COLLOQUIUM MATHEMATICUM

VOL. 119 2010 NO. 1

SEPARATED SEQUENCES IN ASYMPTOTICALLY UNIFORMLY CONVEX BANACH SPACES

BY

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Abstract. We prove that the unit sphere of every infinite-dimensional Banach space X contains an α-separated sequence, for every $0 < \alpha < 1 + \overline{\delta}_X(1)$, where $\overline{\delta}_X$ denotes the modulus of asymptotic uniform convexity of X.

1. Introduction. Elton and Odell [2] proved that the unit sphere S_X of every infinite-dimensional normed linear space X contains a $(1+\varepsilon)$-separated sequence, for some $\varepsilon > 0$ depending on the space X. Recall that a sequence (x_n) in $(X, \|\cdot\|)$ is said to be *α-separated*, for some $\alpha > 0$, if $\|x_n - x_m\| \geq \alpha$ for every $n \neq m$. Diestel [1, p. 254] asked whether this ε can be quantified. Kryczka and Prus [4] answered this question for the class of non-reflexive Banach spaces, proving that the unit sphere of such a space contains a $4^{1/5}$-separated sequence. Van Neerven [6] studied the class of uniformly convex Banach spaces and connected together ε and the modulus of convexity (see comments below).

In this note, we are interested in the class of asymptotically uniformly convex Banach spaces. We connect ε and the modulus of asymptotic uniform convexity. This modulus has been introduced by Milman [5] and investigated by Johnson, Lindenstrauss, Preiss and Schechtman [3]. The *modulus of asymptotic uniform convexity* of an infinite-dimensional Banach space X is given for $t > 0$ by

$$\overline{\delta}_X(t) = \inf_{\|x\|=1} \sup_{\substack{H \subset X \\ \operatorname{codim} H < \infty}} \inf_{\substack{h \in H \\ \|h\| \geq t}} \|x + h\| - 1.$$

The Banach space X is said to be *asymptotically uniformly convex* if $\overline{\delta}_X(t) > 0$ for all $0 < t < 1$. If X is a subspace of ℓ_p, $1 \leq p < \infty$, then $\overline{\delta}_X(t) = (1+t^p)^{1/p} - 1$, and X is asymptotically uniformly convex. If X is a subspace

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2010 *Mathematics Subject Classification*: Primary 46B20.

Key words and phrases: separated sequences, asymptotically uniformly convex Banach spaces, modulus of asymptotic uniform convexity, Elton–Odell theorem.

This work is a part of the author's PhD thesis, submitted in June 2005 at the Institut de Mathématiques de Bordeaux, Université Bordeaux I, and written under the supervision of Robert Deville.

DOI: 10.4064/cm119-1-7

of c_0 then $\overline{\delta}_X(t) = 0$ for every $t \in (0;1]$, so that X is not asymptotically uniformly convex.

2. Statement and proof of the main result

THEOREM 1. *The unit sphere of every infinite-dimensional Banach space X contains an α-separated sequence for every $0 < \alpha < 1 + \overline{\delta}_X(1)$.*

Proof. Fix $0 < \alpha < 1 + \overline{\delta}_X(1)$ and $x_1 \in S_X$. Write $X = H_0$. There exists a finite-codimensional subspace $H_1 \subset H_0$ such that for every $h \in S_{H_1}$, $\|x_1 - h\| = \|x_1 + (-h)\| > \alpha$. Take $x_2 \in S_{H_1}$. It is easy to see that $\overline{\delta}_{H_1}(1) \geq \overline{\delta}_X(1)$. As before, there exists a finite-codimensional subspace $H_2 \subset H_1$ such that for every $h \in S_{H_2}$, $\|x_2 - h\| > \alpha$. Take $x_3 \in S_{H_2}$. As $S_{H_2} \subset S_{H_1}$, we have $\|x_1 - x_3\| > \alpha$ too. Inductively, we construct an α-separated sequence (x_n) in S_X along with a corresponding non-increasing sequence of finite-codimensional subspaces (H_n). These sequences are chosen so that $x_{n+1} \in S_{H_n}$ with $H_n \subset H_{n-1}$ such that $\|x_n - h\| > \alpha$ for every $h \in S_{H_n}$. ■

3. Comments. If X is asymptotically uniformly convex, then $\overline{\delta}_X(1) > 0$ and the ε obtained by Elton and Odell [2] is quantified. In $X = \ell_p$, $1 \leq p < \infty$, the sequence of unit vectors is $2^{1/p}$-separated and $1 + \overline{\delta}_X(1) = 2^{1/p}$. This answers the question raised in [6] whether Theorem 1.2 therein can be improved. This improvement occurs in two ways. Indeed, according to Proposition 2.3(3) in [3], for every infinite-dimensional Banach space X and every $0 < \varepsilon < 1$, we have $\delta_X(\varepsilon) \leq \overline{\delta}_X(\varepsilon)$ (where δ_X denotes the modulus of convexity of X). First, every infinite-dimensional uniformly convex Banach space is asymptotically uniformly convex, but the converse is false (consider ℓ_1). Secondly, quantitatively we have $1 + \overline{\delta}_X(1) \geq 1 + \frac{1}{2}\delta_X(2/3)$ where the right-hand side is the separation constant obtained in [6].

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Received 10 December 2008;
revised 19 August 2009 (5143)