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ON FOURIER ASYMPTOTICS OF A GENERALIZED CANTOR MEASURE

BY

BÉRENGER AKON KPATA, IBRAHIM FOFANA and KONIN KOUA (Abidjan)

Abstract. Let d be a positive integer and μ a generalized Cantor measure satisfying $\mu = \sum_{j=1}^{m} a_j \mu \circ S_j^{-1}$, where $0 < a_j < 1$, $\sum_{j=1}^{m} a_j = 1$, $S_j = \rho R + b_j$ with $0 < \rho < 1$ and R an orthogonal transformation of $\mathbb{R}^d$. Then

$$\begin{cases} 1 < p \le 2 \ \Rightarrow\ \sup\limits_{r>0} r^{d(1/\alpha' - 1/p')} \Big(\displaystyle\int_{J_x^r} |\widehat{\mu}(y)|^{p'}\, dy \Big)^{1/p'} \le D_1 \rho^{-d/\alpha'},\ x \in \mathbb{R}^d, \\ p = 2 \ \Rightarrow\ \inf\limits_{r \ge 1} r^{d(1/\alpha' - 1/2)} \Big(\displaystyle\int_{J_0^r} |\widehat{\mu}(y)|^{2}\, dy \Big)^{1/2} \ge D_2 \rho^{d/\alpha'}, \end{cases}$$

where $J_x^r = \prod_{i=1}^{d} (x_i - r/2, x_i + r/2)$, α' is defined by $\rho^{d/\alpha'} = (\sum_{j=1}^{m} a_j^p)^{1/p}$ and the constants D_1 and D_2 depend only on d and p.

1. Introduction. Let us start with some notations. We will denote by χ_N the characteristic function of the subset N of $\mathbb{R}^d$. Given $1 \le q \le \infty$, $\|\cdot\|_q$ will denote the usual Lebesgue norm and q' the conjugate exponent of q: $1/q + 1/q' = 1$ with the convention $1/\infty = 0$.

Let μ be a non-negative Radon measure on $\mathbb{R}^d$. For $1 \le q < \infty$, $0 < \beta \le d$ and $r > 0$, we define the average

$$H(q, \beta, r) = \frac{1}{r^{d-\beta}} \int_{J_0^r} |\widehat{\mu}(y)|^q \, dy,$$

where $\widehat{\mu}$ is the Fourier transform of the measure μ and $J_x^r = \prod_{i=1}^{d} (x_i - r/2, x_i + r/2)$ for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

We are interested in lower and upper bounds of $H(q, \beta, r)$ when r varies in $(0, \infty)$. This topic has been studied by K. S. Lau and J. Wang [L-W], and R. S. Strichartz ([St1]–[St3]) in the setting of self-similar measures. We recall some results obtained by R. S. Strichartz.

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Suppose that μ is an equicontractive self-similar measure, that is,

$$\mu = \sum_{j=1}^{m} a_j \mu \circ S_j^{-1},$$

where $S_j = \rho R_j + b_j$ with $0 < \rho < 1$, R_j is an orthogonal transformation of $\mathbb{R}^d$, $0 < a_j < 1$ and $\sum_{j=1}^{m} a_j = 1$.

In [St2], R. S. Strichartz proved that if:

(i) μ satisfies the *open set condition*: there exists a bounded, non-empty, open subset U of $\mathbb{R}^d$ such that $S_j U \subset U$ for all j and the sets $S_j U$ are pairwise disjoint,

(ii) the R_j's are either equal or generate a finite group,

then

$$\sup_{r>0} H(2, \beta, r) < \infty$$

where β is defined by

$$\rho^{\beta} = \sum_{j=1}^{m} a_j^2. \tag{1}$$

For any equicontractive self-similar measure satisfying conditions (i) and (ii) above with the R_j's equal and under the following hypothesis:

(H_1) the open set condition holds for the set of similarities $\rho R x + b_j^{(n)}$ where $b_j^{(n)}$ are the frequencies that appear in $(\sum_{j=1}^{m} a_j e^{i b_j . x})^n$,

R. S. Strichartz also obtained

$$\sup_{r>0} \ H(2n, \beta_n, r) < \infty. \tag{2}$$

But he noticed that hypothesis (H_1) will only hold for a finite number of n's (see Corollary 3.4 and the comment following it in [St2]).

An equicontractive self-similar measure where the similarities S_j satisfy the following condition: $1/\rho$ is an integer, for all j, $R_j = R$ is an orthogonal transformation that preserves the integer lattice and $(1/\rho) b_j$ is an integer between 0 and $1/\rho - 1$, is called a *generalized Cantor measure*. For such a measure, the open set condition is clearly satisfied by taking $U = \{x : 0 < x_i < 1 \text{ for } i = 1, \dots, d\}$.

R. S. Strichartz showed that for a generalized Cantor measure such that

(H_2) $(1/\rho) b_i \equiv (1/\rho) b_j \bmod n$ for all i, j,

inequality (2) holds (see Corollary 4.6 in [St2]).

