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A CHARACTERIZATION OF SOBOLEV SPACES VIA LOCAL DERIVATIVES

BY

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Abstract. Let $1 \le p < \infty$, $k \ge 1$, and let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set. We prove a converse of the Calderón–Zygmund theorem that a function $f \in W^{k,p}(\Omega)$ possesses an L^p derivative of order k at almost every point $x \in \Omega$ and obtain a characterization of the space $W^{k,p}(\Omega)$. Our method is based on distributional arguments and a pointwise inequality due to Bojarski and Hajłasz.

1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $1 \le p < \infty$. The Sobolev space $W^{k,p}(\Omega)$, $k \ge 1$, consists of all functions $f \in L^p(\Omega)$ whose distributional partial derivatives of all orders up to and including k are also elements of $L^p(\Omega)$. In other words, for every multi-index σ with $|\sigma| \le k$ there exists $D^\sigma f \in L^p(\Omega)$, termed the *order σ weak derivative* of f, satisfying

$$\int_\Omega f(D^\sigma \varphi)\,dx = (-1)^{|\sigma|} \int_\Omega (D^\sigma f)\varphi\,dx \tag{1.1}$$

for all $\varphi \in C_0^\infty(\Omega)$. The space $W^{k,p}(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{W^{k,p}(\Omega)} = \sum_{|\sigma|\le k} \|D^\sigma f\|_{L^p(\Omega)}. \tag{1.2}$$

Sobolev spaces and their variants are widely used in applications such as the study of partial differential equations and calculus of variations. An important and interesting problem is to determine minimal conditions on f which guarantee the validity of (1.1).

A related concept is that of local differentiability. Given $x \in \mathbb{R}^n$ and $k \ge 1$, the space $T^{k,p}(x)$ consists of all $f \in L^p(\mathbb{R}^n)$ for which there exists a polynomial P_x^{k-1} with degree less than k satisfying

$$\sup_{r>0} r^{-kp} \fint_{B(x,r)} |f(y) - P_x^{k-1}(y)|^p\,dy < \infty. \tag{1.3}$$

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Here we use the notation

$$\fint_E g\,dy = \frac{1}{|E|}\int_E g\,dy \tag{1.4}$$

for the integral average of g over the set E. If $f \in T^{k,p}(x)$, the quantities $f^\sigma(x)$ are defined for $|\sigma| \le k-1$ as the coefficients of P_x^{k-1} when expressed as a polynomial centered at x:

$$P_x^{k-1}(y) = \sum_{|\sigma|\le k-1} f^\sigma(x)\,\frac{(y-x)^\sigma}{\sigma!}. \tag{1.5}$$

A function $f \in T^{k,p}(x)$ is said to have a *local L^p derivative of order k* at x if there exists a polynomial P_x^k with degree not exceeding k with the additional property that

$$\lim_{r\to 0^+} r^{-kp} \fint_{B(x,r)} |f(y) - P_x^k(y)|^p\,dy = 0. \tag{1.6}$$

Calderón and Zygmund [5] showed that if $f \in W^{k,p}(\mathbb{R}^n)$, then f has a local L^p derivative of order k at almost every $x \in \mathbb{R}^n$, where the polynomial P_x^k is precisely the order k Taylor polynomial of f at x. It was later proved by Ziemer [15] that the limit is uniform outside open sets of small Lebesgue measure. In general, there is no converse to either the Calderón–Zygmund or Ziemer theorems: it is possible for a function f to satisfy (1.6) almost uniformly, yet fail to satisfy (1.1). A simple example is the indicator function of the unit ball in $\mathbb{R}^n$.

Many function spaces may be characterized using local derivatives provided that an appropriate integrability condition is placed on the quantity in (1.3); see e.g. [3], [4], [13], [14, Section 1.7]. Such spaces are said to be *definable via local approximations.* Recently, Shvartsman [11] has shown that functions in certain Sobolev spaces $W^{k,p}(\mathbb{R}^n)$, Besov spaces $B_s^{p,q}(\mathbb{R}^n)$, and Lizorkin–Triebel spaces $F_s^{p,q}(\mathbb{R}^n)$ may be characterized via local approximation, and used these characterizations to address the problem of extending a function f defined on a regular subset of $\mathbb{R}^n$ to one of these spaces.

