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MINKOWSKI SUMS OF CANTOR-TYPE SETS

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Abstract. The classical Steinhaus theorem on the Minkowski sum of the Cantor set is generalized to a large class of fractals determined by Hutchinson-type operators. Numerous examples illustrating the results obtained and an application to t-convex functions are presented.

1. Introduction. The celebrated results of Steinhaus [13] and Piccard [10] state that if a set $S \subseteq \mathbb{R}^n$ is either large in the sense of Lebesgue measure theory (i.e., S is of positive Lebesgue measure) or large in the sense of Baire category (i.e., S is of the second Baire category), then the Minkowski sum $S+S := \{x+y : x, y \in S\}$ contains an interior point. An immediate question arises: Do there exist sets $S \subseteq \mathbb{R}^n$ of Lebesgue measure zero and of first Baire category such that $S+S$ contains an interior point? An example of such a set was discovered by Steinhaus [12] in 1917 who proved that, for the classical Cantor set $C \subseteq [0,1]$, the Minkowski sum $C+C$ is surprisingly large, namely

$$C + C = [0,2], \tag{1}$$

which is the same as the sum of the whole intervals $[0,1] + [0,1]$. The above equality can also be expressed in the form

$$\tfrac{1}{2}C + \tfrac{1}{2}C = [0,1], \tag{2}$$

which means that every point of the convex hull of C (i.e., the interval $[0,1]$) is a convex combination of two elements of C with coefficients $1/2$.

The Cantor set C is usually defined in the following way: We remove from the interval $[0,1]$ the open middle third, next we remove the open middle thirds of each of the remaining two intervals, and so on. The remaining set of points is the Cantor set. It is thinly scattered over $[0,1]$. Since the sum of the lengths of all the removed intervals equals

$$(1/3) + 2(1/3)^2 + 2^2(1/3)^3 + \cdots = 1,$$

the Cantor set has Lebesgue measure zero and is nowhere dense and hence it is of first Baire category. In the analysis of the Cantor set, the following

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characterization is very useful: $C \subseteq \mathbb{R}$ is the unique nonempty compact set satisfying the Hutchinson-type identity

$$C = \tfrac{1}{3}C + \{0, 2/3\}.$$

Sets satisfying similar identities are called *fractals* (cf. [1]). The aim of this paper is to generalize the Steinhaus theorem to the class of fractals that are determined by more general Hutchinson-type identities. Our main results characterize those fractals F for which a convex combination $\lambda_1 F + \cdots + \lambda_m F$, with prescribed coefficients $\lambda_1, \ldots, \lambda_m$, is equal to the convex hull of F. Finally, we present numerous examples illustrating the results obtained and give an application to the theory of t-convex functions.

2. The Steinhaus theorem. The starting point of our investigations is the following theorem due to Steinhaus [12].

THEOREM 1. *For the Cantor set C, the identity $C + C = [0, 2]$ holds.*

In this section we will present three short proofs of the Steinhaus theorem. The first one bases on the original geometric method used by Steinhaus (slightly modified by Utz [14]). The second one, which is algebraic, is due to Randolph [11] (see also [7, p. 19]). The third one, proposed by us, is purely set-theoretical. Our aim is to extend the last idea to fractals determined by certain Hutchinson-type operators.

Proof 1. Given an arbitrary closed square $K \subset \mathbb{R}^2$ with sides parallel to the coordinate axes, denote by K_i, $i = 1, 2, 3, 4$, the squares obtained by removing from K the open middle third horizontal and vertical stripes. Note that if a line $y = -x + a$ meets K then it also meets at least one of the squares K_i. Since, for every $a \in [0, 2]$, the line $y = -x + a$ meets the unit square $[0, 1] \times [0, 1]$, it also meets the set $C \times C$, that is, there exist $x, y \in C$ such that $x + y = a$. Hence $C + C = [0, 2]$. ∎

Proof 2. The Cantor set C consists of exactly those points in $[0, 1]$ whose ternary expansions can be written without the use of the digit 1. That is,

$$C = \Big\{x \in [0, 1] : x = \sum_{n\in\mathbb{N}} c_n/3^n,\ c_n \in \{0, 2\}\Big\}.$$

Hence

$$\tfrac{1}{2}C = \Big\{x \in [0, 1] : x = \sum_{n\in\mathbb{N}} c_n/3^n,\ c_n \in \{0, 1\}\Big\}.$$

It easily follows that $\frac{1}{2}C + \frac{1}{2}C = [0, 1]$ and so $C + C = [0, 2]$. Indeed, if $x \in [0, 1]$ is of the form $x = \sum_{n\in\mathbb{N}} c_n/3^n$, where $c_n \in \{0, 1, 2\}$, then set $y := \sum_{n\in\mathbb{N}} a_n/3^n$, $z := \sum_{n\in\mathbb{N}} b_n/3^n$, where $a_n := [c_n/2]$, $b_n := c_n - [c_n/2]$, i.e.,

$$a_n := b_n := 0 \quad \text{if } c_n = 0; \qquad a_n := 0,\ b_n := 1 \quad \text{if } c_n = 1;$$
$$a_n := b_n := 1 \quad \text{if } c_n = 2.$$