He also established that if μ is a generalized Cantor measure satisfying the following hypothesis:

(H_3) there exists a fundamental domain D containing a neighborhood of the origin on which $|\widehat{\mu}(y)|$ is bounded away from zero,

then

$$0 < \inf_{r\geq 1} H(2,\beta,r),$$

where β is still defined by (1) (see Corollary 4.6 in [St2]).

In this paper, we show that if μ is a generalized Cantor measure as defined above, then

$$\sup_{r>0} H(p',\beta_p,r) \leq D = D(p,d), \qquad 1 < p \leq 2, \tag{3}$$

with β_p defined by $\rho^{\beta_p} = (\sum_{j=1}^{m} a_j^p)^{p'/p}$, and

$$0 < \inf_{r\geq 1} H(2,\beta,r) \tag{4}$$

with $\beta = \beta_2$.

We emphasize that for the proof of (3), we do not assume the additional hypotheses (H_1) and (H_2) used by R. S. Strichartz to get inequality (2). Furthermore, (3) contains (2), of course, in the case of the generalized Cantor measure. Moreover, in the proof of (4), we do not use the assumption (H_3). Our approach relies on some properties of generalized Cantor measures (see Section 3 for a detailed exposition of those properties), a reverse Hausdorff–Young inequality (see Theorem 2.2) and a Hausdorff–Young inequality which will be stated below.

We recall that a Radon measure μ on $\mathbb{R}^d$ belongs to the *Wiener amalgam space* M^p ($1 \leq p < \infty$) if ${}_1\|\mu\|_p < \infty$ with

$${}_r\|\mu\|_p = \Big(\sum_{k\in\mathbb{Z}^d} |\mu|(I_k^r)^p\Big)^{1/p}, \qquad r > 0,$$

where $I_k^r = \prod_{i=1}^d [k_i r, (k_i+1)r)$ for $k = (k_1,\ldots,k_d) \in \mathbb{Z}^d$ and $|\mu|$ denotes the total variation of μ.

The Fourier transform on amalgam spaces has been studied by various authors including F. Holland ([Ho1], [Ho2]), J. Stewart [S], J. P. Bertrandias and C. Dupuis [B-D], J. J. F. Fournier ([Fou1], [Fou2], [Fou-S]) and I. Fofana [F].

In [Ho1], the Fourier transform $f \mapsto \widehat{f}$ defined on L^1 by

$$\widehat{f}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(y) e^{-ixy}\, dy, \qquad x \in \mathbb{R}^d,$$

has been extended to the spaces (L^q, l^p) defined for $1 \leq q, p \leq \infty$ as follows:

$$(L^q, l^p) = \{f \in L^0 : {}_1\|f\|_{q,p} < \infty\}$$

where L^0 stands for the space of (equivalence classes modulo the equality Lebesgue almost everywhere of) all complex-valued functions defined on $\mathbb{R}^d$

and for $r > 0$,

$$ {}_r\|f\|_{q,p} = \begin{cases} \Big[\sum\limits_{k \in \mathbb{Z}^d} (\|f\chi_{I_k^r}\|_q)^p \Big]^{1/p} & \text{if } 1 \le p < \infty, \\ \sup\limits_{x \in \mathbb{R}^d} \|f\chi_{J_x^r}\|_q & \text{if } p = \infty. \end{cases} $$

In the same paper, Holland extended to the spaces M^p $(1 < p \le 2)$ the Fourier transform $\mu \mapsto \widehat{\mu}$ defined on the space M^1 of finite Radon measures on $\mathbb{R}^d$ by

$$ \widehat{\mu}(x) = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} e^{-ixy}\, d\mu(y), \qquad x \in \mathbb{R}^d. $$

In fact, he proved that if μ belongs to M^p $(1 < p \le 2)$, then there exists a unique element $\widehat{\mu} \in (L^{p'}, l^\infty)$ such that for any sequence $(r_n)_{n\ge1}$ of positive real numbers increasing to ∞, the sequence $(\widehat{\mu\lfloor J_0^{r_n}})_{n\ge1}$ converges in $(L^{p'}, l^\infty)$ to $\widehat{\mu}$, where $\mu\lfloor J_0^{r_n}$ is the measure defined by $(\mu\lfloor J_0^{r_n})(N) = \mu(J_0^{r_n} \cap N)$ for any Borel subset N of $\mathbb{R}^d$. In addition,

$$ \int\limits_{\mathbb{R}^d} g(x)\widehat{\mu}(x)\, dx = \int\limits_{\mathbb{R}^d} \widehat{g}(x)\, d\mu(x), \qquad g \in (L^p, l^1). \tag{5} $$

In [F], I. Fofana has proved the following Hausdorff–Young inequality:

$$ r^{-d/p'}\; {}_r\|\widehat{\mu}\|_{p',\infty} \le C\; {}_{1/r}\|\mu\|_p, \qquad r > 0, \tag{6} $$

where the real constant C does not depend on μ and r.

The remainder of this paper is organized as follows: in Section 2, we establish a reverse Hausdorff–Young inequality. Using this result and the inequality (6), we examine, in Section 3, bounds for averages of the Fourier transform of a generalized Cantor measure.

