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ON ERGODIC PROPERTIES OF CONVOLUTION OPERATORS ASSOCIATED WITH COMPACT QUANTUM GROUPS

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Abstract. Recent results of M. Junge and Q. Xu on the ergodic properties of the averages of kernels in noncommutative L^p-spaces are applied to the analysis of almost uniform convergence of operators induced by convolutions on compact quantum groups.

The classical ergodic theory was initially concerned with investigating the limits of iterations (or iterated averages) of certain transformations of a measure space. The resulting limit theorems were very quickly seen to have natural generalisations in terms of the evolutions induced by operators acting on the associated L^p-spaces (for an excellent treatment we refer to [Kre]). The noncommutative counterpart of this theory is concerned with investigation of limit properties for the iterations of operators acting on von Neumann algebras (viewed as generalisations of classical L^∞-spaces) or, more generally, on noncommutative L^p-spaces associated with a von Neumann algebra equipped with a faithful normal state. It turned out that, after introducing appropriate counterparts of the classical notion of almost everywhere convergence, one may consider in this generalised context not only mean ergodic theorems, but also "pointwise" ones. This has been investigated intensively in the 70s and 80s by C. E. Lance, F. Yeadon, R. Jajte and others. Several results were obtained for the evolutions on both von Neumann algebras and L^p-spaces associated with a faithful normal trace. Recently M. Junge and Q. Xu in a beautiful paper [JX_2] (whose main results were earlier announced in [JX_1]) proved new noncommutative maximal inequalities and thus extended many ergodic theorems to the context of Haagerup L^p-spaces, which naturally arise when the relevant state is nontracial.

In this paper we apply the results of [JX_2] to obtain ergodic theorems for the evolutions induced by convolution operators on compact quantum

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groups ([Wor$_1$]). Although it is generally natural to view compact quantum groups as C^*-algebras, due to the nature of the problems considered we prefer the von Neumann algebraic framework. It arises naturally as every compact quantum group is equipped with a Haar state and one can pass to the corresponding GNS representation. The importance of this approach, where the Haar functional is a central notion from which in a sense the whole theory is developed, is fully revealed in the context of locally compact quantum groups ([KV$_1$]). Here it provides us both with a von Neumann algebra and with a canonical reference state on it.

The plan of the paper is as follows: after establishing notation and quoting preliminary results in the first section, in Section 2 we introduce convolution operators and obtain the ergodic theorems for their actions on a compact quantum group M. Section 3 contains a discussion of the extensions to the case of Haagerup L^p-spaces associated with the Haar state on M, and in Section 4 we signal possible directions of further investigations.

1. Notations and preliminary results. The symbol $\otimes$ will denote the spatial tensor product of C^*-algebras, $\overline{\otimes}$ the ultraweak tensor product of von Neumann algebras (and relevant extension of the algebraic tensor product of normal maps); $\odot$ will be reserved for the purely algebraic tensor product.

Compact quantum groups. The notion of compact quantum groups has been introduced in [Wor$_1$]. Here we adopt the definition from [Wor$_2$]:

Definition 1.1. A *compact quantum group* is a pair (A, Δ), where A is a unital C^*-algebra, $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital, *-homomorphic map which is coassociative:

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}})\Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta)\Delta,$$

and A satisfies the quantum cancellation properties:

$$\overline{\mathrm{Lin}}((1 \otimes \mathsf{A})\Delta(\mathsf{A})) = \overline{\mathrm{Lin}}((\mathsf{A} \otimes 1)\Delta(\mathsf{A})) = \mathsf{A} \otimes \mathsf{A}.$$

One of the most important features of compact quantum groups is the existence of a dense *-subalgebra $\mathcal{A}$ (the algebra of matrix coefficients of irreducible unitary representations of A), which is in fact a Hopf *-algebra— so for example $\Delta : \mathcal{A} \to \mathcal{A} \odot \mathcal{A}$. As explained in the introduction, for us it is more convenient to work in the von Neumann algebraic context.