Obviously, $x = y + z$ and $y, z \in \frac{1}{2}C$. Hence $x \in \frac{1}{2}C + \frac{1}{2}C$, proving $[0,1] \subseteq \frac{1}{2}C + \frac{1}{2}C$. The reverse inclusion is trivial. ∎

Proof 3. The Cantor set C can be written in the form $C = \bigcap_{n\in\mathbb{N}} C_n$, where

$$C_1 = [0,1], \qquad C_{n+1} = \tfrac{1}{3}C_n + \{0, 2/3\}, \qquad n \in \mathbb{N}.$$

Of course $C_1 + C_1 = [0,2]$, and by induction, for all $n \in \mathbb{N}$,

$$\begin{aligned} C_{n+1} + C_{n+1} &= \tfrac{1}{3}(C_n + C_n) + \{0, 2/3\} + \{0, 2/3\} \\ &= [0, 2/3] + \{0, 2/3, 4/3\} = [0,2]. \end{aligned}$$

Hence, using the identity (cf. [9])

$$\bigcap_{n\in\mathbb{N}} (C_n + C_n) = \bigcap_{n\in\mathbb{N}} C_n + \bigcap_{n\in\mathbb{N}} C_n,$$

we conclude that $C + C = [0,2]$. ∎

3. Minkowski sums of fractals. In this section we present generalizations of the Steinhaus theorem for a large class of fractals determined by certain Hutchinson-type operators.

Given a Banach space X, we denote by $\mathcal{K}(X)$ the family of all nonempty compact subsets of X. It is well-known that $\mathcal{K}(X)$, endowed with the Hausdorff–Pompeiu metric, is a complete metric space, furthermore the multiplication by scalars and the Minkowski sum are continuous operations with respect to this metric (cf. [6]). We shall use the notation

$$\Sigma_k A := \{x_1 + \cdots + x_k : x_1, \ldots, x_k \in A\}$$

for the k-fold Minkowski sum of a set $A \subseteq X$.

Let $\gamma \in (0,1)$ and $P \in \mathcal{K}(X)$ and consider the Hutchinson-type operator $\Phi_{\gamma,P} : \mathcal{K}(X) \to \mathcal{K}(X)$ defined by

$$\Phi_{\gamma,P}(A) := \gamma A + (1-\gamma)P, \qquad A \in \mathcal{K}(X). \tag{3}$$

Note that if P is finite, say $P = \{p_1, \ldots, p_N\}$, then

$$\Phi_{\gamma,P}(A) = \bigcup_{j=1}^{N} (\gamma A + (1-\gamma)p_j) = \bigcup_{j=1}^{N} S_j(A),$$

where $S_j : X \to X$ is a γ-contraction defined by $S_j(x) := \gamma x + (1-\gamma)p_j$. Such operators as well as their fixed points were considered by Hutchinson in his classical paper [5].

The properties of the map $\varPhi_{\gamma,P} : \mathcal{K}(X) \to \mathcal{K}(X)$ and of its fixed point are summarized in the following lemma which is a variant of Theorem 3.1(3) in [5].

LEMMA 2. *Let X be a Banach space, $P \in \mathcal{K}(X)$ and $\gamma \in (0,1)$. Define $\varPhi_{\gamma,P} : \mathcal{K}(X) \to \mathcal{K}(X)$ by (3). Then $\varPhi_{\gamma,P}$ has a unique fixed point $F_{\gamma,P} \in \mathcal{K}(X)$ (i.e., $\varPhi_{\gamma,P}(F_{\gamma,P}) = F_{\gamma,P}$); moreover, for all $A \in \mathcal{K}(X)$,*

$$\lim_{n\to\infty} \varPhi^n_{\gamma,P}(A) = F_{\gamma,P} \tag{4}$$

(where $\varPhi^n_{\gamma,P}$ denotes the nth iterate of $\varPhi_{\gamma,P}$ and the convergence is in the sense of the Hausdorff–Pompeiu metric) and

$$P \subseteq F_{\gamma,P} \subseteq \overline{\operatorname{conv}}\, P, \tag{5}$$

hence also $\overline{\operatorname{conv}}\, F_{\gamma,P} = \overline{\operatorname{conv}}\, P$.

Proof. Since $\varPhi_{\gamma,P} : \mathcal{K}(X) \to \mathcal{K}(X)$ is a contraction with contraction factor $\gamma < 1$, by the Banach contraction principle it has a unique fixed point $F_{\gamma,P} \in \mathcal{K}(X)$ and (4) holds for all $A \in \mathcal{K}(X)$.