In the case of Sobolev spaces, known characterizations of $W^{k,p}(\Omega)$ require that $p > 1$. Nevertheless, certain partial converses do exist in case $p = 1$. Bagby and Ziemer [1] proved that if $f \in T^{k,1}(x)$ for *every* $x \in \mathbb{R}^n$, and $f^\sigma \in L^1(\mathbb{R}^n)$ for every $|\sigma| \le k$, then in fact $f \in W^{k,1}(\mathbb{R}^n)$. Moreover, this fact remains true if $f \in T^{k,1}(x)$ for all x outside a Borel set B whose $n-1$-dimensional integral-geometric measure is zero. See [16, Chapter 3] for further details.

The purpose of the present article is to obtain a characterization of the space $W^{k,p}(\Omega)$ using local derivatives which is valid for arbitrary open sets $\Omega \subset \mathbb{R}^n$ and for all $1 \le p < \infty$. In order to work with functions defined on the open set Ω rather than the whole space $\mathbb{R}^n$ we will use a variant of the space $T^{k,1}(x)$ which coincides with the definition above when $\Omega = \mathbb{R}^n$.

DEFINITION 1.1. Let $x \in \Omega$ and let

$$\delta(x) = \operatorname{dist}(x, \Omega). \tag{1.7}$$

For each $k \geq 1$ we denote by $T^{k,1}_{\Omega}(x)$ the space of all $f \in L^1_{\mathrm{loc}}(\Omega)$ for which there exists a polynomial P^{k-1}_x with degree less than k satisfying

$$\sup_{0<r<\delta(x)} r^{-k} \fint_{B(x,r)} |f(y) - P^{k-1}_x(y)|\, dy < \infty.$$

Given $f \in T^{k,1}_{\Omega}(x)$ we define

$$\Phi^r_f(x) = r^{-k} \fint_{B(x,r)} |f(y) - P^{k-1}_x(y)|\, dy < \infty, \qquad 0 < r < \delta(x). \tag{1.8}$$

The main result is stated next. We write $U \subset\subset \Omega$ to indicate that $\overline{U} \subset \Omega$.

THEOREM 1.2. *Let $1 \leq p < \infty$ and let $f \in L^1_{\mathrm{loc}}(\Omega)$. The following are equivalent:*

(1) $f \in W^{k,p}(\Omega)$,
(2) $f \in T^{k,1}_{\Omega}(x)$ *for almost all* $x \in \Omega$, *where*
  (a) $f^{\sigma} \in L^p(\Omega)$ *for all* $|\sigma| \leq k-1$,
  (b) *there exists* $h \in L^p(\Omega)$ *with the property that*

$$\liminf_{r \to 0^+} \int_Q \Phi^r_f(x)\, dx \leq \int_Q h\, dx$$

for every open n-cube $Q \subset\subset \Omega$.

Moreover, the quantities

$$\|f\|_{W^{k,p}(\Omega)} \quad \textit{and} \quad \sum_{|\sigma| \leq k-1} \|f^{\sigma}\|_{L^p(\Omega)} + \inf \|h\|_{L^p(\Omega)}$$

are equivalent, where the infimum is taken over all $h \in L^p(\Omega)$ satisfying condition (2)(b).

2. Preliminaries. Henceforth we assume that $\Omega \subset \mathbb{R}^n$ is an arbitrary open set and that $1 \leq p < \infty$. We denote by $C_{a,b,\ldots}$ a constant whose particular value depends only on $a, b, \ldots$ and no other parameters. For each $g \in L^1_{\mathrm{loc}}(\Omega)$, we identify g with its representative satisfying

$$g(x) = \lim_{r \to 0^+} \fint_{B(x,r)} g(y)\, dy \tag{2.1}$$

at all points $x \in \Omega$ where the limit exists. It will be convenient to regard g as undefined at all other points.

