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ON SOME NOTIONS OF CHAOS IN DIMENSION ZERO

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Abstract. We compare four different notions of chaos in zero-dimensional systems (subshifts). We provide examples showing that in that case positive topological entropy does not imply strong chaos, strong chaos does not imply complicated dynamics at all, and ω-chaos does not imply Li–Yorke chaos.

1. Introduction. In recent years the notion of chaos has become a subject of interest and a variety of definitions meant to formalize this notion have been introduced by several authors. First results concerning chaotic behavior have been obtained by A. N. Sharkovsky (well known theorem about the order of periods of a map from the closed interval into itself, see [S]) and by Tien-Yien Li and James A. Yorke in [LY]. Many properties of chaotic systems have been investigated mainly in the case of compact interval, where some notions of chaos coincide (see [SS] and [Li] for equivalence of strong chaos, ω-chaos and positive topological entropy; see also [La1], [La2], [SSt] for other properties of ω-chaotic maps). However, already in the case of triangular maps of the square into itself some of the above equivalences no longer hold (see [FP]). Examples of strongly chaotic systems with zero topological entropy are also known for zero-dimensional systems (see [LF]). The importance of zero-dimensional dynamics comes from the fact that Cantor systems are often used in constructions of higher dimensional maps to ensure positive entropy or other chaotic features.

One should also mention that chaos has been studied in general spaces; for instance, in [BGKM] the authors have affirmatively answered the long-standing open question whether positive topological entropy implies Li–Yorke chaos.

In this paper we provide examples to enlarge the collection of implications, between various notions of chaos, which fail in dimension zero.

2. Terminology and notation. By a *dynamical system* we mean a pair (X, f), where X is a compact metric space with metric d, and f is a continuous map from X to itself. A map $\varphi : (X, f) \to (X', f')$, where φ is

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a continuous function from X onto X' such that $f' \circ \varphi = \varphi \circ f$, is called a *factor map*. The system (X, f) is then called an *extension* of (X', f'), and (X', f') is a *factor* of (X, f).

A set $B \subset X$ is called *invariant* under f if $f(B) \subset B$. A set $M \subset X$ is said to be *minimal* if it is nonempty, closed and invariant under f and it does not contain any proper subsets which satisfy these three conditions. A system (X, f) is *minimal* if X is a minimal set.

For $x \in X$, by the *orbit* of x we mean the set $O(x) := \{f^k(x) : k \geq 0\}$, while its closure $\overline{O(x)}$ is called the *orbit closure*. The *ω-limit set* of $x \in X$ is the set

$$\omega(x, f) = \{y \in X : \exists_{\{n_k\}_{k=1}^{\infty},\, n_k \to \infty}\ f^{n_k}(x) \to y\},$$

denoted also by $\omega(x)$ if it causes no ambiguity.

A point x is said to be *recurrent* for f if $x \in \omega(x, f)$.

A point x is said to be *almost periodic* for f if

$$\forall_{\varepsilon>0}\ \exists_{k>0}\ \forall_{q\geq 0}\ \exists_{q\leq r<k+q}\ d(f^r(x), x) < \varepsilon.$$

The following fact is well known.

FACT 2.1. *$\overline{O(x)}$ is minimal if and only if x is an almost periodic point.*

The following definition is based on ideas in [LY].

DEFINITION 2.1. A system (X, f) is said to be *chaotic in the sense of Li–Yorke* if there exists an uncountable set $D \subset X$ such that for any different points $x, y \in D$,

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$ (x, y are *not asymptotic*),

(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$ (x, y are *proximal*).

A much more restrictive version of chaotic behavior was introduced by Schweizer and Smítal in [SS].

DEFINITION 2.2. A system (X, f) is said to be *strongly chaotic* if there exists an uncountable set $D \subset X$ such that for any different points $x, y \in D$,

(1) $\forall_{t>0}\ \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{[0,t)}(d(f^k(x), f^k(y))) = 1,$

(2) $\exists_{t>0}\ \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{[0,t)}(d(f^k(x), f^k(y))) = 0.$

Another definition of chaos was given by Shihai Li in [Li].