To prove (5), we apply (4) in two particular cases. Clearly, $P \subseteq \varPhi_{\gamma,P}(P)$. Hence $\varPhi^n_{\gamma,P}(P) \subseteq \varPhi^{n+1}_{\gamma,P}(P)$ for all $n \in \mathbb{N}$, i.e. the sequence $(\varPhi^n_{\gamma,P}(P))$ of sets is increasing. Thus, applying (4) for $A := P$, we obtain

$$P \subseteq \lim_{n\to\infty} \varPhi^n_{\gamma,P}(P) = F_{\gamma,P}.$$

On the other hand, $\varPhi_{\gamma,P}(\overline{\operatorname{conv}}\, P) \subseteq \overline{\operatorname{conv}}\, P$, so the sequence $(\varPhi^n_{\gamma,P}(\overline{\operatorname{conv}}\, P))$ is decreasing. Thus, applying (4) with $A := \overline{\operatorname{conv}}\, P$, we obtain

$$\overline{\operatorname{conv}}\, P \supseteq \lim_{n\to\infty} \varPhi^n_{\gamma,P}(\overline{\operatorname{conv}}\, P) = F_{\gamma,P},$$

which completes the proof (5). ∎

The set $F_{\gamma,P}$ is called *the fractal determined by $\varPhi_{\gamma,P}$*. The next theorem gives a necessary and sufficient condition for its convex hull to coincide with the set of all convex combinations of m elements of $F_{\gamma,P}$ with prescribed fixed coefficients.

THEOREM 3. *Let X be a Banach space, $P \in \mathcal{K}(X)$, $\gamma \in (0,1)$ and $\lambda_1, \dots, \lambda_m > 0$ with $\lambda_1 + \dots + \lambda_m = 1$. Let $F_{\gamma,P}$ be the fractal determined by the operator $\varPhi_{\gamma,P}$. Then*

$$\lambda_1 F_{\gamma,P} + \dots + \lambda_m F_{\gamma,P} = \overline{\operatorname{conv}}\, P \tag{6}$$

if and only if

$$\gamma\, \overline{\operatorname{conv}}\, P + (1-\gamma)(\lambda_1 P + \dots + \lambda_m P) = \overline{\operatorname{conv}}\, P. \tag{7}$$

Proof. Let $D = \overline{\operatorname{conv}}\, P$.

Sufficiency. Assume that (7) holds. We will show by induction that, for all $n \geq 0$,

$$\lambda_1 \Phi^n_{\gamma,P}(D) + \cdots + \lambda_m \Phi^n_{\gamma,P}(D) = D. \tag{8}$$

By the convexity of D, we have $\lambda_1 D + \cdots + \lambda_m D = D$, which trivially yields (8) for $n = 0$.

Now, assume that (8) holds for some $n \geq 0$. Then, using the definition of $\Phi_{\gamma,P}$ and (7), we obtain

$$\begin{aligned}
&\lambda_1 \Phi^{n+1}_{\gamma,P}(D) + \cdots + \lambda_m \Phi^{n+1}_{\gamma,P}(D) \\
&\quad = \lambda_1(\gamma \Phi^n_{\gamma,P}(D) + (1-\gamma)P) + \cdots + \lambda_m(\gamma \Phi^n_{\gamma,P}(D) + (1-\gamma)P) \\
&\quad = \gamma(\lambda_1 \Phi^n_{\gamma,P}(D) + \cdots + \lambda_m \Phi^n_{\gamma,P}(D)) + (1-\gamma)(\lambda_1 P + \cdots + \lambda_m P) \\
&\quad = \gamma D + (1-\gamma)(\lambda_1 P + \cdots + \lambda_m P) = D,
\end{aligned}$$

which completes the proof of (8).

Finally, letting $n \to \infty$ in (8) and using (4), we get

$$\lambda_1 F_{\gamma,P} + \cdots + \lambda_m F_{\gamma,P} = D = \overline{\mathrm{conv}}\, P,$$

which yields (6).

Necessity. Assume now that (6) is valid. By (5), we have

$$F_{\gamma,P} = \Phi_{\gamma,P}(F_{\gamma,P}) = \gamma F_{\gamma,P} + (1-\gamma)P \subseteq \gamma D + (1-\gamma)P,$$

whence, by (6),

$$\begin{aligned}
D = \overline{\mathrm{conv}}\, P &= \lambda_1 F_{\gamma,P} + \cdots + \lambda_m F_{\gamma,P} \\
&\subseteq \lambda_1(D + (1-\gamma)P) + \cdots + \lambda_m(D + (1-\gamma)P) \\
&= \gamma(\lambda_1 D + \cdots + \lambda_m D) + (1-\gamma)(\lambda_1 P + \cdots + \lambda_m P) \\
&= \gamma D + (1-\gamma)(\lambda_1 P + \cdots + \lambda_m P) \subseteq \overline{\mathrm{conv}}\, P = D,
\end{aligned}$$

which proves (7). ∎

For a given set P, it is not obvious whether or not identity (7) can be satisfied with some convex combination coefficients $\lambda_1, \ldots, \lambda_m > 0$. As we shall see below, in the finite-dimensional case we can always ensure the existence of such coefficients.

We will need the following almost trivial statement.

