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A UNIFIED APPROACH TO THE ARMENDARIZ PROPERTY OF POLYNOMIAL RINGS AND POWER SERIES RINGS

BY

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Abstract. A ring R is called Armendariz (resp., Armendariz of power series type) if, whenever $(\sum_{i\ge 0} a_i x^i)(\sum_{j\ge 0} b_j x^j) = 0$ in $R[x]$ (resp., in $R[[x]]$), then $a_i b_j = 0$ for all i and j. This paper deals with a unified generalization of the two concepts (see Definition 2). Some known results on Armendariz rings are extended to this more general situation and new results are obtained as consequences. For instance, it is proved that a ring R is Armendariz of power series type iff the same is true of $R[[x]]$. For an injective endomorphism σ of a ring R and for $n \ge 2$, it is proved that $R[x;\sigma]/(x^n)$ is Armendariz iff it is Armendariz of power series type iff σ is rigid in the sense of Krempa.

Throughout, unless otherwise stated, all rings are associative with unity and modules are unitary. The ring of polynomials (resp., power series) in indeterminate x over a ring R is denoted by $R[x]$ (resp., $R[[x]]$). For an endomorphism σ of a ring R, we denote by $R[x;\sigma]$ and $R[[x;\sigma]]$ the (left) skew polynomial ring and (left) skew power series ring, in which the multiplication is subject to the condition that $xr = \sigma(r)x$ for all $r \in R$. Following [15] (resp., [14]), a ring R is called *Armendariz* (resp., *Armendariz of power series type*) if, whenever $(\sum_{i\ge 0} a_i x^i)(\sum_{j\ge 0} b_j x^j) = 0$ in $R[x]$ (resp., in $R[[x]]$), then $a_i b_j = 0$ for all i and j. An Armendariz ring of power series type is also called a power-serieswise Armendariz ring in [11].

The two notions have been widely studied. This paper deals with a unified generalization of these rings: For an ideal I of a ring R, the notion of an *I-Armendariz* ring R is defined such that R is Armendariz iff R is 0-Armendariz, and R is Armendariz of power series type iff R is R-Armendariz (see Definition 2). Some known results on Armendariz rings are extended to this more general situation and new results are obtained as consequences. For instance, it is proved that a ring R is Armendariz of power series type iff the same is true of $R[[x]]$. For an injective endomorphism σ of a ring R and for $n \ge 2$, it is proved that $R[x;\sigma]/(x^n)$ is Armendariz iff it is Armendariz of power series type iff σ is rigid in the sense of Krempa.

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1. Definitions and examples. The following ring construction is the general setting in the paper.

DEFINITION 1. Let R be a ring and let I be an ideal of R. We denote by $[R;I][x]$ the subring $R[x]+I[[x]]$ of $R[[x]]$, where $I[[x]]$ denotes the ideal of $R[[x]]$ generated by I. Thus,

$$[R;I][x]=\Big\{\sum_{i\geq 0} r_i x^i \in R[[x]] : \exists 0\leq n\in\mathbb{Z} \text{ such that } r_i\in I,\ \forall i\geq n\Big\}.$$

DEFINITION 2. Let I be an ideal of a ring R. The ring R is called *I-Armendariz* if, whenever

$$\Big(\sum_{i\geq 0} a_i x^i\Big)\Big(\sum_{j\geq 0} b_j x^j\Big)=0 \quad \text{in } [R;I][x],$$

then $a_i b_j = 0$ for all i and j.

For two ideals I_1, I_2 of a ring R with $I_1\subseteq I_2$, if R is I_2-Armendariz then clearly R is I_1-Armendariz. Moreover, a ring R is 0-Armendariz iff R is Armendariz; and R is R-Armendariz iff R an Armendariz ring of power series type. Every *reduced* ring (i.e. a ring containing no nonzero nilpotent elements) is an Armendariz ring of power series type (see [11]). A discussion of commutative Armendariz rings of power series type can be found in the 1975 paper [6] by Gilmer, Grams and Parker.

The purpose of Definition 2 is multi-fold: (1) it gives a unified generalization of Armendariz rings and Armendariz rings of power series type; (2) as shown in Example 3 below, there exist rings R that are not Armendariz of power series type, but I-Armendariz for some nonzero ideals I; (3) as we will see later, a single proof for Armendariz property works for a large class of rings. Sometimes, the arguments of the proof are similar to the polynomial ring case, but other times they are significantly different from the polynomial ring case, in which situation one usually obtains a new result.

We need the notion of a trivial extension in order to give the next example. For an (R,R)-bimodule M, the *trivial extension* of R and M, denoted $R\propto M$, is the subring $\big\{\left(\begin{smallmatrix} a & m\\ 0 & a\end{smallmatrix}\right) : a\in R, m\in M\big\}$ of the formal upper triangular ring $\left(\begin{smallmatrix} R & M\\ 0 & R\end{smallmatrix}\right)$. For convenience, we let $I\propto N=\big\{\left(\begin{smallmatrix} a & m\\ 0 & a\end{smallmatrix}\right) : a\in I, m\in N\big\}$, where I is a subset of R and N is a subset of M. Let $\{\xi_i : i=0,1,\ldots\}$ be a set of generators of the abelian group $\mathbb{Z}_{2^\infty}$ satisfying $2\xi_0=0$ and $2\xi_{i+1}=\xi_i$ for all $i\geq 0$.

EXAMPLE 3. *Let $R=\mathbb{Z}\propto\mathbb{Z}_{2^\infty}$. Then the following hold*:

(1) *R is not Armendariz of power series type.*
(2) *Let $I_s=0\propto\mathbb{Z}\xi_s$ where $s\geq 0$ is an integer. Then R is I_s-Armendariz. In particular, R is Armendariz.*

Proof. (1) For each $i \geq 0$, let

$$a_{2i} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad a_{2i+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b_{2i} = \begin{pmatrix} 0 & -\xi_{2i} \\ 0 & 0 \end{pmatrix}, \quad b_{2i+1} = \begin{pmatrix} 0 & \xi_{2i+1} \\ 0 & 0 \end{pmatrix}.$$

Then $(\sum_{i\geq 0} a_i x^i)(\sum_{i\geq 0} b_i x^i) = 0$ in $R[[x]]$. But $a_0 b_2 \neq 0$. So R is not Armendariz of power series type.

(2) Let $\sum_{i=0}^{\infty} a_i x^i, \sum_{i=0}^{\infty} b_i x^i \in [R; I][x]$ be such that

$$\Big(\sum_{i=0}^{\infty} a_i x^i\Big)\Big(\sum_{i=0}^{\infty} b_i x^i\Big) = 0, \tag{1.1}$$

where $I = I_s$ and $s \geq 0$. We need to show that $a_i b_j = 0$ for all i and j. Suppose that $a_0, \ldots, a_{i-1} \in 0 \propto \mathbb{Z}_{2^\infty}$ but $a_i \notin 0 \propto \mathbb{Z}_{2^\infty}$ and also that $b_0, \ldots, b_{j-1} \in 0 \propto \mathbb{Z}_{2^\infty}$ but $b_j \notin 0 \propto \mathbb{Z}_{2^\infty}$. By (1.1), $a_0 b_{i+j} + \cdots + a_{i-1} b_{j+1} + a_i b_j + a_{i+1} b_{j-1} + \cdots + a_{i+j} b_0 = 0$. Thus, $a_i b_j = -(a_0 b_{i+j} + \cdots + a_{i-1} b_{j+1} + a_{i+1} b_{j-1} + \cdots + a_{i+j} b_0) \in 0 \propto \mathbb{Z}_{2^\infty}$. This is a contradiction. Therefore, we can assume, without loss of generality, that $a_i \notin 0 \propto \mathbb{Z}_{2^\infty}$ for some i but $b_j \in 0 \propto \mathbb{Z}_{2^\infty}$ for all j. Notice that there exists an $m > 0$ such that $a_i, b_i \in I$ for all $i > m$. Write $a_i = \left(\begin{smallmatrix} n_i & r_i \\ 0 & n_i \end{smallmatrix}\right)$ for $i = 0, \ldots, m$ and $b_i = \left(\begin{smallmatrix} 0 & t_i \\ 0 & 0 \end{smallmatrix}\right)$ for $i = 0, 1, \ldots.$ There exist $k \geq 0$ and $l \geq 0$ such that $2^{l+1} \mid n_i$ for $i = 0, \ldots, k-1$, $2^{l+1} \nmid n_k$, and $2^l \mid n_i$ for $i = k, \ldots, m$. For $t \in \mathbb{Z}_{2^\infty}$, we let $o(t)$ be the order of t in the group $\mathbb{Z}_{2^\infty}$. Since $t_i \in \mathbb{Z}\xi_s$ for all $i > m$, there exist $u \geq 0$ and $v \geq 0$ such that $o(t_i) < 2^v$ for all $i < u$, $o(t_u) = 2^v$ and $o(t_i) \leq 2^v$ for all $i > u$. To prove $a_i b_j = 0$ for all i and j, it suffices to show that $l \geq v$. Suppose that $l < v$. Let

