*

$$0 \to W_{\omega+1,4} * P_7 \to P_3 \oplus (W_{\omega+1,4} * \Delta_7) \to T_4 * \Delta_7 \to 0.$$

Furthermore, $P_5 \oplus W_{\omega+1,4} * \Delta_7$ *is the middle term of the relative AR-sequence ending in* $\Delta_5 * \Delta_7$ *which we had already identified in* (I).

The proof is similar to that of Lemma 11.

(IIc) The positions of T_4, P_7**.** It is easy to see that $\tau_\Delta(T_4) = P_7$ and the relative AR-sequence has indecomposable middle term. To locate these in $\Gamma_{p,3}$, we find a few translates,

$$\tau_\Delta(T_4 * P_7) = W_{6,8}, \quad \tau_\Delta(W_{6,8}) = W_{\omega,5} * \Delta_7, \quad \tau_\Delta(W_{\omega,5} * \Delta_7) = \Delta_{\omega+1},$$

and this shows that they connect with the even wing.

(IId) The positions of T_3, Δ_8**.** The only tilting modules and projective modules which have not yet been located are T_3 and Δ_8. We only need to know a few steps. By arguments similar to the ones used before we find

$$\tau_\Delta(T_3) = \Delta_4 * P_7, \quad \tau_\Delta(\Delta_4 * P_7) = T_5 * \Delta_6, \quad \tau_\Delta(T_5 * \Delta_6) = T_3 * P_7.$$

Furthermore, we find a few inverse translates of Δ_8. First, $\tau_\Delta(T_4 * P_6) = \Delta_8$:

Let $M = T_4 * P_6$. Then the usual translate is the uniserial module of length 2 with top composition factor L_8 and socle L_7.

Next, we have $\tau_\Delta(\Delta_5) = T_4 * P_6$. This comes from $\Gamma_{p,2}$ using the fact that Δ_5 does not have non-split extensions by Δ_8. Furthermore, $\tau_\Delta(\Delta_3 * \Delta_8) = \Delta_5$.

We must locate the position of Δ_8.

LEMMA 20. *There is an irreducible map* $\Delta_5 * \Delta_7 \to \Delta_8$.

Proof. Recall that the relative AR-sequence in $\Gamma_{p,2}$ starting with $\Delta_5 * \Delta_7$ is

$$(\zeta) \qquad 0 \to \Delta_5 * \Delta_7 \to U \xrightarrow{f} T_4 * P_6 \to 0$$

with U indecomposable (the module in the position ⊠). Let $0 \to \Delta_5 * \Delta_7 \to X \to Y \to 0$ be the relative AR-sequence in $\Gamma_{p,3}$; we want to show that $X = U \oplus \Delta_8$.

We show first that $Y \cong V$ where V is the middle term of the relative AR-sequence starting with Δ_8, which is $0 \to \Delta_8 \to V \xrightarrow{\pi} T_4 * P_6 \to 0$ and V is indecomposable (one can describe it as $T_4 * (W_{6,8} \oplus \Delta_7)$, via a construction as a pull-back).

One checks that $\mathrm{Ext}^1(T_4 * P_6, \Delta_8) = K$ and $\mathrm{Ext}^1(\Delta_5 * \Delta_7, \Delta_8) = 0$. Furthermore, one shows that $\mathrm{Ext}^1(U, \Delta_8) = 0$.

One then shows using Remark 7 that Y is an extension of $T_4 * P_6$ by copies of Δ_8. We also know that $\mathrm{Ext}^1(T_4 * P_6, \Delta_8) = K$ and hence the (unique) non-split extension must be the sequence in (2), with middle term V. So $Y \cong V$. Furthermore, X is an extension of U by Δ_8. Again by our Ext-calculation we deduce that $X = U \oplus \Delta_8$. ∎

(III) The full sectional path. In (I) we described a sectional path starting with Δ_7. The last term is $W_{\omega+1,4}*\Delta_7$. We also know that $W_{\omega+1,4}*\Delta_7$ is in the middle of a relative AR-sequence together with P_5 and this sequence ends with $\Delta_5 * \Delta_7$. Hence we continue with the sectional path

$$\cdots \to \Delta_5 * \Delta_7 \to \Delta_8$$

and we know that Δ_8 is the start of a relative AR-sequence with indecomposable middle term.

We also know that Δ_7 is close to the edge of the component; we can only extend the sectional path by $\Delta_7 \to P_6$, leading to a singular τ_Δ-orbit.

Proposition 21. *The algebra $\Lambda_{7,3}$ is Δ-finite.*

Proof. After having identified the location of all singular τ_Δ-orbits and knowing a full sectional path, we can proceed by knitting. For $p = 7$, doing this explicitly shows that the algebra is Δ-finite with the Auslander–Reiten quiver given by Figure 1. ∎