2. Reverse Hausdorff–Young inequality. In this section, for any $\eta > 0$, we denote by $\mathrm{Int}(\eta)$ the greatest integer not exceeding η. Throughout this note, we will use the following result:

PROPOSITION 2.1. *Let $1 \le p < \infty$. If μ belongs to M^p and $0 < r < s < \infty$, then*

(a) ${}_s\|\mu\|_p \le (\mathrm{Int}(s/r) + 2)^{d/p'} 2^{d/p}\; {}_r\|\mu\|_p$,
(b) ${}_r\|\mu\|_p \le 3^{d/p'} 2^{d/p}\; {}_s\|\mu\|_p$.

Proof. (a) For all $k \in \mathbb{Z}^d$, set $C(k) = \{l \in \mathbb{Z}^d : |\mu|(I_k^s \cap I_l^r) > 0\}$. Then, for all $k \in \mathbb{Z}^d$, $C(k)$ has at most $(\mathrm{Int}(s/r) + 2)^d$ elements. So for all $k \in \mathbb{Z}^d$,

$$ |\mu|(I_k^s) \le (\mathrm{Int}(s/r) + 2)^{d/p'} \Big(\sum_{l \in C(k)} |\mu|(I_l^r)^p \Big)^{1/p} $$

and

$$ {}_s\|\mu\|_p \le (\mathrm{Int}(s/r)+2)^{d/p'}\Big[\sum_{k\in\mathbb{Z}^d}\Big(\sum_{l\in C(k)}|\mu|(I_l^r)^p\Big)\Big]^{1/p}. $$

Notice that each element $l\in\mathbb{Z}^d$ belongs to at most 2^d subsets $C(k)$. It follows that

$$ {}_s\|\mu\|_p \le (\mathrm{Int}(s/r)+2)^{d/p'}2^{d/p}\,{}_r\|\mu\|_p. $$

(b) Let $m>1$ be the positive integer such that $(m-1)r<s\le mr$. We have

$$ \begin{aligned} {}_r\|\mu\|_p &= \Big[\sum_{l\in\mathbb{Z}^d}\Big(\sum_{\{k\in\mathbb{Z}^d\,:\,I_k^r\subset I_l^{mr}\}}|\mu|(I_k^r)^p\Big)\Big]^{1/p} \le \Big(\sum_{l\in\mathbb{Z}^d}|\mu|(I_l^{mr})^p\Big)^{1/p} \\ &= {}_{mr}\|\mu\|_p. \end{aligned} $$

So, according to (a),

$$ {}_r\|\mu\|_p \le (\mathrm{Int}(mr/s)+2)^{d/p'}2^{d/p}\,{}_s\|\mu\|_p \le 3^{d/p'}2^{d/p}\,{}_s\|\mu\|_p. \ \blacksquare $$

THEOREM 2.2. *There exists a positive constant D such that for all non-negative elements μ of M^2 and $r>0$,*

$$ D\,{}_{1/r}\|\mu\|_2 \le r^{-d/2}\Big(\int_{J_0^r}|\widehat{\mu}(x)|^2\,dx\Big)^{1/2}. $$

Proof. Let μ be a non-negative measure which belongs to M^2. Let $r>0$ and set

$$ f = r^{-d}\chi_{J_0^r}*\chi_{J_0^r}. $$

Then f is non-negative, continuous and satisfies

$$ \|f\|_\infty = 1 = f(0), \qquad x\in\mathbb{R}^d\setminus J_0^{2r} \ \Rightarrow\ f(x)=0. $$

In addition,

$$ \widehat{f}(x) = (2\pi)^{-d}r^d\prod_{j=1}^d\left(\frac{\sin(\frac{r}{2}x_j)}{\frac{r}{2}x_j}\right)^2, \qquad x\in\mathbb{R}^d\setminus\{0\}. $$

Let us consider a finite subset L of $\mathbb{Z}^d$, a sequence $(a_k)_{k\in L}$ of positive real numbers and a positive real number δ.

For all $k\in L$, set $E_k=\prod_{j=1}^d[(k_j-1)\delta,(k_j+2)\delta)$. Then for all $k\in L$ we have

$$ \overset{\vee}{\chi}_{E_k}(x) = \widehat{\chi_{E_k}}(-x) = (2\pi)^{-d/2}(3\delta)^d\prod_{j=1}^d e^{i(k_j-1)\delta x_j}\,\frac{e^{i3\delta x_j}-1}{i3\delta x_j}, \quad x\in\mathbb{R}^d\setminus\{0\}, $$

$$ \widehat{\overset{\vee}{\chi}_{E_k}f} = \chi_{E_k}*\widehat{f}, $$

and

$$\int_{\mathbb{R}^d} \mu(E_k+y)\widehat{f}(y)\,dy = \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\chi_{E_k}(x-y)\,d\mu(x)\Big)\widehat{f}(y)\,dy$$
$$= \int_{\mathbb{R}^d}\chi_{E_k}*\widehat{f}(x)\,d\mu(x),$$

that is, according to (5),

$$\int_{\mathbb{R}^d} \mu(E_k+y)\widehat{f}(y)\,dy = \int_{\mathbb{R}^d} \overset{\vee}{\chi}_{E_k}(x)f(x)\widehat{\mu}(x)\,dx.$$