DEFINITION 2.3. A system (X, f) is said to be *ω-chaotic* if there exists an uncountable set $D \subset X$ such that for any different points $x, y \in D$,

(1) $\omega(x, f) \setminus \omega(y, f)$ is uncountable,

(2) $\omega(x,f)\cap\omega(y,f)$ is nonempty,
(3) $\omega(x,f)$ is not contained in the set of periodic points.

Let $\Sigma=\{0,1\}^{\mathbb{N}}$. Define the metric $d:\Sigma\times\Sigma\to\mathbb{R}$ as follows:

$$d(x,y)=\begin{cases}0 & \text{if } x=y,\\ 1/k & \text{if } x\neq y \text{ and } k=\min\{n\geq 1: x(n)\neq y(n)\}.\end{cases}$$

Define $\sigma:\Sigma\to\Sigma$ by

$$\sigma(x(1)x(2)x(3)\ldots)=x(2)x(3)\ldots.$$

This continuous map is called the *shift* on Σ.

If $X\subset\Sigma$ is closed, nonempty and shift-invariant, then $(X,\sigma|_X)$ is called a *subshift* or a *symbolic system on two symbols*.

If $A=a(1)\ldots a(n)\in\{0,1\}^n$, then A is called a *block* and $|A|:=n$ denotes the *length* of A. If we have two blocks $A=a(1)\ldots a(n)$ and $B=b(1)\ldots$ $\ldots b(m)$ then we can form another block by *concatenation*,

$$AB:=a(1)\ldots a(n)b(1)\ldots b(m).$$

We say that a block B *occurs* in some block A if $A=CBD$, where C and D are some blocks (maybe empty), and we denote it by $B\prec A$. A block A occurs in a point $x\in\Sigma$ if it occurs in some initial block of x.

Let $X\subset\Sigma$. For given $n\geq 1$ denote by $Q_n(X)$ the number of blocks of length n occurring in X, i.e., the cardinality of the set

$$\{A=a(1)\ldots a(n): \text{there is some } x\in X \text{ such that } A\prec x\}.$$

Let (X,σ) be a symbolic system. Let $h(X)$ denote the topological entropy (for the definition see [W]). Then we have the following well known lemma ([W]).

LEMMA 2.1. *Let (X,σ) be a subshift. Then*

$$h(X)=\lim_{n\to\infty}\frac{\log Q_n(X)}{n}.$$

Let $\mathbb{T}=[0,1]$ with the endpoints identified and let $S_\beta:\mathbb{T}\to\mathbb{T}$, where β is an irrational number from the interval $(0,1)$, be given by the formula $S_\beta(t):=t+\beta \pmod 1$. Define $\varphi_{\alpha,\beta}(t)(n)=\mathbf{1}_{[0,\alpha)}(S_\beta^n(t))$, where $\alpha\in(0,1)$. Then $\varphi_{\alpha,\beta}$ is a map from $\mathbb{T}$ to Σ. Now we can define a system $(M_{\alpha,\beta},\sigma):=(\overline{\varphi_{\alpha,\beta}(\mathbb{T})},\sigma)$ which we will call a (two-parameter) *Sturmian system*. The following is a basic fact about Sturmian systems.

FACT 2.2. *For every $\alpha\in(0,1)$ and every irrational number $\beta\in(0,1)$ the system $(M_{\alpha,\beta},\sigma)$ is minimal and it is an extension of $(\mathbb{T},S_\beta)$.*

3. Some technical lemmas. In this section we present a number of lemmas needed later on. The first one is a slight generalization of Lemma 2.2

in [Li]. Let $a = a(1)a(2)\ldots, b = b(1)b(2)\ldots \in \Sigma$. Define the following operation:

$$a \diamond b := a(1)\, b(1)\, a(1)a(2)\, b(1)b(2)\, a(1)a(2)a(3)\, b(1)b(2)b(3)\, \ldots.$$