DEFINITION 2.1. Let $U \subset \mathbb{R}^n$ be an open set and let $\varepsilon > 0$. Define

$$U_\varepsilon = \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\},$$
$$U^\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, U) < \varepsilon\}.$$

DEFINITION 2.2. A *regularizing kernel* is a nonnegative function $\psi \in C_0^\infty(B(0,1))$ with integral 1. Corresponding to a regularizing kernel ψ we define the family $\{\psi_\varepsilon\}_{\varepsilon>0}$ of *regularizers* by

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^n}\,\psi\left(\frac{x}{\varepsilon}\right),$$

and for a locally integrable function $f : \Omega \to \mathbb{R}$ we define the family $\{f_\varepsilon\}$ of *regularizations* of f by

$$f_\varepsilon(x) = f * \psi_\varepsilon(x), \qquad x \in \Omega_\varepsilon.$$

We recall the following standard results concerning the regularizations of f (cf. [9], [16]):

PROPOSITION 2.3. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$.*

(1) *$f_\varepsilon \in C^\infty(\Omega_\varepsilon)$ for each $\varepsilon > 0$ and $f_\varepsilon(x) \to f(x)$ for almost all $x \in \Omega$.*
(2) *If $f \in W^{k,p}(\Omega)$ then $D^\sigma(f_\varepsilon) = (D^\sigma f)_\varepsilon = f * D^\sigma \psi_\varepsilon$ for each multi-index $|\sigma| \le k$ and*

$$\|f_\varepsilon - f\|_{W^{k,p}(\Omega')} \to 0 \qquad \text{as } \varepsilon \to 0^+$$

for every open set $\Omega' \subset\subset \Omega$.

The following result was proved by Calderón and Zygmund (cf. [5], [16]).

PROPOSITION 2.4. *Let $k \ge 1$. There exists a regularizing kernel ψ which commutes with polynomials in the sense that*

$$P_\varepsilon = P * \psi_\varepsilon = P$$

for every polynomial P whose degree does not exceed k.

DEFINITION 2.5. Let $f \in W^{k,p}(\Omega)$ and let $m \le k$. The order m *Taylor polynomial* of f at a point $x \in \Omega$ is

$$T_x^m f(y) = \sum_{|\sigma|\le m} D^\sigma f(x)\,\frac{(y-x)^\sigma}{\sigma!}, \tag{2.2}$$

provided that each $D^\sigma f(x)$ is defined as in (2.1).

A simple consequence of Proposition 2.3 is that if $f \in W^{k,p}(\Omega)$, then $T_x^k f_\varepsilon(y) \to T_x^k f(y)$ as $\varepsilon \to 0^+$ for almost all $x \in \Omega$ and all $y \in \mathbb{R}^n$.

DEFINITION 2.6. Let $B \subset \mathbb{R}^n$ be a ball and let $\alpha > 0$. The *Riesz potential* $I_B^\alpha f$ of a function $f : B \to \mathbb{R}$ is defined for $x \in \mathbb{R}^n$ by

$$I_B^\alpha f(x) = \int_B \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy.$$

Throughout the paper we denote by $D^k f(z)$ the family $\{D^\sigma f(z)\}_{|\sigma|=k}$ and by $|D^k f|(z)$ its Euclidean norm. The following Riesz potential estimate involving $|D^k f|$ is due to Bojarski and Hajłasz [2] and is of fundamental importance.

Proposition 2.7. *Suppose that $f \in W^{k,p}(\Omega)$ and $x \in \Omega$. If x is a point where $T_x^{k-1} f$ is defined and B is a ball of radius r containing x, then*

$$|f(y) - T_x^{k-1} f(y)| \le C_{n,k}\Big(I_B^k |D^k f|(y) + \sum_{m=0}^{k-1} r^m I_B^{k-m} |D^k f|(x)\Big)$$

for almost all $y \in B$.

Finally, we recall the following Riesz potential estimate due to Hedberg [8].

Proposition 2.8. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, $x \in \mathbb{R}^n$, $r > 0$, and $\alpha < n$. Then*

$$I^\alpha_{B(x,r)} f(x) \le C_{\alpha,n} r^\alpha M f(x),$$

where Mf is the Hardy–Littlewood maximal function of f.