Therefore, we have

$$\sum_{k\in L} a_k \int_{\mathbb{R}^d}\mu(E_k+y)\widehat{f}(y)\,dy$$
$$= \int_{\mathbb{R}^d}\sum_{k\in L} a_k\overset{\vee}{\chi}_{E_k}(x)f(x)\widehat{\mu}(x)\,dx$$
$$= \int_{J_0^{2r}}\Big(\sum_{k\in L} a_k\overset{\vee}{\chi}_{E_k}(x)\Big)(f(x)\widehat{\mu}(x))\,dx$$
$$\le \Big(\int_{J_0^{2r}}\Big|\sum_{k\in L} a_k\overset{\vee}{\chi}_{E_k}(x)\Big|^2 dx\Big)^{1/2}\Big(\int_{J_0^{2r}}|\widehat{\mu}(x)|^2\,dx\Big)^{1/2}.$$

Notice that

$$y\in J_0^{\delta} \;\Rightarrow\; I_k^{\delta}\subset E_k+y,\; k\in L,$$

and

$$\Big(\int_{J_0^{2r}}\Big|\sum_{k\in L} a_k\overset{\vee}{\chi}_{E_k}(x)\Big|^2 dx\Big)^{1/2}$$
$$\le (2\pi)^{-d/2}(3\delta)^d\delta^{-d/2}\Big(\sum_{k,t\in L} a_k a_t\int_{J_0^{2r\delta}} e^{-i(k-t)x}\,dx\Big)^{1/2}.$$

Choosing $\delta=\pi r^{-1}$, we obtain

$$\sum_{k\in L} a_k\mu(I_k^{\pi r^{-1}})\int_{J_0^{\pi r^{-1}}}\widehat{f}(y)\,dy \le 3^d(\pi r^{-1})^{d/2}\Big(\sum_{k\in L}a_k^2\Big)^{1/2}\Big(\int_{J_0^{2r}}|\widehat{\mu}(x)|^2\,dx\Big)^{1/2}.$$

Since

$$y\in J_0^{\pi r^{-1}} \;\Rightarrow\; \left(\frac{\sin\big(\frac{r}{2}y_j\big)}{\frac{r}{2}y_j}\right)^2 \ge \frac{8}{\pi^2},\; j\in\{1,\dots,d\},$$

we have

$$\int_{J_0^{\pi r^{-1}}}\widehat{f}(y)\,dy \ge \Big(\frac{2}{\pi}\Big)^{2d},$$

and therefore

$$\left(\frac{2}{\pi}\right)^{2d} \sum_{k\in L} a_k \mu(I_k^{\pi r^{-1}}) \le (9\pi r^{-1})^{d/2} \Big(\sum_{k\in L} a_k^2\Big)^{1/2} \Big(\int_{J_0^{2r}} |\widehat{\mu}(x)|^2\,dx\Big)^{1/2}.$$

Hence, choosing $a_k = \mu(I_k^{\pi r^{-1}})$ for all $k \in L$, it follows that

$$\left(\frac{2}{\pi}\right)^{2d} \Big(\sum_{k\in L} \mu(I_k^{\pi r^{-1}})^2\Big)^{1/2} \le (9\pi r^{-1})^{d/2} \Big(\int_{J_0^{2r}} |\widehat{\mu}(x)|^2\,dx\Big)^{1/2}.$$

Since the previous inequality holds for all finite subsets L of $\mathbb{Z}^d$, we obtain

$$\left(\frac{2}{\pi}\right)^{2d} (18\pi)^{-d/2}\, {}_{2\pi/r}\|\mu\|_2 \le r^{-d/2} \Big(\int_{J_0^{r}} |\widehat{\mu}(x)|^2\,dx\Big)^{1/2}.$$

Applying Proposition 2.1, we obtain the desired result. ■

Let f and E_k ($k \in \mathbb{Z}^d$) be as defined in the proof of Theorem 2.2. We notice that for any $k \in \mathbb{Z}^d$, $\varphi_k = \overset{\vee}{\chi}_{E_k} f \in L^1$ and $\widehat{\varphi_k} = \chi_{E_k} * \widehat{f} \in L^1$. Assume that h is an element of L^1 such that $\widehat{h} \ge 0$ and put $d\mu(x) = \widehat{h}(x)\,dx$. Then μ is a non-negative Radon measure on $\mathbb{R}^d$. For any $k \in \mathbb{Z}^d$, we have by Fubini theorem

$$\int_{\mathbb{R}^d} \widehat{\varphi_k}(x)\widehat{h}(x)\,dx = \int_{\mathbb{R}^d} \widehat{\widehat{\varphi_k}}(x) h(x)\,dx = \int_{\mathbb{R}^d} \varphi_k(-x)h(x)\,dx.$$

Equivalently,

$$\int_{\mathbb{R}^d} \chi_{E_k} * \widehat{f}(x)\ d\mu(x) = \int_{\mathbb{R}^d} \widehat{\chi_{E_k}}(x) f(x) h(x)\,dx.$$

Now, it is easy to see that, with h in place of $\widehat{\mu}$, the proof of Theorem 2.2 remains valid and yields the following result:

Proposition 2.3. *There exists a positive constant D such that for any element h of L^1 with $\widehat{h} \ge 0$, we have*

$$D\, {}_{1/r}\|\widehat{h}\|_{1,2} \le r^{-d/2} \Big(\int_{J_0^{r}} |h(x)|^2\,dx\Big)^{1/2}, \qquad r > 0. \tag{7}$$

Inequality (7) is related to the following result obtained by Fournier (see the proof of Theorem 1.1 in [Fou2]) for $d = 1$.