Lemma 3.1. *Let $a = a(1)a(2)\ldots, b = b(1)b(2)\cdots \in \Sigma$. Then*

(1) $\omega(a \diamond b) \supseteq \overline{O(a)} \cup \overline{O(b)}$,

(2) $\omega(a \diamond b) \subseteq \overline{O(a)} \cup \overline{O(b)} \cup \{a(i)\ldots a(j)b(1)b(2)\ldots : j \geq i \geq 1\}$
$$\cup \{b(i)\ldots b(j)a(1)a(2)\ldots : j \geq i \geq 1\}.$$

Moreover, *if a and b are recurrent points then in* (2) *equality holds.*

Proof. It is obvious that a, b belong to $\omega(a \diamond b)$. This set is closed and invariant, therefore it contains $\overline{O(a)}$ and $\overline{O(b)}$, so (1) is proved.

Let $x \in \omega(a \diamond b)$. There exists a sequence $\{n_k\}$ of positive integers such that $n_k \to \infty$ and $\sigma^{n_k}(a \diamond b) \to x$. It is easy to see that, depending on whether the initial block of $\sigma^{n_k}(a \diamond b)$ belongs to a or to b for infinitely many indices k, and whether its length is bounded or not, x has one of the four possible forms:

$$x = a(i)\ldots a(j)b(1)b(2)b(3)\ldots, \quad \text{where } j \geq i \geq 1,$$
$$x = b(i)\ldots b(j)a(1)a(2)a(3)\ldots, \quad \text{where } j \geq i \geq 1,$$
$$x \in \overline{O(a)} \text{ or } x \in \overline{O(b)}.$$

This proves (2).

Suppose that a is a recurrent point. There exists a sequence $\{n_k\}$ of positive integers such that $n_k \to \infty$ and $\sigma^{n_k}(a) \to a$, which means that any initial block from a occurs in a infinitely many times.

Let $x = a(i)\ldots a(j)b(1)b(2)\ldots$ for some $j \geq i \geq 1$. By the recurrence property of a the block $a(1)\ldots a(j)$, and so $a(i)\ldots a(j)$, occurs in $a \diamond b$ infinitely often, followed by ever longer blocks $b(1)\ldots b(n)$. Therefore, $x \in \omega(a \diamond b)$. The proof for $x = b(i)\ldots b(j)a(1)a(2)\ldots$ is analogous. ∎

Let (Z, f) be a topological dynamical system, let A be a closed subset of Z, and let $z \in Z$. We say that a point $x \in \Sigma$ is *A-compatible* with z if

$$\forall_{k\in\mathbb{N}}\ f^k(z) \in \mathrm{Int}(A) \Rightarrow x(k) = 1 \quad \text{and} \quad f^k(z) \in A^{\mathrm{c}} \Rightarrow x(k) = 0.$$

Let $X_A = \{x : \text{there exists } z \in Z \text{ such that } x \text{ is } A\text{-compatible with } z\}$. Then (X_A, σ) is a subshift.

Lemma 3.2. *Let $A_n, A \subset Z$ be closed subsets of Z such that $A_n \to A$ and $\overline{A_n^{\mathrm{c}}} \to \overline{A^{\mathrm{c}}}$ in the Hausdorff metric. Then*

$$\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} X_{A_n}} \subset X_A.$$

Proof. Let d, dist, d_{H} denote the metric in Z, the distance between a point and a set in Z, and the Hausdorff metric, respectively.

Let $x_j \in X_{A_{n_j}}$, where $n_j \to \infty$ and $x_j \to x \in \Sigma$ as $j \to \infty$. We want to show that $x \in X_A$. Let z_j be a point such that x_j is A_{n_j}-compatible with z_j. We may assume that $z_j \to z \in Z$. We will show that x is A-compatible with z (this fact will be used later in the proof of Lemma 3.3).

