ON THE FOUR VERTEX THEOREM IN PLANES WITH RADIAL DENSITY $e^{\varphi(r)}$

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Abstract. It is shown that in a plane with a radial density the four vertex theorem holds for the class of all simple closed curves if and only if the density is constant. On the other hand, for the class of simple closed curves that are invariant under a rotation about the origin, the four vertex theorem holds for every radial density.

A *manifold with density* is a Riemannian manifold M^n with a positive density function $e^{\varphi(x)}$ used to weight volume and hypersurface area. Such manifolds appeared in many ways in mathematics, for example as quotients of Riemannian manifolds or as the Gauss space. The *Gauss space* G^n is a Euclidean space with Gaussian probability density $(2\pi)^{-n/2}e^{-r^2/2}$ that is very interesting to probabilists. For more details about manifolds with density and some first results in Morgan's grand goal to "generalize all of Riemannian geometry to manifolds with density" we refer the reader to [6], [7], [4], [1]. Following Gromov ([5, p. 213]) the natural generalization of the mean curvature of hypersufaces on a manifold with density e^φ is defined as

$$(1) \qquad H_\varphi = H - \frac{1}{n-1}\frac{d\varphi}{d\mathbf{n}},$$

and therefore the curvature of a curve in a plane with density e^φ is

$$(2) \qquad k_\varphi = k - \frac{d\varphi}{d\mathbf{n}}.$$

We call k_φ the *curvature with density* or *φ-curvature* of the curve.

In this note, we study the four vertex theorem in planes with radial density $e^{\varphi(r)}$, where r is the distance from the origin. Curves and the function φ are assumed to be of class C^3 and C^2, respectively.

It is well known that "every simple closed curve in the Euclidean plane has at least four vertices" (the four vertex theorem). This theorem has a long and interesting history (see [3], [8]).

First we observe that in general, the four vertex theorem does not hold in planes with density.

In the Gauss plane G^2 with density $e^\varphi = \frac{1}{2\pi}e^{-r^2/2}$, let

$$\alpha : [0, 2\pi] \to \mathbb{R}^2, \quad t \mapsto (\cos t, \sin t + b),$$

be a parametrization by arc length of the unit circle with center $I(0, b)$. In the Gauss plane G^2, direct computation shows that

(3) $$k_\varphi = x'y'' - x''y' - xy' + x'y.$$

Applying (3), we get the φ-curvature of the circle

$$k_\varphi = -b\sin t.$$

The equation $k'_\varphi = 0$ has exactly two solutions $\pi/2$ and $3\pi/2$ and hence the circle has exactly two vertices.
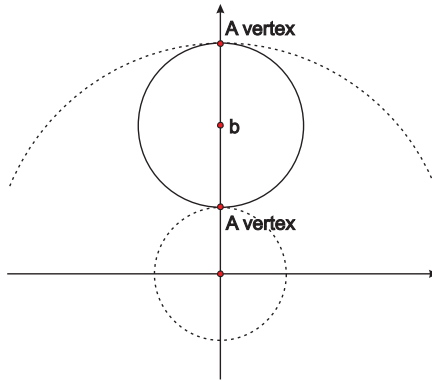

Fig. 1. A circle has two vertices

The above calculation naturally raises two questions.

1. For which classes of simple closed curves does the four vertex theorem hold in planes with arbitrary radial densities?
2. Does there exist a plane with radial density other than the Euclidean one such that the four vertex theorem holds for the class of all closed simple curves?

We can see that under a rotation about the origin the curvature k of a curve and $d\varphi/d\mathbf{n}$ are not changed. Thus, the φ-curvature of a curve in a plane with radial density $e^{\varphi(r)}$ is invariant under a rotation about the origin. As corollaries of this fact, we have the four vertex theorem for the class of simple closed curves that are symmetric about the origin or more generally, for the class of simple closed curves that are invariant under a rotation about the origin.

COROLLARY 1. *In a plane with density $e^{\varphi(r)}$, every simple closed curve α symmetric about the origin has at least four vertices.*

Proof. Since k_φ is continuous on a closed curve, it reaches a minimum and a maximum on the curve. Suppose that k_φ reaches its minimum at t_1 and its maximum at t_2. Obviously, $k_\varphi(t_1) = k_\varphi(t_1 + \pi)$ and $k_\varphi(t_2) = k_\varphi(t_2 + \pi)$. But $\alpha(t_1) \neq \alpha(t_2 + \pi)$ and $\alpha(t_2) \neq \alpha(t_1 + \pi)$, unless $k_\varphi = $ const. ∎

By a similar proof, we have a more general result.