Proposition 2.4. *Let h be an integrable function on $\mathbb{R}^d$ that is square-integrable in some neighborhood of 0 and satisfies $\widehat{h} \ge 0$. Then for $r > 0$ small enough,*

$${}_1\|\widehat{h}\|_{1,2} \le C r^{-d} \|h\phi_r\|_2, \tag{8}$$

where C is a positive constant not depending on h and r, and

$$\phi_r(x) = \begin{cases} \prod_{i=1}^{d}(1 - |x_i|/r) & \text{if } x \in J_0^{2r}, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 2.5. From Proposition 2.1(b) and Proposition 2.3, it is clear that for $0 < r < 1$ we have

$$\begin{aligned} {}_1\|\widehat{h}\|_{1,2} \le 6^{d/2}\, {}_{1/r}\|\widehat{h}\|_{1,2} &\le \frac{6^{d/2}}{D}\, r^{-d/2}\Big(\int_{J_0^r} |h(x)|^2\,dx\Big)^{1/2} \\ &\le \frac{6^{d/2}2^d}{D}\, r^{-d/2}\Big(\int_{\mathbb{R}^d} |h(x)\phi_r(x)|^2\,dx\Big)^{1/2}, \end{aligned}$$

that is,

$${}_1\|\widehat{h}\|_{1,2} \le \Big(\frac{6^{d/2}2^d}{D}\, r^{d/2}\Big) r^{-d}\|h\phi_r\|_2.$$

Therefore, for small values of r, inequality (7) is stronger than (8).

Let us record an immediate consequence of (6):

COROLLARY 2.6. *Suppose that $1 \le \alpha \le p \le 2$. Let C be the constant of* (6). *Then for all $\mu \in M^p$, we have the following inequalities:*

(i) $\sup\limits_{r\ge 1} r^{d(1/\alpha'-1/p')}\, {}_r\|\widehat{\mu}\|_{p',\infty} \le C \sup\limits_{0<r\le 1} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_p$,

(ii) $\limsup\limits_{r\to\infty} r^{d(1/\alpha'-1/p')}\, {}_r\|\widehat{\mu}\|_{p',\infty} \le C \limsup\limits_{r\to 0} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_p$,

(iii) $\inf\limits_{r\ge 1} r^{d(1/\alpha'-1/p')}\, {}_r\|\widehat{\mu}\|_{p',\infty} \le C \inf\limits_{0<r\le 1} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_p$,

(iv) $\liminf\limits_{r\to\infty} r^{d(1/\alpha'-1/p')}\, {}_r\|\widehat{\mu}\|_{p',\infty} \le C \liminf\limits_{r\to 0} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_p$.

The inequalities (i) and (ii) of the previous corollary yield Theorem 3.4 established by K. S. Lau in [L].

A direct consequence of Corollary 2.6 and Theorem 2.2 is the following:

COROLLARY 2.7. *Suppose that $1 \le \alpha \le 2$. Let C be the constant of* (6) *and D the constant of Theorem 2.2. Then for any non-negative measure μ of M^2, we have the following inequalities:*

$$\begin{aligned} \text{(a) } D \sup_{0<r\le 1} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_2 &\le \sup_{r\ge 1} r^{d(1/\alpha'-1/2)}\, {}_r\|\widehat{\mu}\|_{2,\infty} \\ &\le C \sup_{0<r\le 1} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_2, \end{aligned}$$

(b) $D \limsup_{r\to 0} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_2 \le \limsup_{r\to\infty} r^{d(1/\alpha'-1/2)}\, {}_r\|\widehat{\mu}\|_{2,\infty}$
$$\le C \limsup_{r\to 0} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_2,$$

(c) $D \inf_{0<r\le 1} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_2 \le \inf_{r\ge 1} r^{d(1/\alpha'-1/2)}\, {}_r\|\widehat{\mu}\|_{2,\infty}$
$$\le C \inf_{0<r\le 1} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_2,$$

(d) $D \liminf_{r\to 0} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_2 \le \limsup_{r\to\infty} r^{d(1/\alpha'-1/2)}\, {}_r\|\widehat{\mu}\|_{2,\infty}$
$$\le C \liminf_{r\to 0} r^{d(1/\alpha-1)}\, {}_r\|\mu\|_2.$$

The relation (a) yields a converse of Theorem 3.4 of K. S. Lau mentioned above in the case $p = 2$.