$$a = \begin{pmatrix} 2^{v-l-1} & 0 \\ 0 & 2^{v-l-1} \end{pmatrix} \in R.$$

From (1.1), one obtains

$$\begin{aligned} 0 &= a(a_0 b_{k+u} + \cdots + a_{k-1} b_{u+1} + a_k b_u + a_{k+1} b_{u-1} + \cdots + a_{k+u} b_0) \\ &= (aa_0) b_{k+u} + \cdots + (aa_{k-1}) b_{u+1} + (aa_k) b_u \\ &\quad + (aa_{k+1}) b_{u-1} + \cdots + (aa_{k+u}) b_0 \\ &= 0 + \cdots + 0 + (aa_k) b_u + 0 + \cdots + 0 \\ &= \begin{pmatrix} 0 & 2^{v-l-1} n_k t_u \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This shows that $2^{v-1} t_u = 0$, a contradiction. ∎

We remark that, if M is an abelian group containing $\mathbb{Z}_{2^\infty}$ as a subgroup (e.g., $M = \mathbb{Q}/\mathbb{Z}$), then $\mathbb{Z} \propto \mathbb{Z}_{2^\infty}$ is a subring of $\mathbb{Z} \propto M$, and hence $\mathbb{Z} \propto M$ is not Armendariz of power series type because of Example 3(1). But $\mathbb{Z} \propto M$ is Armendariz by [13, Corollary 2.7].

2. Armendariz property. One of the most interesting results on the Armendariz property of rings is the theorem, proved cleverly by Anderson and Camillo in [1], which says that a ring R is Armendariz iff $R[x]$ is Armendariz. With a different proof, we obtain a much more general result which contains the theorem of Anderson and Camillo as a special case and which immediately implies that a ring R is Armendariz of power series type iff $R[[x]]$ is Armendariz of power series type.

LEMMA 4. *Let I be an ideal of a ring R and let $k \geq 1$ be an integer. Then the map $\phi_k : \big[[R;I][x]; I[[x]]\big][y] \to [R;I][x]$, $\sum f_i(x)y^i \mapsto \sum f_i(x)x^{ki}$, is a ring homomorphism.*

Proof. For $\sum f_i(x)y^i \in \big[[R;I][x]; I[[x]]\big][y]$ we have $\sum f_i(x)x^{ki} = \sum_{j\geq 0} c_j x^j$, where if $j < (n+1)k$ then c_j is equal to the well-defined coefficient of the term x^j in $\sum_{i=0}^{n} f_i(x)x^{ik}$. So $\sum f_i(x)x^{ki} \in R[[x]]$. Because there exists an $m > 0$ such that $f_j(x) \in I[[x]]$ for all $j > m$ and $f_j(x) \in [R;I][x]$ for all $0 \leq j \leq m$, there exists an $l > 0$ such that the coefficient of the term x^i in $f_j(x)$ is in I for all $0 \leq j \leq m$ and $i \geq l$. Thus, for all $i \geq mk + l$, $c_i \in I$. So $\sum f_i(x)x^{ki} \in [R;I][x]$. Hence ϕ_k is well defined. It is now routine to verify that ϕ_k is a ring homomorphism. ∎

THEOREM 5. *Let I be an ideal of a ring R. Then R is I-Armendariz if and only if $[R;I][x]$ is $I[[x]]$-Armendariz.*

Proof. One implication is obvious. For the other implication, suppose that R is I-Armendariz and let

$$\Big[\sum_{i\geq 0} f_i(x)y^i\Big]\Big[\sum_{i\geq 0} g_i(x)y^i\Big] = 0 \quad \text{in } \big[[R;I][x]; I[[x]]\big][y]. \tag{2.1}$$

We need to show that $f_i(x)g_j(x) = 0$ for all i and j. It is clear that $f_0(x)g_0(x) = 0$. Assume that, for $k \geq 1$, $f_0(x)g_j(x) = 0$ for $j = 0, \ldots, k-1$. We next prove that $f_0(x)g_k(x) = 0$. For $i \geq 0$, write

$$f_i(x) = a_0^{(i)} + a_1^{(i)}x + \cdots + a_n^{(i)}x^n + \cdots, \qquad g_i(x) = b_0^{(i)} + b_1^{(i)}x + \cdots + b_n^{(i)}x^n + \cdots.$$

If $f_0(x)g_k(x) \neq 0$, then we can assume that, for some $s \geq 0$, $f_0(x)b_j^{(k)} = 0$ for $j < s$ but $f_0(x)b_s^{(k)} \neq 0$. We can further assume that, for some $t \geq 0$, $a_i^{(0)}b_s^{(k)} = 0$ for $i < t$ but $a_t^{(0)}b_s^{(k)} \neq 0$. Take $l > \max\{s,t\}$. By Lemma 4, from (2.1) we obtain

$$\Big[\sum_{i\geq 0} f_i(x)x^{il}\Big]\Big[\sum_{i\geq 0} g_i(x)x^{il}\Big] = 0 \quad \text{in } [R;I][x]. \tag{2.2}$$

Note that

$$\sum_{i\geq 0} f_i(x)x^{il} = (\text{terms of degree } < t) + a_t^{(0)}x^t + (\text{terms of degree } > t),$$

$$\begin{aligned}\sum_{i\geq 0} g_i(x)x^{il} = {} & (\text{terms of degree } < kl+s) \\ & + (b_{kl+s}^{(0)} + b_{(k-1)l+s}^{(1)} + \cdots + b_{l+s}^{(k-1)} + b_s^{(k)})x^{kl+s} \\ & + (\text{terms of degree } > kl+s).\end{aligned}$$

Since R is I-Armendariz, it follows from (2.2) that

$$a_t^{(0)}(b_{kl+s}^{(0)} + b_{(k-1)l+s}^{(1)} + \cdots + b_{l+s}^{(k-1)} + b_s^{(k)}) = 0,$$

and the assumption that $f_0(x)g_j(x) = 0$ for $j = 0,\ldots,k-1$ implies $a_t^{(0)}(b_{kl+s}^{(0)} + b_{(k-1)l+s}^{(1)} + \cdots + b_{l+s}^{(k-1)}) = 0$. Thus, $a_t^{(0)}b_s^{(k)} = 0$. This contradiction shows that $f_0(x)g_k(x) = 0$. By induction, $f_0(x)g_j(x) = 0$ for $j = 0,1,\ldots.$ Thus, we have

$$[f_1(x)y + \cdots + f_n(x)y^n + \cdots][g_0(x) + g_1(x)y + \cdots + g_m(x)y^m + \cdots] = 0$$

and so

$$[f_1(x) + \cdots + f_n(x)y^{n-1} + \cdots][g_0(x) + g_1(x)y + \cdots + g_m(x)y^m + \cdots] = 0.$$

Arguing as above with $f_1(x)$ replacing $f_0(x)$, we obtain $f_1(x)g_j(x) = 0$ for $j = 0,1,\ldots.$ Continuing this process, we see that a simple induction shows that, for each $i \geq 0$, $f_i(x)g_j(x) = 0$ for $j = 0,1,\ldots.$ ∎

Corollary 6 ([1, Theorem 2]). *A ring R is Armendariz iff $R[x]$ is Armendariz.*

Proof. We just apply Theorem 5 to the case where $I = 0$. ∎

Anderson and Camillo's proof of Corollary 6 in [1, Theorem 2] uses the notion of degree of polynomials, which is different from ours. In [11, Proposition 3.1], the same idea of the proof of [1, Theorem 2] was used to show that a ring R is Armendariz of power series type iff the same is true of $R[x]$. This is a quick consequence of the next result.