Definition 1.2. A *von Neumann algebraic* (*vNa*) *compact quantum group* is a pair (M, Δ), where M is a von Neumann algebra, $\Delta : \mathsf{M} \to \mathsf{M} \overline{\otimes} \mathsf{M}$ is a normal unital, *-homomorphic map which is coassociative:

$$(\Delta \overline{\otimes} \mathrm{id}_{\mathsf{A}})\Delta = (\mathrm{id}_{\mathsf{A}} \overline{\otimes} \Delta)\Delta,$$

and there exists a faithful normal state $h \in \mathsf{M}_*$ (called a *Haar state*) such that for all $x \in M$,

$$(h \,\overline{\otimes}\, \mathrm{id}_{\mathsf{M}}) \circ \Delta(x) = (\mathrm{id}_{\mathsf{M}} \,\overline{\otimes}\, h) \circ \Delta(x) = h(x)1.$$

The next lemma and the comments below it should help to understand the connection between these two types of objects.

PROPOSITION 1.3 ([Wor_2]). *Let* A *be a compact quantum group. There exists a unique state* $h \in \mathsf{A}^*$ (*called the* Haar state *of* A) *such that for all* $a \in \mathsf{A}$,

$$(h \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta(a) = (\mathrm{id}_{\mathsf{A}} \otimes h) \circ \Delta(a) = h(a)1.$$

A compact quantum group is said to be *in reduced form* if the Haar state h is faithful. If it is not the case we can always quotient out the null space of h ($\{a \in \mathsf{A} : h(a^*a) = 0\}$). This procedure in particular does not influence the underlying Hopf *-algebra $\mathcal{A}$; in fact, the reduced object may be viewed as the natural completion of $\mathcal{A}$ in the GNS representation with respect to h (as opposed for example to the universal completion of $\mathcal{A}$; for details see [BMT]). We will therefore always assume that our compact quantum groups are in reduced form.

Let A be a compact quantum group and let (π_h, H) be the (faithful) GNS representation with respect to the Haar state of A. Define $\mathsf{M} = \pi_h(\mathsf{A})''$. Then M is a von Neumann algebra, the coproduct has a normal extension to M (denoted by the same symbol) with values in $\mathsf{M} \overline{\otimes} \mathsf{M}$, and by construction the Haar state retains its invariance properties in this new framework—we obtain a vNa compact quantum group. Conversely, given a vNa compact quantum group there is a way of associating to it a C^*-algebraic object, which is a compact quantum group (see [KV_{1-2}] for the details of this construction and the statements which follow). As applying these constructions twice yields the same (i.e. isomorphic) object as the original one, we can without loss of generality assume that whenever a vNa compact quantum group (M, Δ) is considered, it is in its standard form given by a GNS representation with respect to the Haar state and that it has a w^*-dense unital C^*-subalgebra A such that $(\mathsf{A}, \Delta|_{\mathsf{A}})$ is a compact quantum group.

Whenever (M, Δ) is a vNa compact quantum group, there exists a *-anti-automorphism of M (called the *unitary antipode* and denoted by R) and a σ-strongly* continuous one-parameter group τ of *-automorphisms of M (called a *scaling group* of (M, Δ)) such that $\mathrm{Lin}\{(\mathrm{id}_{\mathsf{M}} \,\overline{\otimes}\, h)(\Delta(x)(1 \otimes y)) : x, y \in \mathsf{M}\}$ is contained in the domain of a (densely defined) operator $\mathcal{S} = R\tau_{-i/2}$, called the *antipode*. In fact, the above set is a σ-strong* core for $\mathcal{S}$ and

$$\mathcal{S}((\mathrm{id}_{\mathsf{M}} \,\overline{\otimes}\, h)(\Delta(x)(1 \otimes y))) = (\mathrm{id}_{\mathsf{M}} \,\overline{\otimes}\, h)((1 \otimes x)\Delta(y)), \qquad x, y \in \mathsf{M}.$$

The unimodularity of compact quantum groups is expressed by the condition $h = h \circ R$ — in general the unitary antipode exchanges the left invariant and the right invariant weights. Therefore we also have (by the strong left invariance of the antipode)

$$\mathcal{S}((h \mathbin{\overline{\otimes}} \mathrm{id}_{\mathsf{M}})(\Delta(x)(1 \otimes y))) = (h \mathbin{\overline{\otimes}} \mathrm{id}_{\mathsf{M}})((1 \otimes x)\Delta(y)), \qquad x, y \in \mathsf{M}.$$

Additionally denote by $\mathcal{T}$ the algebra of all analytic elements with respect to the modular group ([Tak]).