3. A criterion for weak differentiability. The following well-known proposition will be useful for localizing our arguments.

Proposition 3.1. *Let $f : \Omega \to \mathbb{R}$. If $f \in W^{k,p}(Q)$ for every open n-cube $Q \subset\subset \Omega$, then $D^\sigma f$ is well defined in Ω for each multi-index $|\sigma| \le k$, and if each $D^\sigma f \in L^p(\Omega)$, then $f \in W^{k,p}(\Omega)$.*

The proof of Theorem 1.2 relies on the following general characterization of the Sobolev space $W^{k,p}(\Omega)$.

Theorem 3.2. *Let $f \in L^1_{\mathrm{loc}}(\Omega)$, let $k \ge 1$, and let $1 \le p < \infty$. Then $f \in W^{k,p}(\Omega)$ if and only if there exists $g \in L^p(\Omega)$ with the property that*

$$\Big|\int_Q f D^\sigma \varphi \, dx\Big| \le \|\varphi\|_\infty \int_Q g \, dx \tag{3.1}$$

for every multi-index σ with $|\sigma| \le k$, open n-cube $Q \subset\subset \Omega$, and $\varphi \in C_0^k(Q)$. Moreover, the quantities

$$\|f\|_{W^{k,p}(\Omega)} \quad \text{and} \quad \inf \|g\|_{L^p(\Omega)}$$

are equivalent, where the infimum is taken over all g satisfying (3.1).

Proof. The necessity follows from (1.1) with

$$g = \sum_{|\sigma| \le k} |D^\sigma f|,$$

in which case we see that

$$\inf \|g\|_{L^p(\Omega)} \le \|f\|_{W^{k,p}(\Omega)}.$$

As for sufficiency, suppose that $g \in L^p(\Omega)$ satisfies (3.1). Let $Q \subset\subset \Omega$ be an open n-cube. In light of Proposition 3.1 it suffices to show that $f \in W^{k,p}(Q)$ and that $|D^\sigma f|$ is comparable to m for each multi-index σ with $|\sigma| \le k$. For all such σ define the functional $L_\sigma : C_0^k(Q) \to \mathbb{R}$ by

$$L_\sigma(\varphi) = \int_Q f D^\sigma \varphi \, dx, \tag{3.2}$$

so that L is a continuous linear functional on $C_0^k(Q)$ satisfying

$$|L_\sigma(\varphi)| \le \|\varphi\|_\infty \int_Q g \, dx. \tag{3.3}$$

Let us write

$$X = \{\varphi \in C(\overline{Q}) : u|_{\partial Q} = 0\}.$$

Since $C_0^k(\Omega)$ is dense in X with respect to the topology of uniform convergence, L_σ may be uniquely extended to a continuous linear functional on X satisfying (3.3) for all $\varphi \in X$. The Riesz representation theorem (cf. [10, Thm. 6.19]) guarantees the existence of a signed Borel measure μ_σ on Q with finite total variation such that

$$L_\sigma \varphi = \int_\Omega \varphi \, d\mu_\sigma$$

for all $\varphi \in X$. The total variation $|\mu_\sigma|(U)$ of μ_σ on an open set $U \subset Q$ is given by

$$\sup\Big\{ \int_Q \varphi \, d\mu_\sigma : \varphi \in X,\ \|\varphi\|_\infty \le 1 \Big\} \\ = \sup\Big\{ \int_U \varphi \, d\mu_\sigma : \varphi \in C_0^k(U),\ \|\varphi\|_\infty \le 1 \Big\},$$

and by the argument just given we see that

$$|\mu_\sigma|(Q') \le \int_{Q'} g \, dx \tag{3.4}$$

for every open subcube $Q' \subset Q$. For an arbitrary open set $U \subset Q$, one may employ a Whitney decomposition of U (cf. [12, p. 169]) to find a sequence $\{Q_j\}$ of open subcubes of U with the property that

$$U = \bigcup_j Q_j \quad \text{and} \quad \sum_j \chi_{Q_j} \le C_n, \tag{3.5}$$