Corollary 7. *A ring R is an Armendariz ring of power series type iff so is $R[[x]]$.*

Proof. This is a special case of Theorem 5 where $I = R$. ∎

In [7], Hirano observed that the Armendariz rings are precisely those rings R for which there is a bijective correspondence between the right annihilators of R and the right annihilators of $R[x]$. Hirano's result well explains the significance of Armendariz rings.

For $U \subseteq R$, the right annihilator of U in R is denoted by $\mathbf{r}_R(U)$, i.e., $\mathbf{r}_R(U) = \{a \in R : Ua = 0\}$. Let I be an ideal of the ring R and $S = [R; I][x]$. For $V \subseteq S$, let C_V be the set of the coefficients of power series in V. Further, let $\mathcal{A}(R) = \{\mathbf{r}_R(U) : U \subseteq R\}$ and $\mathcal{A}(S) = \{\mathbf{r}_S(V) : V \subseteq S\}$. For $J \in \mathcal{A}(R)$, $K \in \mathcal{A}(S)$, let $\Phi(J) = J[x] + (J \cap I)[[x]]$ and $\Psi(K) = K \cap R$. The following statements can be easily verified:

(1) For any $U \subseteq R$, $\mathbf{r}_S(U) = \mathbf{r}_R(U)[x] + (\mathbf{r}_R(U) \cap I)[[x]] = \Phi(\mathbf{r}_R(U))$.
(2) For any $V \subseteq S$, $\mathbf{r}_S(V) \cap R = \mathbf{r}_R(C_V)$.
(3) $\Phi : \mathcal{A}(R) \to \mathcal{A}(S)$ is one-to-one and $\Psi : \mathcal{A}(S) \to \mathcal{A}(R)$ is onto.
(4) $\Psi \circ \Phi = 1_{\mathcal{A}(R)}$.

The next proposition can be proved arguing as in the proof of [7, Proposition 3.1].

Proposition 8. *Let I be an ideal of a ring R. Then R is I-Armendariz $\Leftrightarrow$ Φ is a bijection $\Leftrightarrow$ Ψ is a bijection.*

3. Extensions of rings. In this section, we discuss the Armendariz property for some extensions of rings. For $f \in R[[x]]$ where R is a commutative ring, the *content* A_f of f is the ideal of R generated by the coefficients of f. Following Tsang [16], we call a commutative ring R *Gaussian* if $A_{fg} = A_f A_g$ for all $f, g \in R[x]$. In [1, Theorem 8], Anderson and Camillo proved that a commutative ring R is Gaussian iff every homomorphic image of R is Armendariz, and their proof is still valid for the following more general setting.

Definition 9. Let I be an ideal of a commutative ring R. We say that R is *I-Gaussian* if $A_{fg} = A_f A_g$ for all $f, g \in [R; I][x]$.

Proposition 10. *Let I be an ideal of a commutative ring R. Then R is I-Gaussian iff, for each ideal K of R, R/K is $(K+I)/K$-Armendariz.*

Proof. Write $\overline{R} = R/K$ and, for $f = \sum_{i\ge 0} a_i x^i \in R[[x]]$, define $\bar{f} = \sum_{i\ge 0} \overline{a}_i x^i \in \overline{R}[[x]]$.

"$\Rightarrow$". Let $\bar{f}, \bar{g} \in [R/K; (K+I)/K][x]$ with $\bar{f}\bar{g} = \overline{0}$. Write $\bar{f} = \sum_{i\ge 0} \overline{a}_i x^i$ and $\bar{g} = \sum_{i\ge 0} \overline{b}_i x^i$. We can assume that, for some $n > 0$, $a_i, b_i \in I$ for all $i > n$. Let $f = \sum_{i\ge 0} a_i x^i$, $g = \sum_{i\ge 0} b_i x^i$. Then $f, g \in [R; I][x]$, and so $A_{fg} = A_f A_g$ by hypothesis. Thus, $A_{\bar{f}} A_{\bar{g}} = (A_f + K)/K \cdot (A_g + K)/K = (A_f A_g + K)/K = (A_{fg} + K)/K = A_{\overline{fg}} = A_{\bar{f}\bar{g}} = A_{\overline{0}} = \overline{0}$. Hence R/K is $(K+I)/K$-Armendariz.

"$\Leftarrow$". Let $f, g \in [R; I][x]$. Clearly $\bar{f}\bar{g} = \overline{0}$ in $[R/A_{fg}; (A_{fg} + I)/A_{fg}][x]$. Hence $A_{\bar{f}} A_{\bar{g}} = \overline{0}$ by hypothesis. That is, $(A_f + A_{fg})/A_{fg} \cdot (A_g + A_{fg})/A_{fg} = \overline{0}$. So $A_f A_g \subseteq A_{fg}$ and hence $A_f A_g = A_{fg}$, because $A_{fg} \subseteq A_f A_g$. ■

Letting $I = 0$ in Proposition 10 yields the next result.

Corollary 11 ([1, Theorem 8]). *A commutative ring R is Gaussian if and only if every homomorphic image of R is Armendariz.*

A commutative ring R is called a *Gaussian ring of power series type* if $A_{fg} = A_f A_g$ for all $f, g \in R[[x]]$. The next result is the special case of Proposition 10 where $I = R$.

Corollary 12. *A commutative ring R is a Gaussian ring of power series type if and only if every homomorphic ring of R is an Armendariz ring of power series type.*

In [2], Anderson and Kang proved that a quasi-local integral domain R is Gaussian of power series type if and only if R is a field or a one-dimensional valuation domain. Notice that an integral domain is Gaussian if and only if it is a Prüfer domain (see [16]). Thus, if R is a quasi-local Prüfer domain that is not a valuation domain, then R is a Gaussian ring that is not a Gaussian ring of power series type.

A commutative ring R is called an *arithmetical ring* if the ideals of R form a distributive lattice. This definition dates back to the 1949 paper [5] of L. Fuchs. It is known that arithmetical rings are Gaussian (see [1]). But as shown in the next example, an arithmetical ring need not be Armendariz of power series type. Thus, an arithmetical ring need not be Gaussian of power series type. Note that the ring R in the next example is not Gaussian of power series type, but is I-Gaussian for some nonzero ideal I of R.

Example 13. *Let R be the ring as in Example* 3. *Then*:

(1) *R is not Gaussian of power series type.*
(2) *R is an arithmetical ring.*
(3) *Let $I_s = 0 \propto \mathbb{Z}\xi_s$ where $s \geq 0$ is an integer. Then R is I_s-Gaussian.*

Proof. (1) Since R is not Armendariz of power series type by Example 3, R is not Gaussian of power series type.

(2) Notice that I is an ideal of R if and only if either $I \in \{0 \propto \mathbb{Z}\xi_i : i = 0, 1, \ldots\}$ or $I \in \{\mathbb{Z}n \propto \mathbb{Z}_{2^\infty} : 0 \leq n \in \mathbb{Z}\}$. Since $\mathbb{Z}$ is arithmetical and the set of subgroups of $\mathbb{Z}_{2^\infty}$ is totally ordered, it is easy to see that the set of ideals of R forms a distributive lattice. So R is arithmetical.

(3) By Proposition 10, it suffices to show that R/K is $(K + I_s)/K$-Armendariz for each ideal K of R. We proceed with three cases:

Case 1: $K = n\mathbb{Z} \propto \mathbb{Z}_{2^\infty}$ with $0 \leq n \in \mathbb{Z}$. Then $(K + I_s)/K = \overline{0}$, and $R/K \cong \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n$ is certainly Armendariz (as $\mathbb{Z}_n$ is arithmetical).