The coassociativity of Δ implies that the predual of M equipped with the convolution product

$$\phi \star \psi = (\phi \mathbin{\overline{\otimes}} \psi)\Delta, \qquad \phi, \psi \in \mathsf{M}_*,$$

is a Banach algebra. It contains an important dense subalgebra that may be equipped with the involution relevant for considering noncommutative counterparts of symmetric measures. Define, following [KV$_1$],

$$\mathsf{M}_*^{\#} = \{\omega \in \mathsf{M}_* : \exists_{\theta \in \mathsf{M}_*}\ \theta(x) = \overline{\omega}(\mathcal{S}(x)) \text{ for all } x \in D(\mathcal{S})\}.$$

The involution * in $\mathsf{M}_*^{\#}$ is introduced with the help of the obvious formula: $\omega^* \supset \overline{\omega} \circ \mathcal{S}$.

The modular group of the Haar state will be denoted simply by σ. Let us gather here a few useful commutation relations:

$$(\tau_t \otimes \sigma_t)\Delta = (\sigma_t \otimes \tau_{-t})\Delta = \Delta \circ \sigma_t, \tag{1.1}$$

$$(\tau_t \otimes \tau_t)\Delta = \Delta \circ \tau_t, \tag{1.2}$$

$$R \circ \tau_t = \tau_t \circ R. \tag{1.3}$$

Notions of "pointwise" convergence in the von Neumann algebraic context. Let M be a von Neumann algebra with a faithful normal state $\phi \in \mathsf{M}_*$, called the *reference state*.

Definition 1.4. A sequence $(x_n)_{n=1}^{\infty}$ of operators of M is *almost uniformly* (a.u.) *convergent* to $x \in \mathsf{M}$ if for each $\varepsilon > 0$ there exists $e \in P_M$ such that $\phi(e^{\perp}) < \varepsilon$ and

$$\|(x_n - x)e\|_\infty \xrightarrow{n\to\infty} 0.$$

A sequence $(x_n)_{n=1}^{\infty}$ of operators in M is *bilaterally almost uniformly* (b.a.u.) *convergent* to $x \in \mathsf{M}$ if for each $\varepsilon > 0$ there exists $e \in P_M$ such that $\phi(e^{\perp}) < \varepsilon$ and

$$\|e(x_n - x)e\|_\infty \xrightarrow{n\to\infty} 0.$$

Definition 1.5. A linear map $T : \mathsf{M} \to \mathsf{M}$ is called a *kernel* (or a *positive L^1-L^∞ contraction*) if it is a positive contraction:

$$\forall_{x \in \mathsf{M}} \quad 0 \le x \le I \ \Rightarrow\ 0 \le T(x) \le I,$$

and has the property

$$\forall_{x \in \mathsf{M},\, x \ge 0} \quad \phi(T(x)) \le \phi(x).$$

It is well known that for each kernel T and for each $x \in \mathsf{M}$ the sequence $(M_n(T)(x))_{n=1}^{\infty}$, where

$$M_n(T)(x) = \frac{1}{n}\sum_{k=1}^{n} T^k(x), \tag{1.4}$$

is w^*-convergent to $F(x)$, where $F : \mathsf{M} \to \mathsf{M}$ denotes the w^*-continuous projection on the space of fixed points of T.

The following individual ergodic theorem is due to B. Kümmerer (see also [CD-N]):

THEOREM 1.6 ([Küm]). *If $T : \mathsf{M} \to \mathsf{M}$ is a kernel, then for each $x \in \mathsf{M}$ the sequence $(M_n(T)(x))_{n=1}^{\infty}$ converges to $F(x)$ almost uniformly.*

2. Convolution operators and ergodic theorems on the level of a von Neumann algebra. Let (M, Δ) be a vNa compact quantum group with the Haar state $h \in \mathsf{M}_*$. For any $\phi \in \mathsf{M}_*$, by the convolution operator associated with ϕ we shall understand the map $T_\phi : \mathsf{M} \to \mathsf{M}$ defined by

$$T_\phi = (\mathrm{id}_{\mathsf{M}} \mathbin{\overline{\otimes}} \phi)\Delta. \tag{2.1}$$