where C_n depends only on n. It follows from (3.4) and (3.5) that

$$|\mu_\sigma|(U) \le C_n \int_U g\,dx.$$

The outer regularity of the Lebesgue measure implies that μ_σ is absolutely continuous with respect to the Lebesgue measure on Q. The Radon–Nikodým theorem (cf. [10, Thm. 6.10]) implies in turn the existence of $h_\sigma \in L^1(Q)$ with the property that $d\mu_\sigma = h_\sigma\,dx$, and consequently

$$\int_Q f D^\sigma \varphi\,dx = \int_Q h^\sigma \varphi\,dx \tag{3.6}$$

for all $\varphi \in C^k(Q)$. Therefore, $D^\sigma f$ exists in the weak sense and $D^\sigma f = (-1)^{|\sigma|} h^\sigma$. It follows from (3.2), (3.3), and (3.6) that $|D^\sigma f| \le C_n g$ in Ω, hence $D^\sigma f \in L^p(\Omega)$. Since this holds for all $|\sigma| \le k$, Proposition 3.1 implies that $f \in W^{k,p}(\Omega)$ and that

$$\|f\|_{W^{k,p}(\Omega)} \le C_{n,k}\|g\|_{L^p(\Omega)}$$

as desired. Finally, take the infimum over all g satisfying (3.1). ∎

4. An integral estimate. Recall the definition of Φ_f^r from (1.8) above. The following lemma gives an estimate of Φ_f^r corresponding to $f \in W^{k,p}(\Omega)$ in terms of the highest order derivatives of f. The proof relies on Propositions 2.7 and 2.8.

LEMMA 4.1. *Suppose that $k \ge 1$ and that $f \in W^{k,p}(\Omega)$. Then $f \in T_\Omega^{k-1}(x)$ for almost all $x \in \Omega$, and if Q is an n-cube with $Q \subset\subset \Omega$ then*

$$\int_Q \Phi_f^r(x)\,dx \le C_{n,k} \int_{Q^r} |D^k f(x)|\,dx$$

for all $0 < r < \operatorname{dist}(Q, \partial\Omega)$.

Proof. Suppose that $f \in W^{k,p}(\Omega)$. Let $x \in \Omega$ be a point where $T_x^{k-1} f$ is defined and let $0 < r < \delta(x)$. Proposition 2.7 implies

$$\fint_B |f(y) - T_x^{k-1} f(y)|\,dy \le C_{n,k}\Big(\fint_B I_B^k |D^k f|(y)\,dy + \sum_{m=0}^{k-1} r^m I_B^{k-m}|D^k f(x)|\Big).$$

Tonelli's theorem and Proposition 2.8 imply

$$\begin{aligned}\fint_B I_B^k |D^k f|(y)\,dy &= \fint_B \int_B \frac{|D^k f(z)|}{|y-z|^{n-k}}\,dz\,dy = \fint_B |D^k f(z)|\,dz \int_B \frac{1}{|y-z|^{n-k}}\,dy\,dz \\ &= C_{n,k} r^k \fint_B |D^k f(z)|\,dz,\end{aligned}$$

so we have

$$\text{(4.1)}\quad \Phi_f^r(x) \le C_{n,k}\Big(\fint_{B(x,r)} |D^k f(z)|\,dz + \sum_{m=0}^{k-1} r^{m-k} \int_{B(x,r)} \frac{|D^k f(z)|}{|x-z|^{n-k+m}}\,dz \Big).$$

A second application of Proposition 2.8 implies

$$\sup_{0<r<\delta(x)} \Phi_f^r(x) \le C_{n,k} M|D^k f|(x).$$

Since $|D^k f| \in L^p(\Omega)$, this quantity is finite almost everywhere (including the case $p=1$, in which it fails to be integrable) and thus $f \in T_\Omega^{k,1}(x)$ for almost all $x \in \Omega$.