Case 2: $K = 0 \propto \mathbb{Z}\xi_i$ $(i < s)$. Notice that $\theta : R \to R$ given by $(n, x) \mapsto (n, 2^{i+1}x)$ is an onto ring homomorphism with kernel $\ker(\theta) =$

$0 \propto \mathbb{Z}\xi_i = K$. So it induces a ring isomorphism $\bar{\theta} : R/K \cong R$. Moreover, $\overline{\theta}((K + I_s)/K) = 0 \propto 2^{i+1}\mathbb{Z}\xi_s = 0 \propto \mathbb{Z}\xi_{s-i-1} = I_{s-i-1}$. By Example 3, R is I_{s-i-1}-Armendariz, so R/K is $(K + I_s)/K$-Armendariz.

CASE 3: $K = 0 \propto \mathbb{Z}\xi_i$ $(i \geq s)$. Then $(K + I_s)/K = \overline{0}$, and $R/K \cong R$ is Armendariz. ∎

From now on, all rings R are associative but not necessarily commutative. Next, we consider the Armendariz property for trivial extensions. A necessary and sufficient condition for a trivial extension to be Armendariz is obtained in [13]. For a module M_R, let $M[x]$ (resp., $M[[x]]$) be the set of all formal polynomials (resp., power series) in indeterminate x with coefficients from M. For a submodule N of M, define

$$[M; N][x] = \Big\{ \sum_{i\geq 0} m_i x^i \in M[[x]] : \exists 0 \leq n \in \mathbb{Z} \text{ such that } m_i \in N,\ \forall i \geq n \Big\}.$$

Let I be an ideal of R. It is easy to see that $MI \subseteq N$ iff $[M; N][x]$ is a right $[R; I][x]$-module under usual addition and multiplication of power series.

A bimodule ${}_RM_R$ is called *I-Armendariz* if, whenever $m(x)f(x) = 0$ (resp., $f(x)m(x) = 0$) where $m(x) = \sum_{i\geq 0} m_i x^i \in [M; IM + MI][x]$ and $f(x) = \sum_{j\geq 0} a_j x^j \in [R; I][x]$, then $m_i a_j = 0$ (resp., $a_j m_i = 0$) for all i and j. Notice that a bimodule ${}_RM_R$ is 0-Armendariz iff both ${}_RM$ and M_R are Armendariz modules in the sense of Anderson and Camillo [1].

LEMMA 14. *Let I be an ideal of a ring R, M an (R, R)-bimodule, and $N = IM + MI$. Then*:

1. *$I \propto N$ is an ideal of $R \propto M$ and $[M; N][x]$ is an $([R; I][x], [R; I][x])$-bimodule.*
2. *$[R; I][x] \propto [M; N][x] \cong [R \propto M; I \propto N][x]$, and the isomorphism sends $I[[x]] \propto N[[x]]$ to $(I \propto N)[[x]]$.*

Proof. (1) This is clear.

(2) The map $\phi : [R; I][x] \propto [M; N][x] \to [R \propto M; I \propto N][x]$,

$$\begin{pmatrix} \sum a_i x^i & \sum m_i x^i \\ 0 & \sum a_i x^i \end{pmatrix} \mapsto \sum \begin{pmatrix} a_i & m_i \\ 0 & a_i \end{pmatrix} x^i,$$

is the required isomorphism. ∎

REMARK 15. Suppose that R is I-Armendariz where I is an ideal of R. If S is a subring of R and K is an ideal of S with $K \subseteq I$, then S is K-Armendariz.

THEOREM 16. *Let I be an ideal of a ring R, M an (R, R)-bimodule, and $N = IM + MI$. Then the ring $R \propto M$ is $(I \propto N)$-Armendariz if and only if the following are satisfied*:

(1) *R is I-Armendariz.*
(2) *M is an I-Armendariz bimodule.*
(3) *If $f(x)g(x) = 0$ in $[R; I][x]$, then $f(x)[M; N][x] \cap [M; N][x]g(x) = 0$.*

Proof. "$\Rightarrow$". (1) This follows by Remark 15.

(2) Let $m(x) = \sum_{i\geq 0} m_i x^i \in [M; N][x]$ and $f(x) = \sum_{i\geq 0} a_i x^i \in [R; I][x]$. Suppose $f(x)m(x) = 0$. Then $a_0 m_i + a_1 m_{i-1} + \cdots + a_i m_0 = 0$ for all $i \geq 0$. Hence

$$\sum_{i\geq 0}\begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} x^i \cdot \sum_{i\geq 0}\begin{pmatrix} 0 & m_i \\ 0 & 0 \end{pmatrix} x^i = \sum_{i\geq 0}\left[\sum_{j=0}^{i}\begin{pmatrix} a_j & 0 \\ 0 & a_j \end{pmatrix}\begin{pmatrix} 0 & m_{i-j} \\ 0 & 0 \end{pmatrix}\right] x^i$$
$$= \sum_{i\geq 0}\begin{pmatrix} 0 & \sum_{j=0}^{i} a_j m_{i-j} \\ 0 & 0 \end{pmatrix} x^i = 0.$$

Since $R \propto M$ is $(I \propto N)$-Armendariz, it follows that $\left(\begin{smallmatrix} a_i & 0 \\ 0 & a_i \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & m_j \\ 0 & 0 \end{smallmatrix}\right) = 0$ for all $i, j \geq 0$, i.e., $a_i m_j = 0$ for all $i, j \geq 0$. Similarly, $m(x)f(x) = 0$ implies that $m_j a_i = 0$ for all $i, j \geq 0$.

(3) Since $R \propto M$ is $(I \propto N)$-Armendariz, we see that $[R \propto M; I \propto N][x]$ is $(I \propto N)[[x]]$-Armendariz by Theorem 5. Therefore, $[R; I][x] \propto [M; N][x]$ is $(I[[x]] \propto N[[x]])$-Armendariz by Lemma 14. Now assume $f(x)g(x) = 0$ and $f(x)m(x) = -m'(x)g(x) \neq 0$, where $f(x), g(x) \in [R; I][x]$ and $m(x), m'(x) \in [M; N][x]$. Then

$$\left[\begin{pmatrix} f(x) & 0 \\ 0 & f(x) \end{pmatrix} + \begin{pmatrix} 0 & m'(x) \\ 0 & 0 \end{pmatrix} y\right]\left[\begin{pmatrix} g(x) & 0 \\ 0 & g(x) \end{pmatrix} + \begin{pmatrix} 0 & m(x) \\ 0 & 0 \end{pmatrix} y\right] = 0$$

in $\big[[R; I][x] \propto [M; N][x]; I[[x]] \propto N[[x]]\big][y]$, but $\left(\begin{smallmatrix} f(x) & 0 \\ 0 & f(x) \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & m(x) \\ 0 & 0 \end{smallmatrix}\right) \neq 0$. This is a contradiction.