There is also an obvious left version, given by

$$L_\phi = (\phi \mathbin{\overline{\otimes}} \mathrm{id}_{\mathsf{M}})\Delta. \tag{2.2}$$

The basic properties of the convolution operators are summarised below:

PROPOSITION 2.1. *Let $\phi, \phi_i \in \mathsf{M}_*$ $(i \in \mathcal{I})$. Then the following hold*:

- (i) *if $\phi \in \mathsf{M}_*^+$ then T_ϕ is completely positive*; *if $\phi(1) = 1$ then T_ϕ is unital*;
- (ii) *T_ϕ is normal and decomposable* (*the latter means it can be represented as a linear combination of completely positive maps*);
- (iii) *the map $\phi \mapsto T_\phi$ is a contractive homomorphism between the Banach algebras M_* and $B(\mathsf{M})$*;
- (iv) *$h \circ T_\phi = \phi(1)h$*;
- (v) *if $\phi_i \xrightarrow{i\in\mathcal{I}} \phi$ in norm then $T_{\phi_i} \xrightarrow{i\in\mathcal{I}} T_\phi$ in norm*;
- (vi) *if $\phi_i \xrightarrow{i\in\mathcal{I}} \phi$ weakly then $T_{\phi_i}(x) \xrightarrow{i\in\mathcal{I}} T_\phi(x)$ in w^*-topology for each $x \in \mathsf{M}$.*

Proof. Property (i) is obvious (as positive functionals are automatically CP), and (ii) follows from (i) and the existence of Jordan decomposition of normal functionals. Property (iii) is a consequence of coassociativity, contractivity of Δ and the fact that for each linear functional the completely bounded norm is equal to the standard norm. (iv) follows from the invariance of the Haar state, (v) is a consequence of (iii), and (vi) of the formula

$$\psi(T_\phi(x)) = \phi(L_\psi(x)) \quad \text{for all } x \in \mathsf{M},\ \psi \in \mathsf{M}_*. \ \blacksquare$$

All the above properties have their counterparts for left convolution operators (this time the map $\phi \mapsto L_\phi$ is an antihomomorphism).

For $\phi \in \mathsf{M}_*^+$ we define (for each $n \in \mathbb{N}$)

$$\phi_n = \frac{1}{n}\sum_{k=1}^{n} \phi^{\star k}. \tag{2.3}$$

The properties above in conjunction with Theorem 1.6 imply the following fact (the notation as in the previous subsection); the reference state on M will always be the Haar state.

THEOREM 2.2. *For any* $\phi \in \mathsf{M}_*^+$ *and* $x \in \mathsf{M}$,

$$M_n(T_\phi)(x) = T_{\phi_n}(x) \xrightarrow{n\to\infty} F(x) \quad \textit{almost uniformly.}$$

Properties of compact quantum groups allow us in fact to identify (in most of the cases) the limit in the above theorem. First let us mention the following result due to V. Runde (Corollary 3.5 in [Run]).

THEOREM 2.3. *The Banach algebra* M_* *is an ideal in* M^* (*equipped with the Arens multiplication*).

It is elementary to check that if $\phi \in \mathsf{M}_*$, $\varrho \in \mathsf{M}^*$ the Arens multiplication "$\cdot$" (both left and right version, known to coincide in this situation) may be written in terms of convolution operators:

$$\varrho \cdot \phi = \varrho \circ T_\phi, \qquad \phi \cdot \varrho = \varrho \circ L_\phi.$$

Therefore the above theorem of Runde may be interpreted as the counterpart of the classical fact that for compact groups the convolution of a bounded measure that has a density and any bounded measure is again a measure with a density. In two propositions below we identify the "pointwise" limits whose existence was guaranteed by Theorem 2.2.

PROPOSITION 2.4. *Let* $\phi \in \mathsf{M}_*^+$ *be a faithful state. The fixed point space of* T_ϕ *consists only of scalar multiples of* 1 (*in other words*, T_ϕ *is ergodic*).