Now let $Q \subset\subset \Omega$ and let $0 < r < \operatorname{dist}(Q, \partial\Omega)$. Tonelli's theorem implies

$$\int_Q \fint_{B(x,r)} |D^k f(z)|\,dz\,dx \le \int_{Q^r} |D^k f(z)| \int_Q C_n r^{-n} \chi_{B(x,r)}(z)\,dx\,dz$$

because $x \in Q$ implies $B(x,r) \subset Q^r$. Since $\chi_{B(x,r)}(z) = \chi_{B(z,r)}(x)$ it follows that

$$\int_Q C_n r^{-n}\chi_{B(x,r)}(z)\,dx \le \int_{\mathbb{R}^n} C_n r^{-n} \chi_{B(z,r)}(x)\,dx = 1.$$

Therefore

$$\int_Q \fint_{B(x,r)} |D^k f(z)|\,dz\,dx \le \int_{Q^r} |D^k f(z)|\,dz.$$

Similarly, we have

$$\begin{aligned}\int_Q r^{m-k} \int_{B(x,r)} & \frac{|D^k f(z)|}{|x-z|^{n-k+m}}\,dz\,dx \\ &= \int_{Q^r} |D^k f(z)| \int_Q \frac{r^{m-k}}{|x-z|^{n-k+m}}\,\chi_{B(x,r)}(z)\,dx\,dz \\ &\le C_n \int_{Q^r} |D^k f(z)| \int_{B(z,r)} \frac{r^{m-k}}{|x-z|^{n-k+m}}\,dx\,dz,\end{aligned}$$

where

$$r^{m-k}\int_{\mathbb{R}^n} \frac{1}{|x-z|^{n-k+m}}\,\chi_{B(z,r)}(x)\,dx = r^{m-k} \int_{B(z,r)} \frac{1}{|x-z|^{n-k+m}}\,dz = C_{n,k-m}$$

by Proposition 2.8. Therefore we may integrate (4.1) over U to obtain

$$\int_Q \Phi_f^r(x)\,dx \le C_{n,k} \int_{Q^r} |D^k f(z)|\,dz. \ \blacksquare$$

5. The proof of Theorem 1.2, necessity. Assume that $f \in W^{k,p}(\Omega)$. The fact that $f \in T^{k,1}_{\Omega}(x)$ for almost all $x \in \Omega$ is well known and proved in Lemma 4.1. If $Q \subset\subset \Omega$, then one may apply Lemma 4.1 and e.g. the dominated convergence theorem to obtain

$$\liminf_{r\to 0^+} \int_Q \Phi^r_f(x)\, dx \le C_{n,k} \int_Q |D^k f(z)|\, dz.$$

Writing $h = C_{n,k}|D^k f|$ we deduce that

$$\liminf_{r\to 0^+} \int_Q \Phi^r_f(x)\, dx \le \int_Q h\, dx \tag{5.1}$$

for every open n-cube $Q \subset\subset \Omega$, and the definition of the $W^{k,p}$ norm implies

$$\sum_{|\sigma|\le k-1} \|f^\sigma\|_{L^p(\Omega)} + \inf \|h\|_{L^p(\Omega)} \le C_{n,k}\|f\|_{W^{k,p}(\Omega)}$$

where the infimum is taken over all h satisfying (5.1).

6. The proof of Theorem 1.2, sufficiency. Conversely, assume that $f \in L^1_{\mathrm{loc}}(\Omega)$ satisfies condition (2) of Theorem 1.2. Let ψ be a regularizing kernel as in Proposition 2.4 which satisfies $\psi_\varepsilon * P = P$ for every polynomial P with degree not exceeding k, and let $\{f_\varepsilon\}_{\varepsilon>0}$ be the corresponding family of regularizations of f. We will verify the hypothesis of Theorem 3.2 by adapting the method in [6].