"$\Leftarrow$". Suppose that $\alpha(x)\beta(x) = 0$, where $\alpha(x) = \sum_{i\geq 0}\left(\begin{smallmatrix} a_i & m_i \\ 0 & a_i \end{smallmatrix}\right) x^i$, $\beta(x) = \sum_{i\geq 0}\left(\begin{smallmatrix} b_i & l_i \\ 0 & b_i \end{smallmatrix}\right) x^i \in [R \propto M; I \propto N][x]$. Let

$$f(x) = \sum_{i\geq 0} a_i x^i, \quad g(x) = \sum_{i\geq 0} b_i x^i, \quad m(x) = \sum_{i\geq 0} m_i x^i, \quad l(x) = \sum_{i\geq 0} l_i x^i.$$

Then $f(x), g(x) \in [R; I][x]$ and $m(x), l(x) \in [M; N][x]$. By Lemma 14, it follows from $\alpha(x)\beta(x) = 0$ that

$$0 = \begin{pmatrix} f(x) & m(x) \\ 0 & f(x) \end{pmatrix}\begin{pmatrix} g(x) & l(x) \\ 0 & g(x) \end{pmatrix} = \begin{pmatrix} f(x)g(x) & f(x)l(x) + m(x)g(x) \\ 0 & f(x)g(x) \end{pmatrix}.$$

Thus, $f(x)g(x) = 0$ and $f(x)l(x) + m(x)g(x) = 0$. Since R is I-Armendariz by (1), $a_i b_j =$ for all $i, j \geq 0$. Moreover, by (3),

$$f(x)l(x) = -m(x)g(x) \in f(x)[M; N][x] \cap [M; N][x]g(x) = 0,$$

so $f(x)l(x) = m(x)g(x) = 0$. Then, by (2), $a_i l_j = 0 = m_i b_j$ for all $i, j \geq 0$. Therefore, $\left(\begin{smallmatrix} a_i & m_i \\ 0 & a_i \end{smallmatrix}\right)\left(\begin{smallmatrix} b_j & l_j \\ 0 & b_j \end{smallmatrix}\right) = 0$ for all $i, j \geq 0$. So $R \propto M$ is $(I \propto N)$-Armendariz. ∎

Corollary 17 ([13, Theorem 2.2]). *Let M be an (R, R)-bimodule. Then $R \propto M$ is an Armendariz ring if and only if the following are satisfied*:

(1) *R is an Armendariz ring.*
(2) *M is a 0-Armendariz bimodule.*
(3) *If $f(x)g(x) = 0$ in $R[x]$, then $f(x)M[x] \cap M[x]g(x) = 0$.*

Corollary 18. *Let M be an (R, R)-bimodule. Then $R \propto M$ is an Armendariz ring of power series type if and only if the following are satisfied*:

(1) *R is an Armendariz ring of power series type.*
(2) *M is an R-Armendariz bimodule.*
(3) *If $f(x)g(x) = 0$ in $R[[x]]$, then $f(x)M[[x]] \cap M[[x]]g(x) = 0$.*

If $\{S_\alpha\}_\alpha$ is a chain of Armendariz subrings of a ring R, then it is clear that $\bigcup S_\alpha$ is still Armendariz. But the analog does not hold true for Armendariz subrings of power series type.

Example 19. Let R be the ring as in Example 3. Then R is not Armendariz of power series type. But R is the union of a chain of subrings, each of which is Armendariz of power series type. In fact, let $R_n = \mathbb{Z} \propto \mathbb{Z}\xi_n$ for $n = 0, 1, \ldots$. Then $\{R_n : n = 0, 1, \ldots\}$ is a chain of subrings of R and $R = \bigcup_{n\geq 0} R_n$. It suffices to show that each R_n is Armendariz of power series type. Since $\mathbb{Z}\xi_n \cong \mathbb{Z}_{2^n}$, we only need to show that $\mathbb{Z} \propto \mathbb{Z}_{2^n}$ is Armendariz of power series type. However, $\mathbb{Z}_{2^n}$ is an Armendariz ring of power series type by [11, Proposition 3.2]. Thus, $\mathbb{Z}_{2^n}$ is a $\mathbb{Z}$-Armendariz bimodule over $\mathbb{Z}$. Hence $\mathbb{Z} \propto \mathbb{Z}_{2^n}$ is Armendariz of power series type by Corollary 18.

The last part of this section is about the Armendariz property of the ring $R[x; \sigma]/(x^n)$. Following Krempa [12], an endomorphism σ of R is called *rigid* if, for any element $a \in R$, $a\sigma(a) = 0$ implies $a = 0$. It is easy to prove that any ring with a rigid endomorphism is reduced (see [9, Proposition 5]). It is well known that if $a_1 \cdots a_n = 0$ in a reduced ring R then $a_{\alpha(1)} \cdots a_{\alpha(n)} = 0$ for any permutation α of $\{1, \ldots, n\}$.

In [1, Theorem 5], Anderson and Camillo proved that, for a ring R and $n \geq 2$, $R[x]/(x^n)$ is Armendariz if and only if R is reduced. The following is a generalization of this result, because the identity map of a ring R is rigid if and only if the ring R is reduced.

Theorem 20. *Let σ be an injective endomorphism of R with $\sigma(1) = 1$ and let $n \geq 1$ be an integer. Then the following are equivalent*:

(1) *$R[x; \sigma]/(x^{n+1})$ is an Armendariz ring.*

(2) $R[x;\sigma]/(x^{n+1})$ *is an Armendariz ring of power series type.*
(3) σ *is rigid.*

Proof. (3)⇒(2). Suppose that σ is rigid. Then R is reduced.

CLAIM 1. *For $a, b \in R$, $ab = 0$ iff $a\sigma(b) = 0$; and hence σ^k is rigid for each $k \geq 1$.*

Proof. Because of (3), we have $ab = 0 \Rightarrow ba = 0 \Rightarrow a\sigma(b)\sigma(a\sigma(b)) = a\sigma(ba)\sigma^2(b) = 0 \Rightarrow a\sigma(b) = 0 \Rightarrow ba\sigma(ba) = 0 \Rightarrow ba = 0 \Rightarrow ab = 0$.

CLAIM 2. *If $(a_0 + a_1x + \cdots + a_nx^n)(b_0 + b_1x + \cdots + b_nx^n) \in (x^{n+1})$, then $a_ib_j = 0$ for all $0 \leq i, j \leq n$ with $i + j \leq n$.*

Proof. Clearly, $a_0b_0 = 0$. Suppose that for some $0 < s \leq n$, $a_ib_j = 0$ for all $0 \leq i, j \leq n$ with $i + j < s$. By hypothesis, we have

$$a_0b_s + a_1\sigma(b_{s-1}) + \cdots + a_s\sigma^s(b_0) = 0. \tag{3.1}$$

Multiplying (3.1) by b_0 from the left yields $b_0a_s\sigma^s(b_0) = 0$, because $b_0a_j = 0$ for $j = 0, \ldots, s-1$ by induction assumption. Hence $(b_0a_s)\sigma^s(b_0a_s) = 0$, so $b_0a_s = 0$ by Claim 1. Again by Claim 1, $a_s\sigma^s(b_0) = 0$. Now (3.1) becomes

$$a_0b_s + a_1\sigma(b_{s-1}) + \cdots + a_{s-1}\sigma^{s-1}(b_1) = 0. \tag{3.2}$$

Similarly, multiplying (3.2) by b_1 from the left yields $a_{s-1}b_1 = 0$. By repeating the same argument, one can prove that $a_ib_j = 0$ for all $0 \leq i, j \leq n$ with $i + j = s$. So Claim 2 follows by induction.

CLAIM 3. $\sigma : R[[y]] \to R[[y]]$, $\sum_{i\geq 0} a_iy^i \mapsto \sum_{i\geq 0} \sigma(a_i)y^i$, *is also rigid.*

Proof. If $(\sum_{i\geq 0} a_iy^i)\sigma(\sum_{i\geq 0} a_iy^i) = 0$ where $\sum_{i\geq 0} a_iy^i \in R[[y]]$, then $a_0\sigma(a_0) = 0$. So $a_0 = 0$ since σ is rigid. Thus, $(\sum_{i\geq 1} a_iy^{i-1})\sigma(\sum_{i\geq 1} a_iy^{i-1}) = 0$. A simple induction shows that $a_i = 0$ for $i = 0, 1, \ldots$, proving Claim 3.