We will prove only the first implication in the definition of A-compatibility (the proof of the other one is analogous). Let k be an integer such that $f^k(z) \in \mathrm{Int}(A)$, let $\varepsilon = \mathrm{dist}(f^k(z), \overline{A^{\mathrm{c}}}) > 0$ and let j_0 be an integer such that for every $j > j_0$ the following conditions are satisfied:

1) $d(f^k(z_j), f^k(z)) < \varepsilon/3$,
2) $d_{\mathrm{H}}(\overline{A^{\mathrm{c}}_{n_j}}, \overline{A^{\mathrm{c}}}) < \varepsilon/3$.

Assume that there exists $j_1 \geq j_0$ such that for all $j \geq j_1$ we have

$$f^k(z_j) \in \overline{A^{\mathrm{c}}_{n_j}}.$$

Using 1) we conclude that $\mathrm{dist}(f^k(z_j), \overline{A^{\mathrm{c}}}) > \frac{2}{3}\varepsilon$, hence $d_{\mathrm{H}}(f^k(z_j), \overline{A^{\mathrm{c}}}) > \frac{2}{3}\varepsilon$, but using 2) we deduce that $d_{\mathrm{H}}(f^k(z_j), \overline{A^{\mathrm{c}}}) \leq d_{\mathrm{H}}(\overline{A^{\mathrm{c}}_{n_j}}, \overline{A^{\mathrm{c}}}) < \varepsilon/3$, which gives us a contradiction. Therefore, for infinitely many $j \geq j_0$ we have

$$\mathrm{dist}(f^k(z_j), \overline{A^{\mathrm{c}}_{n_j}}) > 0.$$

Thus, $f^k(z_{j_m}) \in \mathrm{Int}(A_{n_{j_m}})$ for some sequence $\{j_m\}$, which implies that $x_{j_m}(k) = 1$. Since $x_{j_m} \to x$, we obtain $x(k) = 1$. ∎

Let $\alpha_k, \alpha \in (0,1)$ and let β be an irrational number. Let $I_\alpha = [0, \alpha]$. Notice that the sets I_{α_k}, $\overline{I^{\mathrm{c}}_{\alpha_k}}$ tend to I_α, $\overline{I^{\mathrm{c}}_\alpha}$, respectively, in the Hausdorff metric whenever α_k tends to α. We will consider the Sturmian system $M_{\alpha,\beta}$ and the slightly larger system $\widetilde{M}_{\alpha,\beta}$ obtained as the set X_{I_α} of all sequences I_α-compatible with elements of $\mathbb{T}$ with respect to the rotation by β. The system $(\widetilde{M}_{\alpha,\beta}, \sigma)$ is an extension of $(\mathbb{T}, S_\beta)$; the factor map associates to each point $a \in \widetilde{M}_{\alpha,\beta}$ the unique $z \in \mathbb{T}$ with which a is compatible. Each z in $\mathbb{T}$ has one, two or four preimages depending on whether the orbit of z passes through none, one or both endpoints of I_α. Since these preimages differ from each other at at most two coordinates, they are asymptotic. Clearly the set of points having more than one preimage is countable. It is also easily seen that $M_{\alpha,\beta} \subset \widetilde{M}_{\alpha,\beta}$, and that every point $a \in \widetilde{M}_{\alpha,\beta}$ satisfies $\sigma^n(a) \in M_{\alpha,\beta}$ for some integer n.

LEMMA 3.3. *Let $\alpha_k, \alpha \in (0,1)$ and $\alpha_k \to \alpha$. Let $a_k \in M_{\alpha_k,\beta}$ be an extension of $z_k \in \mathbb{T}$. Then*

$$\lim_{k\to\infty} a_k = a \ \Rightarrow\ a \in \widetilde{M}_{\alpha,\beta},\ \lim_{k\to\infty} z_k = z \text{ and } a \text{ is an extension of } z.$$

Proof. This follows directly from Lemma 3.2: in our setup a point $a_k \in \widetilde{M}_{\alpha_k,\beta}$ is compatible with $z_k \in \mathbb{T}$ if and only if it is its extension. The fact that the sequence $\{z_k\}$ converges in $\mathbb{T}$ follows from the proof of Lemma 3.2: a is an extension of any accumulation point of the sequence $\{z_k\}$, so such an accumulation point is unique. ∎

The last lemma in this section is taken from [LF, Lemma 5].