Define

$$g = h + \sum_{|\sigma|\le k-1} f^\sigma.$$

Let $Q \subset\subset \Omega$ be an open n-cube, let $\varphi \in C^k_0(Q)$, and let σ be a multi-index with $|\sigma| \le k$. Choose $\varepsilon_0 > 0$ sufficiently small so that $Q \subset \Omega_{\varepsilon_0}$. Then $0 < \varepsilon < \varepsilon_0$ implies

$$\int_Q f_\varepsilon D^\sigma \varphi\, dx = (-1)^{|\sigma|} \int_Q (D^\sigma f_\varepsilon)\varphi\, dx = (-1)^{|\sigma|} \int_Q (f * D^\sigma \psi_\varepsilon)\varphi\, dx.$$

Since

$$\int_Q f D^\sigma \varphi\, dx = \lim_{\varepsilon\to 0} \int_Q f_\varepsilon D^\sigma \varphi\, dx,$$

it follows that

$$\Big|\int_Q f D^\sigma \varphi\, dx\Big| \le \Big(\liminf_{\varepsilon\to 0^+} \int_Q |f * D^\sigma \psi_\varepsilon|\, dx\Big)\|\varphi\|_\infty. \tag{6.1}$$

Now let $x \in \Omega$ be a point where $f \in T^{k,1}_{\Omega}(x)$ and let P^{k-1}_x be the corresponding polynomial. Since P^{k-1}_x commutes with ψ_ε it follows that

$D^\sigma P_x^{k-1} = D^\sigma(P_x^{k-1} * \psi_\varepsilon) = P_x^{k-1} * D^\sigma \psi_\varepsilon$ and

$$f * D^\sigma \psi_\varepsilon = (f - P_x^{k-1}) * D^\sigma \psi_\varepsilon + D^\sigma P_x^{k-1}.$$

Using the elementary estimate $|D^\sigma \psi_\varepsilon| \le C_n \varepsilon^{-n-|\sigma|} \chi_{B(0,\varepsilon)}$ we have

$$\begin{aligned} |f * D^\sigma \psi_\varepsilon(x)| &\le C_n \varepsilon^{-|\sigma|} \fint_{B(x,\varepsilon)} |f(y) - P_x(y)|\, dy + |D^\sigma P_x^{k-1}(x)| \\ &= C_n \varepsilon^{k-|\sigma|} \Phi_f^\varepsilon(x) + |D^\sigma P_x^{k-1}(x)|, \end{aligned}$$

for almost every $x \in \Omega$. Consequently,

$$\int_Q |f * D^\sigma \psi_\varepsilon(x)|\, dx \le C_n \varepsilon^{k-|\sigma|} \int_Q \Phi_f^\varepsilon(x)\, dx + \int_Q |D^\sigma P_x^{k-1}(x)|\, dx.$$

If $|\sigma| < k$ then $D^\sigma P_x^{k-1}(x) = f^\sigma(x)$, hence

$$\liminf_{\varepsilon \to 0^+} \int_Q |f * D^\sigma \psi_\varepsilon(x)|\, dx \le \int_Q |f^\sigma|\, dx.$$

On the other hand, if $|\sigma| = k$ then $D^\sigma P_x^{k-1} = 0$ and

$$\liminf_{\varepsilon \to 0^+} \int_Q |f * D^\sigma \psi_\varepsilon(x)|\, dx \le C_n \liminf_{\varepsilon \to 0^+} \int_Q \Phi_f^\varepsilon(x)\, dx \le C_n \int_Q h\, dx.$$

In either case it follows that

$$\liminf_{\varepsilon \to 0^+} \int_Q |f * D^\sigma \psi_\varepsilon(x)|\, dx \le C_n \int_Q g\, dx.$$

In light of (6.1), Theorem 3.2 implies that $f \in W^{k,p}(\Omega)$ and

$$\|f\|_{W^{k,p}(\Omega)} \le C_{n,k} \|g\|_{L^p(\Omega)} \le C_{n,k} \Big(\sum_{|\sigma| \le k-1} \|f^\sigma\|_{L^p(\Omega)} + \|h\|_{L^p(\Omega)} \Big).$$

Finally, take the infimum over all admissible h to obtain

$$\|f\|_{W^{k,p}(\Omega)} \le C_{n,k} \Big(\sum_{|\sigma| \le k-1} \|f^\sigma\|_{L^p(\Omega)} + \inf \|h\|_{L^p(\Omega)} \Big)$$

as desired.

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