COROLLARY 2. *In a plane with density $e^{\varphi(r)}$, every simple closed curve that is invariant under a rotation about the origin by an angle $\theta < 2\pi$ has at least four vertices.*

For the second question, we have the following main result of this note.

THEOREM 3. *In a plane with density $e^{\varphi(r)}$, the four vertex theorem holds for the class of all simple closed curves if and only if φ is a constant.*

In order to prove Theorem 3 we need the following.

LEMMA 4. *Let C be the circle with center at $(0, b)$ and radius R in the plane $\mathbb{R}^2$ with density $e^{\varphi(r)}$ and $\alpha : [0, 2\pi] \to \mathbb{R}^2$, $\alpha(t) = (R \cos t, R \sin t + b)$, be its parametrization. Assume that C does not pass through the origin. Denote by k_φ the φ-curvature of C. Then*

$$(4) \qquad \frac{dk_\varphi}{dt} = \frac{b \cos t}{r^3}\left(\varphi'' \frac{r^3 + (R^2 - b^2)r}{2} + \varphi' \frac{r^2 - (R^2 - b^2)}{2} \right).$$

Proof. Direct computation yields

$$k_\varphi = \frac{1}{R} + \varphi' \frac{R + b \sin t}{r}.$$

Therefore,

$$\begin{aligned}
\frac{dk_\varphi}{dt} &= \frac{d}{dt}\left(\frac{1}{R} + \varphi' \frac{R + b \sin t}{r} \right) \\
&= \varphi'' \frac{dr}{dt} \frac{R + b \sin t}{r} + \varphi' \frac{1}{r^2}\left(br \cos t - \frac{dr}{dt}(R + b \sin t) \right) \\
&= \varphi'' \frac{bR \cos t}{r} \frac{R + b \sin t}{r} + \varphi' \frac{1}{r^2}\left(br \cos t - \frac{bR \cos t}{r}(R + b \sin t) \right) \\
&= \frac{b \cos t}{r^3}[\varphi''(R^2 r + bRr \sin t) + \varphi'(b^2 + bR \sin t)].
\end{aligned}$$

Since on the circle we have $r = \sqrt{R^2 + b^2 + 2Rb \sin t}$, replacing $bR \sin t$ by $(r^2 - R^2 - b^2)/2$, we obtain (4). ∎

Lemma 4 has some useful corollaries.

COROLLARY 5. *In a plane with density $e^{\varphi(r)}$, a circle about the origin has constant φ-curvature.*

Proof. This is immediate since $b = 0$. ∎

COROLLARY 6. *For every positive integer $n \geq 1$, there exists a radial density $e^{\varphi(r)}$ in the plane such that a given circle containing the origin in its interior, but not as the center, has exactly $2n$ vertices.*

Proof. We can assume that the center of the circle is $(0, b)$, $b > 0$, and the radius is $R > 0$. Since the circle contains the origin, we have $R > b$ and therefore $r^2 + R^2 - b^2 > 0$. Let $r_1, \ldots, r_{n-1}$ be positive real numbers such that $R - b < r_1 < \cdots < r_{n-1} < R + b$. Set

$$p(r) = (r - r_1) \ldots (r - r_{n-1}).$$

It is easy to see that the ODE

$$\varphi''(r^3 + (R^2 - b^2)r) + \varphi'(r^2 - R^2 + b^2) = r^2 p(r)$$

has a family of solutions of the form

$$\varphi = \int \frac{(\int p(r)\,dr + c_1)r}{r^2 + R^2 - b^2}\,dr + c_2.$$

Therefore, the equation $dk_\varphi/dt = 0$ has exactly $2n$ solutions and it follows that the circle has exactly $2n$ vertices. ■
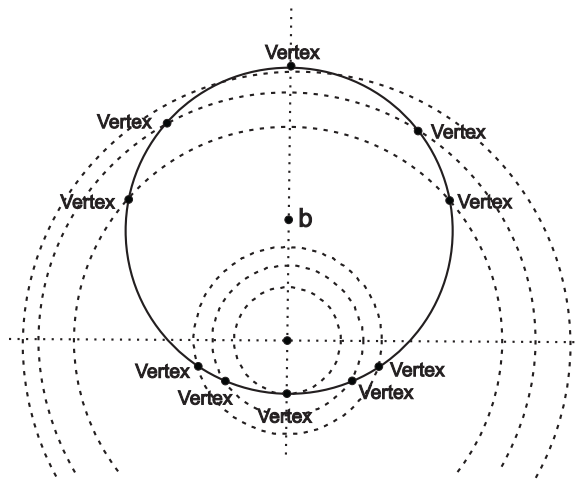