LEMMA 3.4. *There exists an uncountable subset $E \subset \Sigma$ such that for any different points $x = x(1)x(2)\ldots$, $y = y(1)y(2)\ldots$ in E we have $x(n) = y(n)$ for infinitely many n and $x(m) \neq y(m)$ for infinitely many m.*

4. The main results. In this section we give examples of dynamical systems in which one kind of chaotic behavior does not imply another.

The first example based on Example 3.4 in [GW] shows that there exist positive entropy systems which are not strongly chaotic.

EXAMPLE 4.1. Let $p \geq 3$ be an integer. Let w be a point from (Σ, σ) such that $w(n) = 0$ for every positive integer n from the set

$$A = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \{mp^k, mp^k + 1, \ldots, mp^k + k - 1\},$$

and $w(n) = 1$ for $n \notin A$.

First of all, notice that since $mp^k + i = (mp^{k-i-1})p^{i+1} + i$ for $i < k$, the set A may be rewritten as

$$A = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \{mp^k + k - 1\} = \bigcup_{k=1}^{\infty} \{mp^k + k - 1 : m \in \mathbb{N}\}.$$

Thus, for $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{\#\{j \in A : j \leq n\}}{n} &\leq \sum_{k=1}^{\infty} \frac{\#\{j \in \{mp^k + k - 1 : m \in \mathbb{N}\} : j \leq n\}}{n} \\ &< \sum_{k=1}^{\infty} \frac{1}{p^k} = \frac{1}{p-1}. \end{aligned}$$

The above inequality implies that the number of zeros which occur in the block $w(1)\ldots w(n)$ is smaller than $n/(p-1)$, for $n \in \mathbb{N}$. Thus for a chosen $j \in \mathbb{N}$ there are infinitely many blocks, among the blocks of the form $w(mp^j)\ldots w((m+1)p^j - 1)$, where $m \in \mathbb{N}$, in which the number of zeros is not greater than $p^j/(p-1)$.

We will now modify w by changing some of its ones to zeros. Firstly, notice that among the above blocks of length p^j with no more than $p^j/(p-1)$ zeros, infinitely many must be identical. Therefore, we may choose some finite but large enough number of these blocks (e.g. 2 to the power $(1-1/(p-1))p^j$)

and change some of its ones to zeros so that after this modification all these blocks are distinct. Secondly, notice that the above change does not interfere with the property $w(n) = 0$ for $n \in A$ and that it affects only some initial block of w, so there is enough "free space" for further modifications in the next steps, which are, in fact, inductive steps with respect to j.

Finally, we obtain an estimation of $Q_{p^j}(\{w\})$, the number of blocks of length p^j which occur in w after all the above modifications:

$$Q_{p^j}(\{w\}) \geq 2^{(1-1/(p-1))p^j} \quad \text{for } j \in \mathbb{N}.$$

Let $(X, \sigma) := (\overline{O(w)}, \sigma)$. Applying Lemma 2.1 we can estimate the topological entropy:

$$h(X) \geq \left(1 - \frac{1}{p-1}\right) \log(2) > 0.$$

Now we will show that for any two different points in (X, σ) the second condition in the definition of strong chaos is not satisfied. Recall that this condition is

$$\exists_{t>0} \ \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{[0,t)}(d(\sigma^k(x), \sigma^k(y))) = 0,$$

where x, y $(x \neq y)$ belong to some uncountable set $D \subset X$.