3. Estimate on the average of the Fourier transform of a generalized Cantor measure. Let us consider:

(i) two integers m and $1/\rho$ such that $m > 1$ and $1/\rho > 1$,

(ii) m distinct elements b_j $(1 \le j \le m)$ of $\mathbb{R}^d$ such that the coordinates of each point $(1/\rho)b_j$ are integers denoted by $(1/\rho)b_{ji}$ $(1 \le i \le d)$ satisfying $0 \le (1/\rho)b_{ji} \le (1/\rho) - 1$,

(iii) a permutation σ of $\{1, \ldots, d\}$, R the orthogonal transformation defined by $R(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(d)})$ for all $x \in \mathbb{R}^d$, m contractive similarities $S_j = \rho R + b_j$,

(iv) m elements a_j such that $0 < a_j < 1$ for $j = 1, \ldots, m$ and $\sum_{j=1}^m a_j = 1$.

J. E. Hutchinson proved in [Hu] that for the system (S_j, a_j, m), there exists a unique non-negative Borel measure $\mu = \mu(S_j, a_j, m)$ with compact support such that $\mu(\mathbb{R}^d) = 1$ and $\mu = \sum_{j=1}^m a_j \mu \circ S_j^{-1}$. So μ is the self-similar measure on $\mathbb{R}^d$ associated to the system (S_j, a_j, m).

Denote by Q the closure of I_0^1, O the set of interior points of I_0^1, and E the support of $\mu = \mu(S_j, a_j, m)$. For any positive integer n, we set
$$\varLambda_n = \{J = (j_1, \ldots, j_n) : j_i \in \{1, \ldots, m\},\ i = 1, \ldots, n\}$$
and $S_J = S_{j_n} \circ \cdots \circ S_{j_1}$ for $J \in \varLambda_n$.

Notice that the sequence of contractive similarities $(S_j)_{1\le j\le m}$ satisfies the following open set condition: O is an open bounded subset of $\mathbb{R}^d$, $S_j(O) \subset O$ for $j = 1, \ldots, m$ and $S_j(O) \cap S_k(O) = \emptyset$ for $j \ne k$. Therefore, we have the following remark:

Remark 3.1 (see [Hu], [St2] or [L-W]). Let n be a positive integer. Then:

(i) $E \subset \bigcup_{J\in\varLambda_n} S_J(Q) \subset Q$,

(ii) for $J \in \varLambda_n$, $S_J(O)$ is an open subset of $\mathbb{R}^d$ and $S_J(O) \subset O$,

(iii) $S_J(O) \cap S_{J''}(O) = \emptyset$ for $J, J'' \in \Lambda_n$ such that $J \neq J''$,
(iv) $\mu(S_J(O)) = \prod_{i=1}^n a_{j_i}\mu(O)$ for $J \in \Lambda_n$.

Thus, μ is a generalized Cantor measure.

PROPOSITION 3.2. *Let $\mu = \mu(S_j, a_j, m)$ be a generalized Cantor measure. If*

$$K = \{k = (k_1, \dots, k_d) : k_i \in \{0, 1\},\ i = 1, \dots, d\}$$

and $K_0 = \{k \in K : \mu(I_k^1) > 0\}$, then K_0 is a singleton.

Proof. Particular case: Suppose that R is the identity map of $\mathbb{R}^d$. Each element k of K_0 has a length $|k| = \sum_{i=1}^d k_i \le d$. Let $\widetilde{k}$ be an element of maximal length of K_0.

(a) Suppose that all the coordinates of $\widetilde{k}$ are equal to 1. Then $I_{\widetilde{k}}^1 \cap Q = \{\widetilde{k}\}$ and there exists an element j of $\{1, \dots, m\}$ such that $\widetilde{k} \in S_j(Q)$. Since for $j = 1, \dots, m$, $S_j(Q) = \{x \in Q : b_{ji} \le x_i \le b_{ji} + \rho,\ i = 1, \dots, d\}$, we have

$$\widetilde{k} \in S_j(Q) \ \Rightarrow\ b_{ji} = 1 - \rho,\ i = 1, \dots, d.$$

Therefore, there exists a unique element $\widetilde{j}$ of $\{1, \dots, m\}$ such that $\widetilde{k} \in S_{\widetilde{j}}(Q)$. In addition, $\widetilde{k} = S_{\widetilde{j}}(\widetilde{k})$. It follows that

$$0 \neq \mu(\{\widetilde{k}\}) = \sum_{j=1}^m a_j \mu(S_j^{-1}(\{\widetilde{k}\})) = a_{\widetilde{j}}\mu(\{\widetilde{k}\}) < \mu(\{\widetilde{k}\}).$$

We arrive at a contradiction. So $\widetilde{k}$ has at least one coordinate different from 1.

(b) Set

$$\begin{aligned}
&A = \{i \in \{1, \dots, d\} : \widetilde{k}_i = 1\}, \qquad B = \{1, \dots, d\} \setminus A,\\
&T = \{x \in Q : i \in A \Rightarrow x_i = 1,\ \text{there exists } i \in B \text{ with } x_i = 1\},\\
&P = \{x \in Q : i \in A \Rightarrow x_i = 1,\ i \in B \Rightarrow 0 \le x_i < 1\},\\
&M = \{j \in \{1, \dots, m\} : \mu(P \cap S_j(Q)) \neq 0\}.
\end{aligned}$$

Let $\overline{P}$ denote the closure of P. Notice that $P = I_{\widetilde{k}}^1 \cap Q$, $\overline{P} = P \cup T$ and $T \subset \bigcup_{|k| > |\widetilde{k}|} I_k^1 \cap Q$. It follows that $\mu(P) \neq 0$, $\mu(T) = 0$ and $M \neq \emptyset$.