Proof. Consider the restriction of ϕ to the w^*-dense compact quantum group A. As the restriction is also a faithful state, a remark ending Section 2 of [Wor$_2$] implies that for each $a \in \mathsf{A}$ we have $\phi_n(a) \to h(a)$ as $n \to \infty$. It follows (see the proof of Proposition 2.1(iii)) that for each $a \in \mathsf{A}$ the sequence $(M_n(T_\phi)(a))_{n=1}^\infty$ converges to $T_h(a) = h(a)1$ in w^*-topology. Let now $\varrho \in \mathsf{M}^*$ be any w^*-accumulation point of $(\phi_n)_{n=1}^\infty$ in the unit ball of M^*. It is easy to check that (for each $x \in \mathsf{M}$)

$$\varrho(x) = \varrho(T_\phi(x)) = \varrho(L_\phi(x)).$$

Theorem 2.3 yields normality of ϱ, and as the first part of the proof shows that $\varrho|_{\mathsf{A}} = h$ and A is dense, we must have $\varrho = h$. Therefore the projection on the fixed point space is given by the formula $F(x) = h(x)1$ ($x \in \mathsf{M}$). ∎

Note that in fact we did not need the theorem of Runde; it was enough to conclude by recalling the w^*-continuity of F. The next corollary, however, makes essential use of Theorem 2.3.

PROPOSITION 2.5. *Let $\phi \in \mathsf{M}_*^+$. The sequence $(\phi_n)_{n=1}^\infty$ is weakly convergent to a normal functional ϱ. In particular, for each $x \in \mathsf{M}$,*

$$M_n(T_\phi)(x) \xrightarrow{n\to\infty} T_\varrho(x) \quad \textit{almost uniformly.}$$

Proof. We can assume that ϕ is a state. Choosing this time two, potentially different, accumulation points ϱ_1, ϱ_2 of the sequence $(\phi_n)_{n=1}^\infty$ in the unit ball of M^* we deduce as above that both ϱ_1, ϱ_2 are normal. Theorem 1.6 and Properties 2.1 imply that in fact $T_{\varrho_1} = F = T_{\varrho_2}$. Further, the cancellation properties of A yield the implication

$$T_{\varrho_1} = T_{\varrho_2} \;\Rightarrow\; \varrho_1|_{\mathsf{A}} = \varrho_2|_{\mathsf{A}},$$

and the density of A in M gives the equality $\varrho_1 = \varrho_2$. ∎

3. Extensions to L^p-spaces and iterates of symmetric convolution operators. This section will only briefly introduce bits of notation and terminology—for a precise treatment of Haagerup L^p-spaces we refer for example to [JX$_2$]. The "density" operator of the Haar state will be denoted by D, the canonical trace-like functional on $L^1(\mathsf{M})$ by τ, and p' will be the exponent conjugate to p. For each $\phi \in \mathsf{M}_*$ the operator defined by

$$T_\phi^{(p)}(D^{1/2p}xD^{1/2p}) = D^{1/2p}T_\phi(x)D^{1/2p}, \quad x \in \mathsf{M},$$

extends uniquely to a continuous operator on $L^p(\mathsf{M})$. This follows from the fact that each T_ϕ may be written (in a canonical way) as a linear combination of four kernels, and from the results of [JX$_2$]. One of the main theorems of the latter paper assert the almost sure convergence of ergodic averages in L^p-spaces. Recall first the definition, due to R. Jajte.

DEFINITION 3.1. Let $p \in [1,\infty)$, $x_n, x \in L^p(\mathsf{M}), n \in \mathbb{N}$. The sequence $(x_n)_{n=1}^\infty$ is said to *converge almost surely* (a.s.) to x if for each $\varepsilon > 0$ there exists a projection $e \in \mathsf{M}$ and a family $(a_{n,k})_{n,k=1}^\infty$ of operators in M such that

$$\phi(e^\perp) < \varepsilon, \quad x_n - x = \sum_{k=1}^\infty a_{n,k}D^{1/p}, \qquad \lim_{n\to\infty}\Big\|\sum_{k=1}^\infty a_{n,k}e\Big\| = 0.$$

Analogously the sequence $(x_n)_{n=1}^\infty$ is said to converge *bilaterally almost surely* (b.a.s.) to x if for each $\varepsilon > 0$ there exists a projection $e \in \mathsf{M}$ and a family $(a_{n,k})_{n,k=1}^\infty$ of operators in M such that

$$\phi(e^\perp) < \varepsilon, \qquad x_n - x = \sum_{k=1}^\infty D^{1/2p}a_{n,k}D^{1/2p}, \qquad \lim_{n\to\infty}\Big\|\sum_{k=1}^\infty e a_{n,k} e\Big\| = 0.$$