Let x and y be any two distinct points in X and let $t \in (0,1)$. Let $k_0 = [1/t] + 1$. From the construction of (X, σ) we can deduce the following facts:

1) in x there is a block of zeros of length k_0 occurring periodically with period p^{k_0},
2) in y there is a block of zeros of length $p^{k_0} + 2k_0$ occurring periodically with period $p^{p^{k_0}+2k_0}$, which is a multiple of p^{k_0}.

The above properties imply that there exists an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers, which is, in fact, an arithmetic progression with difference $p^{p^{k_0}+2k_0}$, such that the initial blocks of both points $\sigma^{n_k}(x), \sigma^{n_k}(y)$ consist of at least k_0 consecutive zeros. Thus

$$d(\sigma^{n_k}(x), \sigma^{n_k}(y)) < \frac{1}{k_0} < t \quad \text{for each } k \geq 1.$$

Hence

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbf{1}_{[0,t)}(d(\sigma^k(x), \sigma^k(y))) \geq \frac{1}{p^{p^{k_0}+2k_0}} > 0,$$

which shows that (X, σ) is not strongly chaotic.

The above example combined with the fact that strong chaos does not imply positive topological entropy ([LF], see also Example 4.2 below) shows that these two notions are independent in dimension zero.

The aim of the next example (whose idea is based on [LF]) is to construct a system which is formally strongly chaotic, but whose dynamics is almost trivial; every orbit "bounces" back and forth between two fix-points or "sinks" in one of them. Not only does this system have zero topological entropy, but its complexity is easily seen to be linear.

EXAMPLE 4.2. Define $\varphi : \Sigma \to \Sigma$ by $\varphi(x) = B_1 B_2 \ldots$ for all $x \in \Sigma$, where

$$B_i = \begin{cases} \underbrace{00\ldots0}_{2^{2^i}\text{ times}} & \text{if } x(i) = 0, \\ \underbrace{11\ldots1}_{2^{2^i}\text{ times}} & \text{if } x(i) = 1, \end{cases} \qquad \text{for } i = 1, 2, \ldots.$$

By Lemma 3.4 there exists an uncountable set E such that for any distinct points $x = x(1)x(2)\ldots, y = y(1)y(2)\ldots \in E$, we have $x_n = y_n$ for infinitely many n and $x_m \neq y_m$ for infinitely many m. Since φ is injective, the set $\varphi(E)$ is uncountable. Define

$$X = \overline{\bigcup_{x\in\Sigma} O(\varphi(x))}.$$

We have $\varphi(E) \subset X$ and it is easy to see that every element of X is either eventually constant or it is built of constant blocks (alternately blocks of zeros and blocks of ones) whose lengths tend to infinity. Thus every ω-limit set consists of one or two fix-points and at most countably many points which become fixed after a finite number of iterations. Obviously the entropy of (X, σ) is zero. We will show that this is a strongly chaotic system. Indeed, it is enough to show that any two distinct points in $\varphi(E)$ satisfy the conditions from the definition of strong chaos.

Let $a = A_1 A_2 \ldots$, $b = B_1 B_2 \ldots$ be points in $\varphi(E)$, where A_i, B_i are blocks of consecutive zeros or ones of length 2^{2^i}. By the definition of $\varphi(E)$ there exist increasing sequences $\{p_i\}$ and $\{q_i\}$ of positive integers such that $A_{p_i} = B_{p_i}$, and A_{q_i} is the negation of B_{q_i} for all $i \in \mathbb{N}$.

Let $r_k = \sum_{j=1}^{k} 2^{2^j}$. Then the first r_{p_i-1} symbols of $\sigma^{m_1}(a)$ and $\sigma^{m_1}(b)$ are the same for all m_1 such that $r_{p_i-1} < m_1 \le r_{p_i} - r_{p_i-1}$ and the first r_{q_i-1} symbols of $\sigma^{m_2}(a)$ and $\sigma^{m_2}(b)$ are all distinct for all m_2 such that $r_{q_i-1} < m_2 \le r_{q_i} - r_{q_i-1}$ (there are such m_1, m_2 because $r_n < \frac{1}{2} r_{n+1}$ for $n \in \mathbb{N}$). Thus, for given $0 < t \le 1$, the distance between $\sigma^{m_1}(a)$ and $\sigma^{m_1}(b)$ is not greater than $1/r_{p_i-1} < t$ for $r_{p_i-1} < m_1 \le r_{p_i} - r_{p_i-1}$, provided p_i is large enough. Therefore the expression