Lemma 4. *Let X be a linear space, $D \subseteq X$ be convex, $S \subseteq D$ be nonempty and $0 \leq \beta \leq \gamma \leq 1$. Then*

$$\beta D + (1-\beta)S \subseteq \gamma D + (1-\gamma)S. \tag{9}$$

Proof. If $\beta = \gamma$, then there is nothing to prove. Thus, we may assume that $\gamma > \beta \geq 0$.

Let x be an element of the left hand side of (9). Then $x = \beta u + (1-\beta)v$ for some $u \in D$, $v \in S$. Hence,

$$x = \beta u + (1-\beta)v = \gamma\Big(\frac{\beta}{\gamma}\,u + \frac{\gamma-\beta}{\gamma}\,v\Big) + (1-\gamma)v \in \gamma D + (1-\gamma)S,$$

where $\frac{\beta}{\gamma}u + \frac{\gamma-\beta}{\gamma}v \in D$ by the convexity of D and the inclusion $S \subseteq D$. ∎

The next result brings two conditions equivalent to (7) when $D = \overline{\operatorname{conv}}\, P$.

THEOREM 5. *Let X be a linear space, $D \subseteq X$ be a convex set, $P \subseteq D$ be non-empty, and $\gamma \in (0,1)$ be a constant. Then the following conditions are equivalent*:

(i) *There exists $\alpha \in (0,1)$ such that*

$$\alpha D + (1-\alpha)P = D. \tag{10}$$

(ii) *There exists $m \in \mathbb{N}$ such that*

$$\gamma D + (1-\gamma)\Big(\frac{1}{m}\,P + \cdots + \frac{1}{m}\,P\Big) = D. \tag{11}$$

(iii) *There exist $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m > 0$ with $\lambda_1 + \cdots + \lambda_m = 1$ such that*

$$\gamma D + (1-\gamma)(\lambda_1 P + \cdots + \lambda_m P) = D. \tag{12}$$

Proof. The implication (ii)⇒(iii) is obvious.

(i)⇒(ii). Assume that (10) is satisfied by some $\alpha \in (0,1)$. Using Lemma 4, it follows for $\alpha \le \alpha' \le 1$ that

$$D = \alpha D + (1-\alpha)P \subseteq \alpha' D + (1-\alpha')P \subseteq D,$$

hence we may assume that α is a rational number.

We show by induction that, for all $k \in \mathbb{N}$,

$$\alpha^k D + (1-\alpha)(\alpha^{k-1}P + \cdots + P) = D. \tag{13}$$

The case $k = 1$ follows from condition (ii). Now, assume that (13) has been verified for some k. Then

$$\begin{aligned}
&\alpha^{k+1} D + (1-\alpha)(\alpha^k P + \cdots + P)\\
&\qquad\qquad = \alpha(\alpha^k D + (1-\alpha)(\alpha^{k-1}P + \cdots + P)) + (1-\alpha)P\\
&\qquad\qquad = \alpha D + (1-\alpha)P = D,
\end{aligned}$$

proving that (13) is valid.

Now, choose $k \in \mathbb{N}$ so that $\alpha^k \le \gamma$. Using the convexity of D and applying Lemma 4 with $\beta := \alpha^k$ and $S := \frac{1-\alpha}{1-\alpha^k}(\alpha^{k-1}P + \cdots + P) \subseteq D$, we obtain

$$\begin{aligned} D &= \alpha^k D + (1-\alpha)(\alpha^{k-1}P + \cdots + P) \\ &\subseteq \gamma D + \frac{(1-\gamma)(1-\alpha)}{1-\alpha^k}\,(\alpha^{k-1}P + \cdots + P) \subseteq D. \end{aligned}$$

Therefore,

$$\gamma D + (1-\gamma)\biggl(\frac{\alpha^{k-1}-\alpha^k}{1-\alpha^k}\,P + \cdots + \frac{1-\alpha}{1-\alpha^k}\,P\biggr) = D. \tag{14}$$

Since $\alpha = p/q$ for some $p, q \in \mathbb{N}$, (14) can be rewritten as

$$\gamma D + (1-\gamma)\biggl(\frac{p^{k-1}q-p^k}{q^k-p^k}\,P + \cdots + \frac{q^k-pq^{k-1}}{q^k-p^k}\,P\biggr) = D. \tag{15}$$

As obviously $m_iP \subseteq \Sigma_{m_i}P$ with $m_i := p^{k-i}q^i - p^{k-i+1}q^{i-1}$ $(i = 1, \ldots, k)$ (15) shows that (11) is satisfied with $m := m_1 + \cdots + m_k = q^k - p^k$.