To prove (2), we now let $A(y)B(y) = 0$ in $(R[x;\sigma]/(x^{n+1}))[[y]]$, where

$$A(y) = (a_{00} + a_{01}x + \cdots + a_{0n}x^n) + (a_{10} + a_{11}x + \cdots + a_{1n}x^n)y + \cdots,$$
$$B(y) = (b_{00} + b_{01}x + \cdots + b_{0n}x^n) + (b_{10} + b_{11}x + \cdots + b_{1n}x^n)y + \cdots.$$

We need to show that

$$(a_{i0} + a_{i1}x + \cdots + a_{in}x^n)(b_{j0} + b_{j1}x + \cdots + b_{jn}x^n) \in (x^{n+1}) \tag{3.3}$$

for all $i, j \geq 0$. Note that σ extends to a rigid endomorphism of $R[[y]]$ by Claim 3, and $(R[x;\sigma]/(x^{n+1}))[[y]] \cong R[[y]][x;\sigma]/(x^{n+1})$ canonically. Let

$$C(x) = f_0(y) + f_1(y)x + \cdots + f_n(y)x^n,$$
$$D(x) = g_0(y) + g_1(y)x + \cdots + g_n(y)x^n,$$

where

$$f_l(y) = \sum_{i\geq 0} a_{il}y^i, \quad g_{l'}(y) = \sum_{j\geq 0} b_{jl'}y^j \quad \text{for all } 0 \leq l, l' \leq n.$$

Thus, it follows from $A(y)B(y) = 0$ that $C(x)D(x) \in (x^{n+1})$, i.e.,

$$(f_0(y) + f_1(y)x + \cdots + f_n(y)x^n)(g_0(y) + g_1(y)x + \cdots + g_n(y)x^n) \in (x^{n+1}).$$

So, by Claim 2, we obtain

$$f_l(y)g_{l'}(y) = 0 \quad \text{ in } R[[y]] \text{ for all } 0 \le l, l' \le n \text{ with } l + l' \le n.$$

That is, $(\sum_{i\ge 0} a_{il}y^i)(\sum_{j\ge 0} b_{jl'}y^j) = 0$. Since R is reduced, R is Armendariz of power series type, and thus $a_{il}b_{jl'} = 0$ for all $i, j \ge 0$. So (3.3) holds, as desired.

(2)⇒(1). This is obvious.

(1)⇒(3). Suppose that $R[x;\sigma]/(x^{n+1})$ is an Armendariz ring.

(i) For any $a \in R$, $\sigma^n(a)a = 0$ implies $a = 0$: In fact, if $\sigma^n(a)a = 0$ then $(\sigma^n(a) + x^n y)(a - x^n y) = 0$ in $(R[x;\sigma]/(x^{n+1}))[y]$. Hence $\sigma^n(a)x^n \in (x^{n+1})$. So $\sigma^n(a) = 0$. Since σ is injective, $a = 0$.

(ii) R is reduced: If $a^2 = 0$ in R, then $\sigma^n(\sigma^n(a)a)\sigma^n(a)a = \sigma^{2n}(a)\sigma^n(a^2)a = 0$. By what was proved above, $\sigma^n(a)a = 0$, and hence $a = 0$.

Now we prove that σ is rigid. Let $a\sigma(a) = 0$ in R. Then $(\sigma(a) - \sigma(a)xy)(a + \sigma(a)xy) = 0$ in $(R[x;\sigma]/(x^{n+1}))[y]$. Thus $\sigma(a)^2x \in (x^{n+1})$ by our assumption. Hence $\sigma(a)^2 = 0$, so $a = 0$. ∎

It is easy to exhibit reduced rings R with $\sigma(a)a = 0$ but $\sigma(a) \ne 0$: Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\sigma : R \to R$, $(r, s) \mapsto (s, r)$. Then R is reduced and σ is an automorphism of R with $\sigma(1) = 1$. Let $a = (1, 0)$. Then $\sigma(a)a = 0$ but $\sigma(a) = (0, 1)$. So $R[x;\sigma]/(x^2)$ is not Armendariz.

If $R[x;\sigma]/(x^2)$ is Armendariz, then $\sigma(a)a = 0$ implies $\sigma(a) = 0$: In this case, $(\sigma(a)+xy)(a-xy) = 0$ in $(R[x;\sigma]/(x^2))[y]$. Hence $\sigma(a)x = 0$, implying $\sigma(a) = 0$. But σ need not be rigid even if $R[x;\sigma]/(x^2)$ is Armendariz of power series type.

Example 21. *Let D be a (not necessarily commutative) domain and let $R = D[t]$ be the polynomial ring. Define $\sigma : R \to R$ by $f(t) \mapsto f(0)$. Then σ is an endomorphism of R and the following hold*:

(1) *σ is not injective (so not rigid).*

(2) *$R[x;\sigma]/(x^2)$ is Armendariz of power series type.*

Proof. (1) Clearly, σ is an endomorphism of R and it is not injective.

(2) Let $S = R[x;\sigma]/(x^2)$. Then S can be identified with the ring $S = \{f + gx : f, g \in R\}$, with multiplication defined by $x^2 = 0$ and $xf = \sigma(f)x$ for all $f \in R$. Suppose that

$$(3.4) \qquad (\alpha_0 + \alpha_1 y + \alpha_2 y^2 + \cdots)(\beta_0 + \beta_1 y + \beta_2 y^2 + \cdots) = 0 \quad \text{ in } S[[y]].$$

We need to show that $\alpha_i\beta_j = 0$ for all i and j. Without loss of generality, we may assume that $\alpha_0 \ne 0$ and $\beta_0 \ne 0$. Write $\alpha_i = f_i + f_i'x$ and $\beta_i = g_i + g_i'x$, where $f_i, f_i', g_i, g_i' \in R$ for $i \ge 0$.

CLAIM 1. *$f_i = 0$ for all $i \geq 0$.*

Proof. By (3.4), $\alpha_0\beta_0 = 0$. It follows that $f_0g_0 = 0$ and $f_0g_0' + f_0'\sigma(g_0) = 0$ in R. Since $g_0 + g_0'x \neq 0$ and R is a domain, it must be that $f_0 = 0$. Suppose that $f_0 = \cdots = f_{k-1} = 0$ and $f_k \neq 0$, where $k \geq 1$. We show that this leads to a contradiction. Again by (3.4), $\alpha_0\beta_k + \alpha_1\beta_{k-1} + \cdots + \alpha_k\beta_0 = 0$ in S, i.e.,

$$(3.5)\qquad f_kg_0 + [f_0'\sigma(g_k) + f_1'\sigma(g_{k-1}) + \cdots + f_k'\sigma(g_0) + f_kg_0']x = 0 \quad \text{in } S.$$

So $f_kg_0 = 0$ in R, and hence $g_0 = 0$. Assume that $g_0 = \cdots = g_{l-1} = 0$, where $l \geq 1$. By (3.4), $\alpha_0\beta_{k+l} + \alpha_1\beta_{k+l-1} + \cdots + \alpha_{k+l}\beta_0 = 0$ in S, that is,

$$\begin{aligned} f_kg_l + [f_0'\sigma(g_{k+l}) + \cdots + f_k'\sigma(g_l') \\ + f_kg_l' + k_{k+1}g_{l-1}' + \cdots + f_{k+l}g_0']x = 0 \quad \text{in } S. \end{aligned}$$

Thus $f_kg_l = 0$, so $g_l = 0$. A simple induction shows that $g_i = 0$ for $i = 0, 1, \ldots$. It then follows from (3.5) that $f_kg_0' = 0$ in R. So $g_0' = 0$ and hence $\beta_0 = 0$. This contradiction shows that Claim 1 holds.

CLAIM 2. *$\sigma(g_i) = 0$ for all $i \geq 0$.*

Proof. Since $\alpha_0 \neq 0$, we have $f_0' \neq 0$. Because of Claim 1, $\alpha_0\beta_0 = 0$ gives $f_0'\sigma(g_0) = 0$, so $\sigma(g_0) = 0$. Assume that $\sigma(g_0) = \cdots = \sigma(g_{k-1}) = 0$, where $k \geq 1$. As before, $\alpha_0\beta_k + \alpha_1\beta_{k-1} + \cdots + \alpha_k\beta_0 = 0$ in S. This gives $f_0'\sigma(g_k) = 0$, so $\sigma(g_k) = 0$. Hence $\sigma(g_i) = 0$ for all $i \geq 0$ by induction.