Fig. 2. There exists a radial density such that a given circle containing the origin has $2n$ vertices.

COROLLARY 7. *There exists a radial density $e^{\varphi(r)}$ in the plane such that a given circle containing the origin in its interior has constant density curvature.*

Proof. We can assume that the center of the circle is $(0, b)$, $b > 0$ (the case of $b = 0$ follows from Corollary 5) and the radius is $R > 0$. Since the

circle contains the origin, we have $R > b$ and therefore $r^2 + R^2 - b^2 > 0$. It is easy to see that the ODE

$$\varphi''(r^3 + (R^2 - b^2)r) + \varphi'(r^2 - R^2 + b^2) = 0$$

has a family of solutions of the form

$$\varphi = c_1 \ln(r^2 + R^2 - b^2) + c_2. \quad \blacksquare$$

Proof of Theorem 3. We prove Theorem 3 by showing that if φ is not a constant, then there exists a circle with center $(0, b)$ and radius R that has exactly two vertices. Now suppose $\varphi'(b) \neq 0, b \neq 0$.

1. If $\varphi'(b) > 0$, then there exists $0 < \varepsilon < b$ such that $\varphi'(r) > 0$ for all $r \in [b - \varepsilon, b + \varepsilon]$. Let $N = \min_{r \in [b-\varepsilon,b+\varepsilon]} \varphi'(r) > 0$. Since φ'' is continuous in $[b - \varepsilon, b + \varepsilon]$ there exists a positive real number M with $M > |\varphi''(r)|$ for all $r \in [b - \varepsilon, b + \varepsilon]$.

Choose $R > 0$ so that $0 < R < \min\{\varepsilon, b(b - \varepsilon)N/(M(b + \varepsilon)^2)\}$. We have

$$|\varphi'' Rr(R + b \sin t)| < MR(b + \varepsilon)|R + b \sin t| \leq MR(b + \varepsilon)(R + b)$$

$$< MR(b + \varepsilon)^2 < M \frac{b(b - \varepsilon)(b + \varepsilon)^2 N}{M(b + \varepsilon)^2}$$

$$= b(b - \varepsilon)N < b(b - R)N \leq \varphi' b(b + R \sin t).$$

Recall that

$$\varphi'' Rr(R + b \sin t) = \varphi'' \frac{r^3 + (R^2 - b^2)r}{2}, \quad \varphi' b(b + R \sin t) = \varphi' \frac{r^2 - (R^2 - b^2)}{2}$$

(see the proof of Lemma 4); we have

$$\varphi'' \frac{r^3 + (R^2 - b^2)r}{2} + \varphi' \frac{r^2 - (R^2 - b^2)}{2} > 0$$

and hence the circle with center $(0, b)$ and radius R has exactly two vertices.

2. If $\varphi'(b) < 0$, then there exists $0 < \varepsilon < b$ such that $\varphi'(r) < 0$ for all $r \in [b - \varepsilon, b + \varepsilon]$. Let $N = \min_{r \in [b-\varepsilon,b+\varepsilon]} |\varphi'(r)| > 0$. We have $\varphi'(r) \leq -N$ for all $r \in [b - \varepsilon, b + \varepsilon]$. Let M, R be as in the proof of the first part; by a similar proof, we obtain

$$|\varphi'' Rr(R + b \sin t)| < b(b - R)N \leq -b(b + R \sin t)\varphi'.$$

Therefore,

$$\varphi'' Rr(R + b \sin t) + b(b + R \sin t)\varphi' < 0,$$

or

$$\varphi'' \frac{r^3 + (R^2 - b^2)r}{2} + \varphi' \frac{r^2 - (R^2 - b^2)}{2} < 0.$$

Thus, the circle with center $(0, b)$ and radius R has exactly two vertices. $\blacksquare$

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