(iii)⇒(i). Assume that (12) holds for some $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m > 0$ with $\lambda_1 + \cdots + \lambda_m = 1$. If $m = 1$ then (10) is satisfied with $\alpha := \gamma$. In the case $m > 1$, using the convexity of D, we obtain

$$\begin{aligned} D &= \gamma D + (1-\gamma)(\lambda_1 P + \cdots + \lambda_{m-1}P + \lambda_m P) \\ &\subseteq \gamma D + (1-\gamma)(\lambda_1 D + \cdots + \lambda_{m-1}D + \lambda_m P) \\ &= (\gamma + (1-\gamma)(\lambda_1 + \cdots + \lambda_{m-1}))D + (1-\gamma)\lambda_m P \subseteq D. \end{aligned}$$

Therefore, (10) is satisfied with $\alpha := 1 - (1-\gamma)\lambda_m$. ∎

Remark 6. For (10) to be valid for some $\alpha \in (0,1)$, it is necessary that $\operatorname{extr} D \subseteq P$, where $\operatorname{extr} D$ denotes the set of extremal points of D. Indeed, let $x \in D$ be an extremal point of D. Then, by (10), there exist $p \in P$ and $q \in D$ such that $\alpha q + (1-\alpha)p = x$. By the extremality of x, it follows that $x = q = p$, so $x \in P$.

Assuming that D is also compact, by the celebrated Krein–Milman theorem we have $D = \overline{\operatorname{conv}}(\operatorname{extr} D)$. Hence, in this case, $D = \overline{\operatorname{conv}}\,P$ is also necessary for (10) to be valid for some $\alpha \in (0,1)$. As we shall see in Lemma 4 below, if X is also finite-dimensional then $D = \overline{\operatorname{conv}}\,P$ is also sufficient for (10) to hold with $\alpha = \dim X/(\dim X + 1)$.

On the other hand, we shall show that, in an infinite-dimensional Hilbert space X, there exists a compact set P such that

$$\alpha\,\overline{\operatorname{conv}}\,P + (1-\alpha)P = \overline{\operatorname{conv}}\,P \tag{16}$$

is not valid with any $\alpha \in (0,1)$. Let $(p_n)_{n\in\mathbb{N}}$ be an orthogonal system in X such that $\sum_{n=1}^{\infty} \|p_n\|^2 < \infty$. Let $P := \{0, p_1, p_2, p_3, \ldots\}$. Then P is compact since (p_n) tends to zero. One can easily prove that

$$\overline{\operatorname{conv}}\, P = \Big\{ \sum_{n=1}^{\infty} t_n p_n \,\Big|\, t_n \geq 0,\ \sum_{n=1}^{\infty} t_n \leq 1 \Big\}. \tag{17}$$

Assume that there exists $\alpha \in (0,1)$ such that (16) holds. Choose $n \in \mathbb{N}$ so that $\alpha < n/(n+1)$. Then

$$\frac{p_1 + \cdots + p_n}{n+1} \in \operatorname{conv} P \subseteq \overline{\operatorname{conv}}\, P,$$

hence, by (16),

$$\begin{aligned}\frac{p_1 + \cdots + p_n}{n+1} &\in \alpha\, \overline{\operatorname{conv}}\, P + (1-\alpha) P \\ &= \alpha\, \overline{\operatorname{conv}}\, P + (1-\alpha)\{p_0, p_1, p_2, p_3, \dots\},\end{aligned}$$

where p_0 denotes 0. Therefore, using (17), we can find $t_1, t_2, \ldots \geq 0$ with $t_1 + t_2 + \cdots \leq 1$ and $k \in \mathbb{N} \cup \{0\}$ such that

$$\frac{p_1 + \cdots + p_n}{n+1} = \alpha(t_1 p_1 + t_2 p_2 + \cdots) + (1-\alpha) p_k.$$

If $k = 0$, then $n/(n+1) = \alpha(t_1 + \cdots + t_n) \leq \alpha$, which contradicts the choice of n. If $k \geq 1$, then from the comparison of the coefficients of p_k, it follows that $k \leq n$ and $1/(n+1) = \alpha t_k + (1-\alpha) \geq 1 - \alpha$, which again contradicts the inequality $\alpha < n/(n+1)$.

Thus (16) cannot be valid for any $\alpha \in (0,1)$.

The following is a direct consequence of Theorems 3 and 5.

THEOREM 7. *Let X be a Banach space, $P \in \mathcal{K}(X)$ and $\gamma \in (0,1)$. Denote by $F_{\gamma,P}$ the fractal determined by the operator $\Phi_{\gamma,P}$. Then there exists $m \in \mathbb{N}$ such that $\Sigma_m F_{\gamma,P} = m\, \overline{\operatorname{conv}}\, P$, i.e.,*

$$\frac{1}{m} F_{\gamma,P} + \cdots + \frac{1}{m} F_{\gamma,P} = \overline{\operatorname{conv}}\, P, \tag{18}$$

if and only if there exists $\alpha \in (0,1)$ such that

$$\alpha\, \overline{\operatorname{conv}}\, P + (1-\alpha) P = \overline{\operatorname{conv}}\, P. \tag{19}$$

Proof. If (19) holds, then condition (i) of Theorem 5 is satisfied with $D := \overline{\operatorname{conv}}\, P$. Thus, by (ii), we can find $m \in \mathbb{N}$ such that

$$\gamma\, \overline{\operatorname{conv}}\, P + (1-\gamma)\Big(\frac{1}{m} P + \cdots + \frac{1}{m} P\Big) = \overline{\operatorname{conv}}\, P. \tag{20}$$

Therefore, (7) holds with $\lambda_1 = \cdots = \lambda_m = 1/m$ and Theorem 3 yields (18).