$$\frac{1}{r_{p_i} - r_{p_i-1}} \sum_{m_1=1}^{r_{p_i}-r_{p_i-1}} \mathbf{1}_{[0,t)}(d(\sigma^{m_1}(a), \sigma^{m_1}(b)))$$

is bounded from below by

$$\frac{1}{r_{p_i}-r_{p_i-1}}\sum_{m_1=r_{p_i-1}+1}^{r_{p_i}-r_{p_i-1}}\mathbf{1}_{[0,t)}(d(\sigma^{m_1}(a),\sigma^{m_1}(b)))\geq\frac{r_{p_i}-2r_{p_i-1}}{r_{p_i}-r_{p_i-1}}\to 1,$$

which means that

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{[0,t)}(d(\sigma^j(a),\sigma^j(b)))=1,$$

so the first condition in the definition of strong chaos is satisfied.

Furthermore, for all m_2 such that $r_{q_i-1}<m_2\leq r_{q_i}-r_{q_i-1}$, the distance between $\sigma^{m_2}(a)$ and $\sigma^{m_2}(b)$ equals 1 and is greater than t. Therefore the expression

$$\frac{1}{r_{q_i}-r_{q_i-1}}\sum_{m_2=1}^{r_{q_i}-r_{q_i-1}}\mathbf{1}_{[0,t)}(d(\sigma^{m_2}(a),\sigma^{m_2}(b)))$$

is bounded from above by

$$\frac{1}{r_{q_i}-r_{q_i-1}}\sum_{m_2=1}^{r_{q_i-1}}\mathbf{1}_{[0,t)}(d(\sigma^{m_2}(a),\sigma^{m_2}(b)))\leq\frac{r_{q_i-1}}{r_{q_i}-r_{q_i-1}}\to 0,$$

which means that

$$\liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{[0,t)}(d(\sigma^j(a),\sigma^j(b)))=0,$$

so the second condition in the definition of strong chaos is satisfied.

The last example of this paper is the construction of an ω-chaotic system which is not chaotic in the sense of Li–Yorke. Below we use the notation introduced prior to Lemma 3.3.

Example 4.3. Let $J:=[r,s]$ where $0<\beta<r<s<1$. For $\alpha\in J$ let $b_\alpha\in M_{\beta,\beta}$ be an extension of α and let $a_\alpha\in M_{\alpha,\beta}$ be an extension of 0 (we emphasize that, in spite of similar notation, the points b_α and a_α come from different systems and extend different points of $\mathbb{T}$). Define

$$c_\alpha:=a_\alpha\diamond b_\alpha,\qquad W:=\overline{\bigcup_{\alpha\in J}O(c_\alpha)}.$$

Firstly, we show that (W,σ) is an ω-chaotic system. Let $\alpha_1,\alpha_2\in J$, $\alpha_1\neq\alpha_2$. Using Lemma 3.1 we obtain

$$|\omega(c_{\alpha_1})\setminus\omega(c_{\alpha_2})|\geq|M_{\alpha_1,\beta}|>\aleph_0,\qquad \omega(c_{\alpha_1})\cap\omega(c_{\alpha_2})\supset M_{\beta,\beta}\neq\emptyset.$$

Hence $\{c_\alpha:\alpha\in J\}$ is an uncountable ω-chaotic set.

Secondly, we show that (W,σ) is not chaotic in the sense of Li–Yorke. To do this we need to investigate the structure of Li–Yorke chaotic sets, i.e., the sets in which any two points are proximal and not asymptotic (see Def. 2.1).