Thus, by Claims 1 and 2, $\alpha_i\beta_j = (f_i + f_i'x)(g_j + g_j'x) = (f_i'x)(g_j + g_j'x) = f_i'\sigma(g_j)x = 0$ for all i and j. So S is Armendariz of power series type. ∎

4. An extension of Theorem 5. When considering the Armendariz property of a skew polynomial ring $R[x;\sigma]$, the authors of [8] were naturally led to the notion of a σ-skew Armendariz ring. In [8, Theorem 6], they proved that, for an endomorphism σ of R with $\sigma^k = 1_R$ for some $k \geq 1$, R is σ-skew Armendariz if and only if $R[x]$ is σ-skew Armendariz. (There is a gap in the proof of [8, Theorem 6], as noted in [4, Remark, p. 1139].) This result is a consequence of a much more general result proved in this section.

DEFINITION 22. Let I be an ideal of a ring R and let σ be an endomorphism of R with $\sigma(I) \subseteq I$. We denote by $[R;I][x;\sigma]$ the subring $R[x;\sigma] + I[[x;\sigma]]$ of $R[[x;\sigma]]$, where $I[[x;\sigma]]$ is the ideal of $R[[x;\sigma]]$ generated by I.

DEFINITION 23. Let I be an ideal of R and let σ be an endomorphism of R with $\sigma(I) \subseteq I$. The ring R is called (σ, I)-*Armendariz* if, whenever $(\sum_{i\geq 0} a_ix^i)(\sum_{j\geq 0} b_jx^j) = 0$ in $[R;I][x;\sigma]$, then $a_i\sigma^i(b_j) = 0$ for all i and j.

LEMMA 24. *Let I be an ideal of a ring R and let σ be an endomorphism of R with $\sigma(I) \subseteq I$. Then the mapping $[R;I][x;\sigma] \to [R;I][x;\sigma]$, $\sum a_i x^i \mapsto \sum \sigma(a_i)x^i$, is a ring homomorphism, still denoted by σ, and $\sigma(I[[x;\sigma]]) \subseteq I[[x;\sigma]]$.*

Proof. The verification is straightforward. ∎

LEMMA 25. *Let I be an ideal of a ring R and let σ be an endomorphism of R with $\sigma(I) \subseteq I$, and let $k \geq 1$ be an integer. Define*

$$\phi_k : \big[[R;I][x;\sigma]; I[[x;\sigma]]\big][y;\sigma] \to [R;I][x;\sigma], \qquad \sum f_i y^i \mapsto \sum f_i x^{ki}.$$

Then:

(1) *ϕ_k is a well-defined mapping.*
(2) *ϕ_k is a ring homomorphism iff $\sigma^k = \sigma$.*

Proof. (1) Because of Lemma 24, (1) can be proved arguing as in the proof of Lemma 4.

(2) "$\Rightarrow$". Suppose that ϕ_k is a ring homomorphism. Then, for all $r \in R$, $0 = \phi_k(yr) - \phi_k(y)\phi_k(r) = \phi_k(\sigma(r)y) - x^k r = \sigma(r)x^k - \sigma^k(r)x^k = (\sigma(r) - \sigma^k(r))x^k$. This shows that $\sigma(r) = \sigma^k(r)$ for all $r \in R$. That is, $\sigma^k = \sigma$.

"$\Leftarrow$". The mapping ϕ_k clearly preserves addition. Direct calculation shows that the condition $\sigma^k = \sigma$ implies that ϕ_k preserves multiplication. ∎

Theorem 5 is a special case of the next theorem where $\sigma = 1_R$. The hypothesis that σ is periodic occurs here because of Lemma 25(2).

THEOREM 26. *Let I be an ideal of a ring R and let σ be an endomorphism of R with $\sigma(I) \subseteq I$. Suppose that σ is periodic (i.e., there exists $n_0 > 1$ such that $\sigma^{n_0} = \sigma$). Then R is (σ, I)-Armendariz if and only if $[R;I][x;\sigma]$ is $(\sigma, I[[x;\sigma]])$-Armendariz.*

Proof. One implication is clear. So, suppose that R is (σ, I)-Armendariz. Let

$$\Big(\sum_{i\geq 0} f_i y^i\Big)\Big(\sum_{i\geq 0} g_i y^i\Big) = 0 \quad \text{in } \big[[R;I][x;\sigma]; I[[x;\sigma]]\big][y;\sigma]. \tag{4.1}$$

We need to show that $f_i\sigma^i(g_j) = 0$ for all i and j. It is clear that $f_0 g_0 = 0$. Assume that, for $k \geq 1$, $f_0 g_j = 0$ for $j = 0, \ldots, k-1$. We next prove that $f_0 g_k = 0$. For $i \geq 0$, write

$$f_i = a_0^{(i)} + a_1^{(i)}x + \cdots + a_n^{(i)}x^n + \cdots,$$
$$g_i = b_0^{(i)} + b_1^{(i)}x + \cdots + b_n^{(i)}x^n + \cdots.$$

If $f_0 g_k \neq 0$, then we can assume that, for some $s \geq 0$, $f_0 b_j^{(k)} = 0$ for $j < s$ but $f_0 b_s^{(k)} \neq 0$. We can further assume that, for some $t \geq 0$, $a_i^{(0)}\sigma^i(b_s^{(k)}) = 0$ for $i < t$ but $a_t^{(0)}\sigma^t(b_s^{(k)}) \neq 0$. Let $n_1 > \max\{s, t\}$. Since $n_0 > 1$, there exists

$k_0 \geq 1$ such that $k_0(n_0 - 1) \geq n_1 - n_0$. Let $l = (k_0 + 1)n_0 - k_0$. Then $l \geq n_1$ and $\sigma^l = \sigma$. (Indeed, $\sigma^{(i+1)n_0 - i} = \sigma$ for all $i \geq 0$.) By Lemma 25,

$$\phi_l : \big[[R; I][x; \sigma]; I[[x; \sigma]]\big][y; \sigma] \to [R; I][x; \sigma], \qquad \sum f_i y^i \mapsto \sum f_i x^{li},$$

is a ring homomorphism. Thus, it follows from (4.1) that

$$\Big(\sum_{i\geq 0} f_i x^{il}\Big)\Big(\sum_{i\geq 0} g_i x^{il}\Big) = 0 \quad \text{ in } [R; I][x; \sigma].$$

Now arguing exactly as in the proof of Theorem 5, we deduce that, for each $i \geq 0$, $f_i \sigma^i(g_j) = 0$ for $j = 0, 1, \ldots$. ∎

The authors of [8] call a ring R *σ-skew Armendariz* if R is $(\sigma, 0)$-Armendariz. We call R *σ-skew Armendariz of power series type* if R is (σ, R)-Armendariz.

COROLLARY 27. *Let σ be a periodic endomorphism of R. Then R is σ-skew Armendariz iff $R[x; \sigma]$ is σ-skew Armendariz.*

Proof. We just apply Theorem 26 to the case where $I = 0$. ∎

COROLLARY 28. *Let σ be a periodic endomorphism of R. Then R is σ-skew Armendariz of power series type iff $R[[x; \sigma]]$ is σ-skew Armendariz of power series type.*

Proof. This is the special case of Theorem 26 where $I = R$. ∎

It is known that reduced rings are Armendariz of power series type, and Armendariz rings of power series type are Armendariz, but neither of the implications is reversible. It can also be shown that R being σ-rigid implies R is σ-skew Armendariz of power series type, and R being σ-skew Armendariz of power series type implies R is σ-skew Armendariz. Below, we present a ring R with an endomorphism $1_R \neq \sigma = \sigma^3$ such that R is σ-skew Armendariz but not σ-skew Armendariz of power series type, and a ring R with an endomorphism $1_R \neq \sigma = \sigma^2$ such that R is σ-skew Armendariz of power series type but not σ-rigid.

EXAMPLE 29. *Let $S = R \propto R$ where R is a* (*not necessarily commutative*) *domain. Define $\sigma : S \to S$, $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$. Then*:

(1) *σ is an endomorphism of S, it is neither injective nor surjective, and $\sigma^2 = \sigma$.*
(2) *σ is not rigid.*
(3) *S is σ-skew Armendariz of power series type.*

Proof. (1) and (2) are clear.