Conversely, if (18) is valid then Theorem 3 implies (7) with $\lambda_1 = \cdots = \lambda_m = 1/m$, i.e., (20) holds. Applying Theorem 5, it follows that, for some $\alpha \in (0,1)$, condition (19) must be satisfied. ∎

The next lemma is deduced by using Carathéodory's theorem; it also follows from [3, Remark on p. 191]. It shows that condition (19) is automatically satisfied in finite-dimensional spaces.

LEMMA 8. *Let X be a finite-dimensional space and $P \in \mathcal{K}(X)$. Then*

$$\frac{\dim X}{\dim X + 1} \operatorname{conv} P + \frac{1}{\dim X + 1} P = \operatorname{conv} P. \tag{21}$$

Proof. The inclusion $\subseteq$ in (21) is obvious. To prove the reverse inclusion, let $x \in \operatorname{conv} P$. By Carathéodory's theorem, there exist $p_0, p_1, \ldots, p_n \in P$ and $\lambda_0, \lambda_1, \ldots, \lambda_n \in [0,1]$ with $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1$ such that $\lambda_0 p_0 + \lambda_1 p_1 + \cdots + \lambda_n p_n = x$, where $n = \dim X$. We may assume that $\lambda_0 \geq \lambda_i$ for all i. Then $\lambda_0 \geq 1/(n+1)$. Hence

$$\beta := 1 - \lambda_0 \leq \frac{n}{n+1} =: \gamma.$$

Applying Lemma 4 for the sets $D := \operatorname{conv} P$ and $S := \{p_0\}$, we get

$$\begin{aligned} x = \lambda_0 p_0 + \lambda_1 p_1 + \cdots + \lambda_n p_n \in \lambda_0 p_0 + (1 - \lambda_0) D = \beta D + (1-\beta) p_0 \\ \subseteq \gamma D + (1-\gamma) p_0 \subseteq \frac{n}{n+1} D + \frac{1}{n+1} P, \end{aligned}$$

proving the inclusion $\subseteq$ in (21). ■

REMARK 9. Note that the coefficient $\alpha = n/(n+1)$ (where $n = \dim X$) in Lemma 8 is the smallest one for which (21) is valid, i.e., for $\alpha < n/(n+1)$, the condition

$$\alpha \operatorname{conv} P + (1-\alpha) P = \operatorname{conv} P \tag{22}$$

may not hold for some compact set $P \subset X$. In fact, more generally, we show that if P is the set of vertices of a k-dimensional simplex (where $k \leq n$), then (22) does not hold if $\alpha < k/(k+1)$.

Assume that $P := \{p_0, \ldots, p_k\}$, where $p_0, \ldots, p_k$ are affinely independent vectors in X. Let $\alpha \in (0,1)$ be a constant such that (22) holds. Then

$$\frac{1}{k+1}(p_0 + \cdots + p_k) \in \operatorname{conv} P = \alpha \operatorname{conv} P + (1-\alpha) P.$$

Therefore, there exist an index i and constants $t_0, \ldots, t_k \geq 0$ with $t_0 + \cdots + t_k = 1$ such that

$$\frac{1}{k+1}(p_0 + \cdots + p_k) = \alpha(t_0 p_0 + \cdots + t_k p_k) + (1-\alpha) p_i.$$

By the affine independence of $p_0, p_1, \ldots, p_k$, the coefficients of p_i on both sides coincide, i.e., $1/(k+1) = \alpha t_i + 1 - \alpha$. Thus, $t_i = 1 - k/(\alpha(k+1))$. Since $t_i \geq 0$, this yields $\alpha \geq k/(k+1)$.

As an immediate consequence of Lemma 8, we obtain the following result which shows that in finite-dimensional spaces every fractal of the type $F_{\gamma,P}$ is sufficiently large.

COROLLARY 10. *Let X be a finite-dimensional space, $P \in \mathcal{K}(X)$ and $\gamma \in (0,1)$. Let $F_{\gamma,P}$ be the fractal determined by the operator $\Phi_{\gamma,P}$. Then there exists $m \in \mathbb{N}$ such that $\Sigma_m F_{\gamma,P} = m \operatorname{conv} P$, i.e.,* (18) *holds.*

Proof. By Lemma 8, condition (19) of Theorem 7 is satisfied with $\alpha := \dim X/(\dim X + 1)$. Thus, by that theorem, (18) must be valid for some $m \in \mathbb{N}$. ∎