Fix $j \in M$. Notice that $P \cap S_j(Q) \neq \emptyset$ because $\mu(P \cap S_j(Q)) \neq 0$. Furthermore, for any $x \in P \cap S_j(Q)$ we have $1 \le x_i \le b_{ji} + \rho$ for all $i \in A$ and therefore

$$b_{ji} + \rho = 1, \quad i \in A, \quad \text{and} \quad x \in S_j(P \cup T).$$

So $b_{ji} + \rho = 1$ for all $i \in A$ and

$$P \cap S_j(Q) = P \cap S_j(P \cup T) = (P \cap S_j(P)) \cup (P \cap S_j(T)).$$

Fix $y \in P$. Let $x = S_j(y)$. Then, for all $i \in A$, $y_i = 1$ and so $x_i = b_{ji} + \rho$. In addition, for all $i \in B$, $0 \le y_i < 1$ and so $b_{ji} \le x_i = b_{ji} + \rho y_i < b_{ji} + \rho \le 1$.

It follows that $S_j(y) \in P$. Thus, $S_j(P) \subset P$ and we obtain

$$P \cap S_j(Q) = S_j(P) \cup (P \cap S_j(T)). \tag{9}$$

We have

$$\mu(P) = \sum_{j=1}^{n} a_j \mu(S_j^{-1}(P)) = \sum_{j=1}^{n} a_j \mu(S_j^{-1}(P \cap S_j(Q))).$$

Notice that

$$j \notin M \;\Rightarrow\; \mu(P \cap S_j(Q)) = 0 \;\Rightarrow\; \mu(S_j^{-1}(P \cap S_j(Q))) = 0.$$

Since $\mu(P) \neq 0$, it follows from these implications and (9) that

$$\begin{aligned}\mu(P) &= \sum_{j \in M} a_j \mu(S_j^{-1}(P \cap S_j(Q))) = \sum_{j \in M} a_j (\mu(P) + \mu(S_j^{-1}(P) \cap T)) \\ &= \sum_{j \in M} a_j \mu(P).\end{aligned}$$

Therefore $M = \{1, \ldots, m\}$ and

$$S(Q) = \bigcup_{j=1}^{m} S_j(Q) \subset P_\rho = \{x \in Q : d(x, P) \leq \sqrt{d}\,\rho\},$$

where $d(x, P)$ denotes the distance between x and P. According to (9), we have $S_j(P) \subset P \cap S_j(Q)$ for $j = 1, \ldots, m$. It follows that

$$S_{j_2} \circ S_{j_1}(P) \subset S_{j_2}(P) \cap S_{j_2} \circ S_{j_1}(Q) \subset P \cap S_{j_2} \circ S_{j_1}(Q)$$

for all $(j_1,\ j_2) \in \Lambda_2$. By iterating this process, we have $S_J(P) \subset P \cap S_J(Q)$ for all positive integers n and all $J \in \Lambda_n$. So for all positive integers n,

$$S^n(Q) = \bigcup_{J \in \Lambda_n} S_J(Q) \subset P_{\rho^n} = \{x \in Q : d(x, P) \leq \sqrt{d}\,\rho^n\}.$$

Since $(S^n(Q))_{n \geq 1}$ and $(P_{\rho^n})_{n \geq 1}$ are two decreasing sequences of bounded subsets of $\mathbb{R}^d$ which converge respectively to E and P, we have

$$1 = \mu(E) = \mu(\overline{P}) = \mu(P) = \mu(I_{\widetilde{k}}^{1}).$$

So $K_0 = \{\widetilde{k}\}$.

General case: For a given permutation σ of $\{1, \ldots, d\}$, there exists an integer $s > 1$ such that $\sigma^s = \sigma \circ \cdots \circ \sigma = \mathrm{id}_{\mathbb{R}^d}$. Notice that for all $J \in \Lambda_s$, $S_J = \rho^s \mathrm{id}_{\mathbb{R}^d} + b_J$ where

$$\frac{1}{\rho^s}\, b_J = \frac{1}{\rho^s} \sum_{i=1}^{s} \rho^{s-i} R^{s-i} b_{ji}$$

has all its coordinates in $\{0, 1, \ldots, 1/\rho^s - 1\}$. In addition, if J and J'' are two distinct elements of Λ_s then b_J and $b_{J''}$ are distinct. We have

$$\mu = \sum_{J \in \Lambda_s} a_J \mu \circ S_J^{-1}$$

with $a_J = \prod_{i=1}^s a_{j_i} \in (0,1)$ for all $J \in \Lambda_s$ and $\sum_{J \in \Lambda_s} a_J = 1$. So μ is the self-similar measure associated to the system $(S_J,\ a_J,\ m^s)$ of m^s contractive similarities S_J and weights a_J $(J \in \Lambda_s)$. We apply the particular case to conclude that K_0 is a singleton. ■

The construction of the generalized Cantor measure allows us to make the following remarks.