Note the following fact, which can be easily deduced from the properties of the modular action described in the introduction (see formula (1.1)):

PROPOSITION 3.2. *Let $\phi \in \mathsf{M}_*$. The operator T_ϕ commutes with the modular action of the Haar state if and only if $\phi \circ \tau_t = \phi$ for each $t \in \mathbb{R}$.*

The set of all normal states satisfying the equivalent conditions above will be denoted by M_*^τ. It is easy to check that it is closed under convolution multiplication of M_*. Moreover, $\mathsf{M}_*^\tau \cap \mathsf{M}_*^\#$ is a *-subsemigroup of $\mathsf{M}_*^\#$, by the commutation relations (1.2)–(1.3).

Corollary 7.12 of [JX$_2$] therefore yields the following theorem:

THEOREM 3.3. *Let $\phi \in \mathsf{M}_*^\tau$ be a state and $x \in L^p(\mathsf{M})$. The sequence $(M_n(T_\phi^{(p)})(x))_{n=1}^\infty$ is b.a.s. (and for $p > 2$ even a.s.) convergent to $F^{(p)}(x)$, where $F^{(p)} : L^p(\mathsf{M}) \to L^p(\mathsf{M})$ denotes the projection on the fixed points of $T_\phi^{(p)}$. If ϕ is faithful, then $F^{(p)}(x) = \tau(D^{1/p'}x)D^{1/p}$.*

The classical Stein theorem ([Ste]) and its noncommutative generalisation ([JX$_2$]) allow one to deduce the convergence of the iterates (as opposed to averages) of T_ϕ if it induces a symmetric operator on the L^2-space. The states whose associated convolution operators have this property correspond to "symmetric" measures and can be characterised by an invariance property with respect to the antipode. This is the context of the next proposition.

PROPOSITION 3.4. *Let $\omega \in \mathsf{M}_*^\# \cap \mathsf{M}_*^\tau$. Then $(T_\omega^{(2)})^* = T_{\omega^*}^{(2)}$.*

Proof. Assume that ω is as above and $a, b \in \mathcal{T}$. Note that Proposition 3.2 implies in particular that $T_\omega(a) \in \mathcal{T}$. Moreover,

$$\begin{aligned}
&\langle T_\omega^{(2)}(D^{1/4}aD^{1/4}), D^{1/4}bD^{1/4}\rangle = \tau(D^{1/4}(T_\omega(a))^*D^{1/4}D^{1/4}bD^{1/4})\\
&= \tau(\sigma_{i/2}(T_\omega(a)^*)bD) = h(\sigma_{i/2}(T_\omega(a)^*)b) = h(\sigma_{i/2}((\mathrm{id}_{\mathsf{M}} \overline{\otimes}\, \overline{\omega})\Delta(a^*))b)\\
&= h((\mathrm{id}_{\mathsf{M}} \overline{\otimes}\, \overline{\omega})\Delta(\sigma_{i/2}(a^*))b) = \overline{\omega}((h \overline{\otimes} \mathrm{id}_{\mathsf{M}})(\Delta(\sigma_{i/2}(a^*))(b \otimes 1)))\\
&= \overline{\omega} \circ \mathcal{S}((h \overline{\otimes} \mathrm{id}_{\mathsf{M}})((\sigma_{i/2}(a^*) \otimes 1)\Delta(b))) = \omega^*((h \overline{\otimes} \mathrm{id}_{\mathsf{M}})((\sigma_{i/2}(a^*) \otimes 1)\Delta(b)))\\
&= h(\sigma_{i/2}(a^*)(\mathrm{id}_{\mathsf{M}} \overline{\otimes}\, \omega^*)\Delta b) = \tau(\sigma_{i/2}(a^*)T_{\omega^*}(b)D)\\
&= \tau(D^{1/2}a^*D^{1/2}(T_{\omega^*}(b))) = \langle D^{1/4}aD^{1/4}, T_{\omega^*}^{(2)}(D^{1/4}bD^{1/4})\rangle.
\end{aligned}$$

The claim now follows from the density of $\mathcal{T}$ in M. ∎

Therefore the Stein theorem in our context implies the following result:

THEOREM 3.5. *Let $\phi \in \mathsf{M}_*^\# \cap \mathsf{M}_*^\tau$ be a state with $\phi = \phi^*$. For $p \in (1, \infty)$ and $x \in L^p(\mathsf{M})$ the sequence $((T_\phi^{(p)})^{2n}(x))_{n=1}^\infty$ is b.a.s. (and for $p > 2$ even a.s.) convergent to $F^{(p)}(x)$, where $F^{(p)} : L^p(\mathsf{M}) \to L^p(\mathsf{M})$ denotes* *the projection on the fixed points of* $(T_\phi^{(p)})^2$. *If* $x \in \mathsf{M}$ *then the sequence* $((T_\phi)^{2n}(x))_{n=1}^\infty$ *converges almost uniformly.*

Continuous semigroups. The theorems stated above, exactly as in [JX$_2$], have their multi-parameter versions and counterparts for continuous semigroups. We mention for example the following (F denotes this time a projection on the space of fixed points of the semigroup in question):

THEOREM 3.6. *Let* $(\phi_t)_{t>0}$ *be a (weakly continuous) convolution semigroup of normal states on* M. *Then for each* $x \in \mathsf{M}$,

$$M_t(x) = \frac{1}{t}\int_0^t T_{\phi_s}(x)\,ds \xrightarrow{t\to\infty} F(x) \quad \textit{almost uniformly.}$$

If $\phi_t \in \mathsf{M}_*^\tau$ *for all* $t \geq 0$ *then for every* $p \in [1,\infty)$ *and* $x \in L^p(\mathsf{M})$,

$$M_t^{(p)}(x) = \frac{1}{t}\int_0^t T_{\phi_s}^{(p)}(x)\,ds \xrightarrow{t\to\infty} F^{(p)}(x) \quad \textit{b.a.s.}$$

(*and a.s. if* $p > 2$). *If additionally* $\phi_t \in \mathsf{M}_*^\# \cap \mathsf{M}_*^\tau$ *and* $\phi_t = \phi_t^*$ $(t \geq 0)$ *then for every* $p \in (1,\infty)$ *and* $x \in L^p(\mathsf{M})$,

$$T_{\phi_t}^{(p)}(x) \xrightarrow{t\to\infty} F^{(p)}(x) \quad \textit{b.a.s.}$$

(*and a.s. if* $p > 2$).

4. Questions and comments. The first natural question to consider is the following: what are the limit properties of the sequence $(T_\phi^n = T_{\phi^{\star n}})_{n=1}^\infty$ if no symmetry properties of ϕ are assumed? In the classical case a general answer is given by the Itô–Kawada theorem. Suppose that G is a subgroup generated by the support of the measure in question. Then the limit exists if and only if the support is not contained in a nonzero coset of any closed normal subgroup of G (as otherwise a "periodicity effect" arises), and is the Haar measure on G (see for example [Gre]). Commutative proofs suggest that the way to obtain results of such type probably leads through the Fourier analysis, which is also available for compact quantum groups. The quantum answer is however clearly more complicated, as the example of A. Pal ([Pal]) shows the existence of atypical idempotent states (i.e. idempotent states which are not Haar measures on a quantum subgroup) on a Kac–Palyutkin quantum group. For more examples of this type and characterisation of atypical states on various types of compact quantum groups we refer to the forthcoming paper [FrS].

The second question concerns the ergodic properties of convolution operators on locally compact (but noncompact) quantum groups. One difference lies in the fact that one has to deal with the left and right invariant weights (and not states), which in general will not be equal. If discrete quantum groups are considered, the invariant weights are strictly normal (that is, arise as sums of normal states with orthogonal supports), as M is a direct sum of matrix algebras. There is, however, no reason to expect that the convolution operators would respect the underlying decomposition; their behaviour is governed by the fusion rules for unitary (co)representations. Satisfactory general results seem to be currently out of reach, and in all probability even the consideration of concrete examples (such as, say, convolution operators on the quantum deformation of the Lorentz group) should involve the extensive use of von Neumann algebraic techniques and exploit certain compatibility between the modular theory of the Haar weights and the behaviour of the convolution operator in question. We hope that the introductory results of this note may provide motivation and framework for further investigations in this area.

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