4. Examples and applications. The following examples show some applications of our main results.

EXAMPLE 1. The Cantor set C is the fractal determined by the operator $\Phi_{1/3,\{0,1\}} : \mathcal{K}(\mathbb{R}) \to \mathcal{K}(\mathbb{R})$, i.e., $C = F_{1/3,\{0,1\}}$. Since

$$\begin{aligned}\tfrac{1}{3}[0,1] + \tfrac{2}{3}\big(\tfrac{1}{2}\{0,1\} + \tfrac{1}{2}\{0,1\}\big) &= [0,1/3] + \{0,1/3,2/3\}\\ &= [0,1/3] \cup [1/3,2/3] \cup [2/3,1] = [0,1],\end{aligned}$$

applying Theorem 3, we obtain

$$\tfrac{1}{2}C + \tfrac{1}{2}C = [0,1].$$

Now, we can raise a more general question: For which values of $\lambda \in (0,1)$ do we have $\lambda C + (1-\lambda)C = [0,1]$? In view of Theorem 3, it suffices to determine all $\lambda \in (0,1)$ such that

$$\tfrac{1}{3}[0,1] + \tfrac{2}{3}(\lambda\{0,1\} + (1-\lambda)\{0,1\}) = [0,1]. \tag{23}$$

In the case $0 < \lambda \le 1/2$, we have

$$\begin{aligned}&\tfrac{1}{3}[0,1] + \tfrac{2}{3}(\lambda\{0,1\} + (1-\lambda)\{0,1\}) = [0,1/3] + \{0, 2\lambda/3, (2-2\lambda)/3, 2/3\}\\ &= [0,1/3] \cup [2\lambda/3, (1+2\lambda)/3] \cup [(2-2\lambda)/3, (3-2\lambda)/3] \cup [2/3,1]\\ &= [0,(1+2\lambda)/3] \cup [(2-2\lambda)/3, 1].\end{aligned}$$

Thus, condition (23) holds if and only if $1 + 2\lambda \ge 2 - 2\lambda$, i.e., if $\lambda \ge 1/4$. In the case $1/2 \le \lambda < 1$, we can similarly obtain the inequality $\lambda \le 3/4$. Therefore,

$$\lambda C + (1-\lambda)C = [0,1]$$

holds if and only if $1/4 \le \lambda \le 3/4$.

EXAMPLE 2. Consider now another set $\widetilde{C} \subseteq [0,1]$ constructed similarly to the Cantor set. We remove from $[0,1]$ the open middle half, next the open middle half of each of the remaining two pieces, etc. Iterating this process, we obtain a set $\widetilde{C}$ homeomorphic to C. However, $\frac{1}{2}\widetilde{C} + \frac{1}{2}\widetilde{C}$ is not equal to $[0,1]$. To show this, first observe that $\widetilde{C}$ is the fractal determined by the operator $\Phi_{1/4,\{0,1\}}$, i.e., $\widetilde{C} = F_{1/4,\{0,1\}}$. We have

$$\tfrac{1}{4}[0,1] + \tfrac{3}{4}(\tfrac{1}{2}\{0,1\} + \tfrac{1}{2}\{0,1\}) = [0,1/4] + \{0,3/8,3/4\} \neq [0,1],$$

hence condition (7) of Theorem 3 is violated, implying that $\frac{1}{2}\widetilde{C}+\frac{1}{2}\widetilde{C} \neq [0,1]$. On the other hand,

$$\tfrac{1}{4}[0,1] + \tfrac{3}{4}(\tfrac{1}{3}\{0,1\} + \tfrac{2}{3}\{0,1\}) = [0,1/4] + \{0,1/4,2/4,3/4\} = [0,1],$$

which implies that

$$\tfrac{1}{3}\widetilde{C} + \tfrac{2}{3}\widetilde{C} = [0,1]. \tag{24}$$

One can also easily check that $\lambda = 1/3$ is the only possible parameter in $(0,1/2]$ for which $\lambda\widetilde{C} + (1-\lambda)\widetilde{C} = [0,1]$. The equality (24) also gives

$$[0,3] = \widetilde{C} + 2\widetilde{C} \subseteq \widetilde{C} + \widetilde{C} + \widetilde{C} \subseteq [0,3].$$

Hence

$$\Sigma_3\widetilde{C} = \widetilde{C} + \widetilde{C} + \widetilde{C} = [0,3].$$

EXAMPLE 3 (An extension of Examples 1 and 2). For $\gamma \in (0,1)$, denote by C_γ the fractal $F_{\gamma,\{0,1\}} \subseteq [0,1]$. (Note that $C_{1/3} = C$ and $C_{1/4} = \widetilde{C}$.) Then, for every $m \in \mathbb{N}$, the identity

$$\Sigma_m C_\gamma = [0,m] \tag{25}$$

holds if and only if $m \geq (1-\gamma)/\gamma$. Indeed, by Theorem 3, (25) holds if and only if

$$\gamma[0,1] + (1-\gamma)\left(\frac{1}{m}\{0,1\} + \cdots + \frac{1}{m}\{0,1\}\right) = [0,1]. \tag{26}$$

Obviously,

$$\gamma[0,1] + (1-\gamma)\left(\frac{1}{m}\{0,1\} + \cdots + \frac{1}{m}\{0,1\}\right) = [0,\gamma] + \frac{1-\gamma}{m}\{0,1,\ldots,m\}, \tag{27}$$

which is the union of $m+1$ translates of the interval $[0,\gamma]$. Therefore, this set can cover $[0,1]$ only if $(m+1)\gamma \geq 1$. Conversely, if $(m+1)\gamma \geq 1$, then the consecutive intervals of the right hand side of (27) are not disjoint and hence they cover $[0,1]$, i.e., (26) holds.