REMARK 3.3. Let $\mu = \mu(S_j, a_j, m)$ be a generalized Cantor measure and $K_0 = \{\widetilde{k}\}$ defined as in Proposition 3.2.

(1) Let n be a positive integer and $K_n = \{k \in \mathbb{N}^d\ :\ \mu(I_k^{\rho^n}) \neq 0\}$. Then:
   (a) K_n has m^n elements,
   (b) to each element k of K_n is associated an element J of Λ_n such that

$$I_k^{\rho^n} = S_J(I_{\widetilde{k}}^1) \quad \text{and} \quad \mu(I_k^{\rho^n}) = \prod_{i=1}^n a_{j_i}.$$

(2) For $1 \le p < \infty$ we have

$$ {}_{\rho^n}\|\mu\|_p = \Big(\sum_{j=1}^m a_j^p\Big)^{n/p}, \qquad n \in \mathbb{N}. \tag{10}$$

Now we state and prove the main result of this note.

THEOREM 3.4. *Let* $1 < p \le 2$ *and* μ *be a generalized Cantor measure. Suppose that* $\alpha > 1$ *satisfies* $\rho^{d/\alpha'} = (\sum_{j=1}^m a_j^p)^{1/p}$. *Then:*

(i) *There exists a constant* $D_1 = D_1(p, d)$ *such that*

$$\sup_{r>0} r^{d(1/\alpha' - 1/p')}\Big(\int_{J_x^r} |\widehat{\mu}(y)|^{p'}\, dy\Big)^{1/p'} \le D_1 \rho^{-d/\alpha'}, \qquad x \in \mathbb{R}^d. \tag{11}$$

(ii) *For* $p = 2$, *there exists a constant* $D_2 = D_2(d)$ *such that*

$$\begin{aligned} D_2 \rho^{d/\alpha'} &\le \inf_{r \ge 1} r^{d(1/\alpha' - 1/2)}\Big(\int_{J_0^r} |\widehat{\mu}(y)|^2\, dy\Big)^{1/2} \\ &\le \sup_{r\ge1}\sup_{x \in \mathbb{R}^d} r^{d(1/\alpha' - 1/2)}\Big(\int_{J_x^r} |\widehat{\mu}(y)|^2\, dy\Big)^{1/2} \le D_1 \rho^{-d/\alpha'}. \end{aligned}$$

Proof. Since $\|\widehat{\mu}\|_\infty \le 1$, we have

$$r^{d(1/\alpha'-1/p')}\Big(\int_{J_x^r} |\widehat{\mu}(y)|^{p'}\,dy\Big)^{1/p'} \le 1, \quad x\in\mathbb{R}^d,\ r\le 1. \tag{12}$$

Let $r \ge 1$. Then there exists a non-negative integer n such that $\rho^{n+1} < 1/r \le \rho^n$. By Proposition 2.1 and (10), we have

$$r^{d/\alpha'}\ {}_{1/r}\|\mu\|_p \le (\rho^{n+1})^{-d/\alpha'} 3^{d/p'} 2^{d/p}\ {}_{\rho^n}\|\mu\|_p = 3^{d/p'}2^{d/p}\rho^{-d/\alpha'} \tag{13}$$

and

$$r^{d/\alpha'}\ {}_{1/r}\|\mu\|_p \ge (\rho^{n})^{-d/\alpha'} 3^{-d/p'} 2^{-d/p}\ {}_{\rho^{n+1}}\|\mu\|_p = 3^{-d/p'}2^{-d/p}\rho^{d/\alpha'}. \tag{14}$$

Therefore, combining (13) and (6), we obtain

$$r^{d(1/\alpha'-1/p')}\ {}_r\|\widehat{\mu}\|_{p',\infty} \le C\,3^{d/p'}2^{d/p}\rho^{-d/\alpha'}, \quad r\ge 1. \tag{15}$$

From (12) and (15), we deduce (11) by taking $D_1 = \max\{1, C\,3^{d/p'}2^{d/p}\}$.

Suppose now that $p=2$. Combining (14) and Theorem 2.2, we obtain the first inequality of assertion (ii) of Theorem 3.4. The other inequalities are simple consequences of (11). ■

REMARK 3.5. Note that for $p\ge 1$, if $\alpha \ge 1$ satisfies $\rho^{d/\alpha'} = (\sum_{j=1}^m a_j^p)^{1/p}$, then $1<\alpha\le p$ since $\rho^{d(p-1)} \le 1/m^{p-1} \le \sum_{j=1}^m a_j^p < 1$. Therefore, from Theorem 3.4, we deduce the inequalities (3) and (4) displayed in the introduction by taking $\beta_p = dp'/\alpha'$.

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Bérenger Akon Kpata, Ibrahim Fofana, Konin Koua
UFR Mathématiques et Informatique
Université de Cocody
22 BP 582 Abidjan 22, Côte d'Ivoire
E-mail: kpata_akon@yahoo.fr
fofana_ib_math_ab@yahoo.fr
kroubla@yahoo.fr

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