(3) Let $(\sum_{i\geq 0} \alpha_i x^i)(\sum_{j\geq 0} \beta_j x^j) = 0$ in $S[[x; \sigma]]$, where $\alpha_i = \left(\begin{smallmatrix} a_i & r_i \\ 0 & a_i \end{smallmatrix}\right)$ and $\beta_j = \left(\begin{smallmatrix} b_j & s_j \\ 0 & b_j \end{smallmatrix}\right)$. We prove that $\alpha_0 \beta_j = 0$ for all j. Then it follows by

induction that $\alpha_i\sigma^i(\beta_j)=0$ for all i and j. We can assume that $\alpha_0\beta_j=0$ for $j=0,1,\dots,k-1$. We next show that $\alpha_0\beta_k=0$.

Suppose that $a_0\neq 0$. Since R is a domain, $\alpha_0\beta_j=0$ implies that $\beta_j=0$ (for $j=0,\dots,k-1$). Thus, it follows from $(\sum_{i\geq 0}\alpha_i x^i)(\sum_{j\geq 0}\beta_j x^j)=0$ that $\alpha_0\beta_k=0$.

Suppose that $a_0=0$ but $r_0\neq 0$. Again since R is a domain, $\alpha_0\beta_j=0$ implies that $b_j=0$ (for $j=0,\dots,k-1$). Thus, from $(\sum_{i\geq 0}\alpha_i x^i)(\sum_{j\geq 0}\beta_j x^j)$ $=0$, one obtains $\alpha_0\beta_k+\alpha_1\sigma(\beta_{k-1})+\dots+\alpha_k\sigma^k(\beta_0)=0$. That is, $\alpha_0\beta_k+\alpha_1 0+\dots+\alpha_k 0=0$. So $\alpha_0\beta_k=0$.

Hence in any case, $\alpha_0\beta_k=0$. By induction, $\alpha_0\beta_j=0$ for all j. ∎

EXAMPLE 30. *Let $R=\mathbb{Z}\propto\mathbb{Z}_{2^\infty}$. Define $\sigma:R\to R$, $\left(\begin{smallmatrix} n & a\\ 0 & n\end{smallmatrix}\right)\mapsto\left(\begin{smallmatrix} n & -a\\ 0 & n\end{smallmatrix}\right)$. Then*:

(1) *σ is an automorphism of R, $\sigma\neq 1_R$ and $\sigma^2=1_R$ (so $\sigma^3=\sigma$).*
(2) *R is not σ-skew Armendariz of power series type.*
(3) *R is σ-skew Armendariz.*

Proof. (1) is clear.

(2) Let $\{\xi_i : i=0,1,\dots\}$ be a set of generators of the abelian group $\mathbb{Z}_{2^\infty}$ such that $2\xi_0=0$, and $2\xi_{i+1}=\xi_i$ for all $i\geq 0$. For each $i\geq 0$, let

$$a_{2i}=\begin{pmatrix}2&0\\0&2\end{pmatrix},\qquad a_{2i+1}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad b_i=\begin{pmatrix}0&\xi_i\\0&0\end{pmatrix}.$$

Then $(\sum_{i\geq 0}a_i x^i)(\sum_{i\geq 0}b_i x^i)=0$ in $R[[x;\sigma]]$. But $a_0b_2\neq 0$. So R is not σ-skew Armendariz of power series type.

(3) This can be proved arguing as in the proof of Example 3(2). ∎

PROPOSITION 31. *Let I be an ideal of a ring R and let σ be an endomorphism of R with $\sigma(I)\subseteq I$. Suppose that $\sigma^k=1_R$ for some $k\geq 1$. Then R is (σ,I)-Armendariz if and only if $[R;I][x]$ is $(\sigma,I[[x]])$-Armendariz.*

Proof. If $k=1$, this is Theorem 5. So assume that $k>1$, and write

$$[R;I][x^k]=\{f(x^k): f(x)\in[R;I][x]\},$$
$$[R;I][x^k;\sigma]=\{f(x^k): f(x)\in[R;I][x;\sigma]\},$$
$$I[[x^k]]=\{f(x^k): f(x)\in I[[x]]\},$$
$$I[[x^k;\sigma]]=\{f(x^k): f(x)\in I[[x;\sigma]]\}.$$

Then $[R;I][x^k]$ is a subring of $[R;I][x]$ and $I[[x^k]]$ is an ideal of $[R;I][x^k]$, and $[R;I][x^k;\sigma]$ is a subring of $[R;I][x;\sigma]$ and $I[[x^k;\sigma]]$ is an ideal of $[R;I][x^k;\sigma]$. The following two claims can be easily verified.

CLAIM 1. *The mapping $\big[[R;I][x];I[[x]]\big][y;\sigma]\to\big[[R;I][x^k];I[[x^k]]\big][y;\sigma]$, $\sum_{i\geq 0}f_i(x)y^i\mapsto\sum_{i\geq 0}f_i(x^k)y^i$, is a ring homomorphism.*

CLAIM 2. *$\theta : [R,I][x^k] \cong [R,I][x^k;\sigma]$ and $\theta(I[[x^k]]) = I[[x^k;\sigma]]$, where $\theta(\sum_{i\geq 0} a_i x^{ik}) = \sum_{i\geq 0} a_i x^{ik}$; and $\bar{\theta} : \big[[R,I][x^k], I[[x^k]]\big][y;\sigma] \cong \big[[R,I][x^k;\sigma], I[[x^k;\sigma]]\big][y;\sigma]$, where $\bar{\theta}(\sum_{i\geq 0} f_i y^i) = \sum_{i\geq 0} \theta(f_i) y^i$.*

Now let

$$\Big(\sum_{i\geq 0} f_i(x)y^i\Big)\Big(\sum_{i\geq 0} g_i(x)y^i\Big) = 0 \quad \text{in } \big[[R;I][x], I[[x]]\big][y;\sigma].$$

We need to show that $f_i(x)\sigma^i(g_j(x)) = 0$ for all i and j. By Claim 1,

$$\Big(\sum_{i\geq 0} f_i(x^k)y^i\Big)\Big(\sum_{i\geq 0} g_i(x^k)y^i\Big) = 0 \quad \text{in } \big[[R;I][x^k], I[[x^k]]\big][y;\sigma].$$

This shows, by Claim 2, that

$$\Big(\sum_{i\geq 0} f_i(x^k)y^i\Big)\Big(\sum_{i\geq 0} g_i(x^k)y^i\Big) = 0 \quad \text{in } \big[[R;I][x^k;\sigma], I[[x^k;\sigma]]\big][y;\sigma].$$

Thus, $(\sum_{i\geq 0} f_i(x^k)y^i)(\sum_{i\geq 0} g_i(x^k)y^i) = 0$ in $\big[[R;I][x;\sigma], I[[x;\sigma]]\big][y;\sigma]$. But by Theorem 26, $[R;I][x;\sigma]$ is $(\sigma, I[[x;\sigma]])$-Armendariz, and hence $f_i(x^k)\sigma^i(g_j(x^k)) = 0$ in $[R;I][x;\sigma]$ for all i and j. Hence $f_i(x^k)\sigma^i(g_j(x^k)) = 0$ in $[R;I][x^k;\sigma]$ for all i and j. Thus, by Claim 2, $f_i(x^k)\sigma^i(g_j(x^k)) = 0$ in $[R;I][x^k]$ for all i and j. This clearly shows that $f_i(x)\sigma^i(g_j(x)) = 0$ in $[R;I][x]$. ∎

COROLLARY 32 ([4, Proposition 7], [8, Theorem 6]). *Let σ be an endomorphism of R. Suppose that $\sigma^k = 1_R$ for some $k \geq 1$. Then R is σ-skew Armendariz if and only if $R[x]$ is σ-skew Armendariz.*

COROLLARY 33. *Let σ be an endomorphism of R. Suppose that $\sigma^k = 1_R$ for some $k \geq 1$. Then R is σ-skew Armendariz of power series type if and only if $R[[x]]$ is σ-skew Armendariz of power series type.*

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