Proposition 22. *The algebra $\Lambda_{9,3}$ is Δ-infinite.*

Proof. As in the previous corollary, knowing a full sectional path as well as the location of all singular τ_Δ-orbits means we can again proceed by knitting. We obtain a full sectional path on the left of the wing spanned by the singular orbit $P_2, \ldots, \Delta_{-3}$, given by

$$\begin{array}{ccccccccccccccc} A_6 & \longrightarrow & A_5 & \longrightarrow & A_4 & \longrightarrow & A_3 & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & Y & \longrightarrow & B \\ &&&&&&&&&& \big\downarrow &&&& \\ &&&&&&&&&& X &&&& \end{array}$$

where

$$\begin{aligned} A_6 &= W_{-3,3} * \Delta_8, \\ A_5 &= W_{-3,3} * W_{6,8}, \\ A_4 &= (W_{-3,3} \oplus T_5) * (P_7 \oplus W_{6,8}), \\ A_3 &= (W_{-3,3} \oplus T_5 \oplus \Delta_3) * (P_7 \oplus W_{6,8}), \\ A_2 &= (W_{-3,3} \oplus T_5 \oplus W_{1,3}) * (P_7 \oplus W_{6,8}), \\ A_1 &= (W_{-3,3} \oplus T_5 \oplus W_{-1,3}) * (P_7 \oplus W_{6,8}), \end{aligned} \qquad \begin{aligned} X &= W_{-3,3} * P_7, \\ Y &= (W_{-3,3} \oplus T_5) * W_{6,8}, \\ B &= \Delta_4 * W_{6,8}. \end{aligned}$$

We denote their dimension vectors by $a_1, \ldots, a_6, x, y, b$. Simple arithmetic shows that the relative Auslander–Reiten translates of $A_1, \ldots A_6, X, Y, B$ have dimension vectors given by $a_1 + a_2 - x - b, a_1 + a_3 - x - b, \ldots, a_1 + a_6 - x - b, a_1 - x - b, a_1 - x, a_1 - b, y - b$ respectively. We collect this into the matrix

$$B := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & & & & & & & & \\ 0 & 1 & & & & & & & \\ 0 & & 1 & & & & & & \\ 0 & & & 1 & & & & & \\ 0 & & & & 1 & & & & \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & & & & & & & & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 \end{pmatrix}$$

where all other entries are zero. Given that we know the -3-components in $a_1, a_2, \ldots, a_5, a_6,\ x, y, b$ are $1,1,1,1,1,1,1,1,0$ respectively, the multiplicity of the simple -3 in $\tau^n a_1$ is given by $v \cdot B^n \cdot w$ where $v = (1,1,1,1,1,1,1,1,0)$ and $w = (1,0,0,0,0,0,0,0,0)^T$. The rational canonical form of B is

$$C := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

with base change matrix

$$Q := \begin{pmatrix} 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}$$

so that $Q^{-1}BQ = C$. Hence $v \cdot B^n \cdot w = v \cdot Q \cdot C^n \cdot Q^{-1} \cdot w$. We compute

$$\tilde{v} := v \cdot Q = (0, 1, -1, 1, 0, 0, 1, 0, 0)$$

and

$$\tilde{w} := Q^{-1} \cdot w = (1, 2, 2, 1, 1, 1, 0, -1, -1)^T.$$

Now write n as $n = 5k + j$ where $0 \leq j \leq 4$ and write

$$v \cdot Q \cdot C^n \cdot Q^{-1} \cdot w = v \cdot Q \cdot C^j \cdot C^{5k} \cdot Q^{-1} \cdot w.$$

By induction, it is easy to see that

$$C^{5k} \cdot Q^{-1} \cdot w \\ = (-(k-1), -2(k-1), -2(k-1), -(k-1), 1, k+1, 2k, 2k-1, k-1)^T.$$

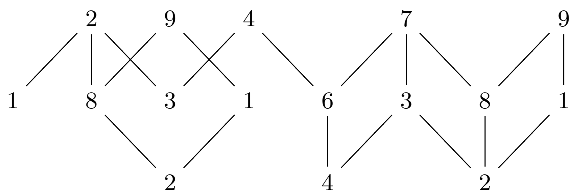
Now we compute $v \cdot Q \cdot C^j$ for $j = 0, \ldots, 4$ and then see that for $j = 0, \ldots, 3$ the final result for $v \cdot B^n \cdot w$ is $k + 1$, and for $j = 4$ it is $k + 2$. Hence the dimension of $\tau^n((W_{-3,3} \oplus W_{-1,3} \oplus T_5) * (P_7 \oplus W_{6,8}))$ becomes arbitrarily large as n increases, contradicting the assumption that the quiver is finite. ∎

This completes the proof of the theorem. ∎

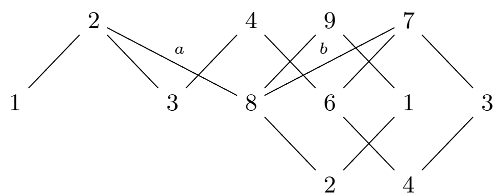
3.5. The algebras $\Lambda_{p,4}$. We first note that $\mathrm{Ext}_{\Lambda_{5,4}}(W_{2,4} * \Delta_7, \Delta_9)$ is two-dimensional. Then we can prove the following:

THEOREM 23. *$\Lambda_{p,4}$ is Δ-infinite.*

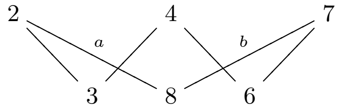
Proof. For $p = 5$, we need to check that the two-dimensional Ext-space indeed yields an infinite family. The module with two copies of Δ_9 has a composition structure given by



where the number j stands for the simple module with index j. Factoring out one copy of Δ_9 can be done with a choice of a and b in



Inequivalent $(a : b) \in \mathbb{P}^1(K)$ obviously produce non-isomorphic modules since the isomorphism classes of the factor modules



are already different for inequivalent $(a:b) \in \mathbb{P}^1(K)$.

Again, for $p > 5$, notice that there is an embedding of $\Lambda_{5,4}$ into $\Lambda_{p,4}$ which respects the quasi-hereditary structure, hence there is an embedding of the category of Δ-filtered $\Lambda_{5,4}$-modules into the category of Δ-filtered $\Lambda_{p,4}$-modules, proving the claim. ∎

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