Notice that if in the system (W,σ) there were an uncountable Li–Yorke chaotic set then for any covering of W by countably many sets, at least one of them would also contain an uncountable Li–Yorke chaotic set.

Let w belong to W. Then there exist sequences $n_k \geq 0$ and $\alpha_k \in J$ with

$$w = \lim_{k\to\infty} \sigma^{n_k}(c_{\alpha_k}).$$

Of course, we may assume that the sequence $\{\alpha_k\}$ converges in $\mathbb{T}$ to some α. Consider two major cases: n_k increases to infinity or it remains bounded.

In the first case, recall that $c_{\alpha_k} = a_{\alpha_k} \diamond b_{\alpha_k}$, so, as in the proof of Lemma 3.1, depending on whether the initial block of $\sigma^{n_k}(c_{\alpha_k})$ belongs to a_{α_k} or to b_{α_k} and whether its length is bounded or not, and using Lemma 3.3, we deduce that the point w may have one of the following two forms:

1) $w = Ba$, where B is some block occurring in $M_{\beta,\beta}$ (or the empty block) and $a \in \widetilde{M}_{\alpha,\beta}$.
2) $w = Ab$, where A is some block occurring in $M_{\alpha,\beta}$ (or the empty block) and $b \in M_{\beta,\beta}$.

Denote the collection of all $w \in W$ satisfying 1) by W_a and those satisfying 2) by W_b.

If n_k is bounded we may assume it is a constant integer $p \geq 0$. Then, by Lemma 3.3, $w = \sigma^p(a'_\alpha \diamond b'_\alpha)$, where a'_α is some extension of 0 in $\widetilde{M}_{\alpha,\beta}$ and b'_α is an extension of α in $M_{\beta,\beta}$. For given p denote the corresponding set of points w by W_p.

Consider the covering of W by countably many sets: W_a, W_b and W_p ($p \geq 0$). We conclude the proof by showing that none of them contains an uncountable Li–Yorke chaotic set. Take a pair of points w_i $(i = 1, 2)$ in W_a. Then $\sigma^n(w_i) = v_i \in \widetilde{M}_{\alpha,\beta}$ for some n. If v_i are extensions of different points in $\mathbb{T}$ then they are not proximal, because $\mathbb{T}$ contains no proximal pairs. On the other hand, any two extensions of the same point are asymptotic. So, W_a contains no Li–Yorke pairs at all. An identical argument applies to W_b. Finally, consider a pair $w_1 = \sigma^p(a'_{\alpha_1} \diamond b'_{\alpha_1})$ and $w_2 = \sigma^p(a''_{\alpha_2} \diamond b''_{\alpha_2})$ in W_p and suppose these points are proximal. Then either a'_{α_1} is proximal to a''_{α_2} or b'_{α_1} is proximal to b''_{α_2}. In the first case there is an n such that $\sigma^n(a'_{\alpha_1}) \in M_{\alpha_1,\beta}$ and $\sigma^n(a'_{\alpha_2}) \in M_{\alpha_2,\beta}$. Because these two sets are minimal, they are disjoint, hence proximality implies $\alpha_1 = \alpha_2$. In the other case, $\alpha_1 = \alpha_2$ follows from the fact that b'_{α_1} and b''_{α_2} are extensions in the same system $M_{\beta,\beta}$ of α_1 and α_2, respectively (recall that $\mathbb{T}$ has no proximal pairs). In either case $\alpha_1 = \alpha_2$. Thus there are only finitely many points proximal to w_1 (with at most four choices of a''_{α_1} and at most two choices of b''_{α_1}). Hence the only possible Li–Yorke chaotic sets in W_p are finite, which concludes the construction.

Notice, however, that the above construction does not imply that all ω-chaotic sets are finite. In fact, certain such sets are countably infinite.

REMARK 4.1. In [La2] the author considers among other things a relaxed version of ω-chaos and shows the existence of a zero-dimensional ω-chaotic system with a countably infinite ω-chaotic set for which any Li–Yorke chaotic set consists of only two points.

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