For other examples of this type, see Ger [4].

EXAMPLE 4. Let S be the classical Sierpiński carpet defined in the following way: We divide the unit square $[0,1]\times[0,1]$ into the 3-by-3 grid and remove the central square; next we repeat this process for each of the remaining eight squares, etc. Clearly, S is the fractal determined by the operator

$\Phi_{1/3,P} : \mathcal{K}(\mathbb{R}^2) \to \mathcal{K}(\mathbb{R}^2)$, where

$$P = \{(0,0),(0,1),(1,0),(1,1),(0,1/2),(1/2,0),(1,1/2),(1/2,1)\}.$$

It can be easily shown that

$$\tfrac{1}{3}([0,1]\times[0,1]) + \tfrac{2}{3}(\tfrac{1}{2}P + \tfrac{1}{2}P) = [0,1]\times[0,1].$$

Therefore, by Theorem 3, we also have $\frac{1}{2}S + \frac{1}{2}S = [0,1]\times[0,1]$.

Example 5. Consider two other Sierpiński-type sets: T is determined by $\Phi_{1/3,P_1}$, where $P_1 = \{(0,0),(0,1),(1,0),(1,1),(1/2,1/2)\}$, and U is determined by $\Phi_{1/3,P_2}$, where $P_2 = \{(0,0),(0,1),(1,0),(1,1)\}$. By Theorem 3, we obtain

$$\tfrac{1}{2}T + \tfrac{1}{2}T = \tfrac{1}{2}U + \tfrac{1}{2}U = [0,1]\times[0,1].$$

Finally, we present an application of our results to regularity of t-convex functions.

Let D be a convex open subset of a normed space and $t \in (0,1)$. A function $f : D \to \mathbb{R}$ is called *t-convex* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y), \qquad x, y \in D;$$

f is *Jensen-convex* if it is $1/2$-convex; f is *convex* if it is t-convex for all $t \in (0,1)$. It is known that a convex function defined on an infinite-dimensional space need not be continuous; also, Jensen-convex functions may be discontinuous even if they are defined on an open interval in $\mathbb{R}$. However, if a Jensen-convex function is bounded from above on a set with nonempty interior then, by the Bernstein–Doetsch theorem [2] (see also [7]), it is continuous. Even a stronger result holds: If f is Jensen-convex and bounded from above on a set A such that $A + A$ has nonempty interior then f is continuous (cf. [4], [7]). We will generalize this result to t-convex functions. Given a function $f : D \to \mathbb{R}$ set

$$K(f) := \{t \in (0,1) : f \text{ is } t\text{-convex on } D\}.$$

By Kuhn's theorem [8], if f is t-convex, then $K(f) = [K(f)] \cap (0,1)$, where $[K(f)]$ denotes the subfield of $\mathbb{R}$ generated by $K(f)$. In particular, $\mathbb{Q}\cap(0,1) \subset K(f)$, and consequently every t-convex function is Jensen-convex.

Theorem 11. *Let D be an open convex subset of a normed space X and let $A \subset D$ be such that for some $t_1, \dots, t_m \in K(f)$ with $t_1 + \dots + t_m = 1$ the set $t_1A + \dots + t_mA$ has nonempty interior. Let $f : D \to \mathbb{R}$ be t-convex and bounded from above on A. Then f is a continuous convex function on D.*

Proof. By Kuhn's theorem, f is Jensen-convex. One can prove easily by induction that if f is t-convex then

$$f(s_1x_1 + \dots + s_nx_n) \le s_1f(x_1) + \dots + s_nf(x_n), \tag{28}$$

for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in D$ and $s_1, \dots, s_n \in K(f)$ such that $s_1 + \cdots + s_n = 1$. Now, if $f \le M$ on A for some constant $M < \infty$, then, by (28), the same inequality holds on $t_1 A + \cdots + t_m A$. Since this latter set has nonempty interior, the Bernstein–Doetsch theorem implies that f is continuous. ∎

COROLLARY 12. *Let D be an open convex subset of a finite-dimensional normed space X, let $\gamma \in (0,1)$ and let $P \in \mathcal{K}(X)$ with $P \subset D$ be such that* conv P *has nonempty interior. Let $f : D \to \mathbb{R}$ be t-convex and bounded from above on the fractal $F_{\gamma,P}$ determined by* (3). *Then f is a continuous convex function on D.*

Proof. Apply the previous theorem with $A := F_{\gamma,P}$ and Corollary 10. ∎

EXAMPLE 6. As an immediate consequence of the above corollary and Example 2 we see, for instance, that every $1/3$-convex function $f : I \to \mathbb{R}$ (where I is an open interval containing $[0,1]$) bounded from above on the set $\widetilde{C}$ is continuous.

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