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TRISECTIONS OF MODULE CATEGORIES

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Abstract. Let A be a finite-dimensional algebra over a field k. We discuss the existence of *trisections* $(\operatorname{mod}_+ A, \operatorname{mod}_0 A, \operatorname{mod}_- A)$ of the category of finitely generated modules $\operatorname{mod} A$ satisfying exactness, standardness, separation and adjustment conditions. Many important classes of algebras admit trisections. We describe a construction of algebras admitting a trisection of their module categories and, in special cases, we describe the structure of the components of the Auslander–Reiten quiver lying in $\operatorname{mod}_0 A$.

Let A be a finite-dimensional algebra over a field k. One of the main problems of the representation theory of algebras is to describe the finite-dimensional indecomposable left A-modules. Since this is possible only for algebras of tame representation type, in the general situation other types of structural results are considered. In certain important cases the category $\operatorname{mod} A$ of finitely generated left modules admits partitions which determine the direction of morphisms between indecomposable modules. This is the case for tilted algebras, tubular algebras, concealed-canonical algebras and other important examples.

Let A be a finite-dimensional algebra. A *trisection* $(\operatorname{mod}_+ A, \operatorname{mod}_0 A, \operatorname{mod}_- A)$ of $\operatorname{mod} A$ satisfies the following conditions:

- (*Exactness*) Each of $\operatorname{mod}_+ A$, $\operatorname{mod}_0 A$, $\operatorname{mod}_- A$ is a full subcategory of $\operatorname{mod} A$ closed under direct sums, direct summands, extensions and isomorphisms.
- (*Standardness*) The indecomposable modules in $\operatorname{mod}_0 A$ yield a family of generalized standard (pairwise orthogonal) components of the Auslander–Reiten quiver Γ_A of A, that is, for $X, Y \in \operatorname{mod}_0 A$, we have $\bigcap_{n\geq 1} \operatorname{rad}_A^n(X,Y) =: \operatorname{rad}_A^\infty(X,Y) = 0$.
- (*Separation*) Every indecomposable A-module belongs to $\operatorname{mod}_+ A$, $\operatorname{mod}_0 A$ or $\operatorname{mod}_- A$. If $X \in \operatorname{mod}_+ A$ and $Y \in \operatorname{mod}_- A$ then

$$\operatorname{Hom}_A(\operatorname{mod}_0 A, X) = 0 = \operatorname{Hom}_A(Y, \operatorname{mod}_0 A).$$

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- (*Adjustment*) For every $X \in \operatorname{mod}_+ A$ and $Y \in \operatorname{mod}_- A$ we have
$$\operatorname{Hom}_A(X, \operatorname{mod}_0 A) \neq 0 \quad \text{and} \quad \operatorname{Hom}_A(\operatorname{mod}_0 A, Y) \neq 0.$$

Many important classes of algebras admit a trisection of their module categories, notably the tilted algebras and the algebras with a separating tubular family of stable tubes. Recall that an algebra A is *tilted* if $A = \operatorname{End}_H(T)$, where H is a hereditary algebra and T is a tilting H-module (that is, $\operatorname{Ext}^1_H(T,T) = 0$ and T is a direct sum of $n = \operatorname{rank}_{\mathbb{Z}} K_0(H)$ pairwise non-isomorphic indecomposable modules); in that case $\operatorname{mod}_0 A = \operatorname{add} \mathcal{C}$ is the additive closure of the modules lying in the connecting component $\mathcal{C}$ of the Auslander–Reiten quiver Γ_A (see [20]). On the other hand, the finite-dimensional algebras with a *separating tubular family of stable tubes* (in the sense of [12]) include the tame hereditary algebras [20], the tubular algebras [6, 20], the canonical algebras [11, 20] and more generally the concealed-canonical algebras [12, 13]; in that case, $\operatorname{mod}_0 A = \operatorname{add} \mathcal{T}_0$, where $\mathcal{T}_0$ is the separating tubular family.

Generalizing some of the above examples, the class of quasitilted algebras was introduced in [10]. An algebra A is *quasitilted* if $A = \operatorname{End}_{\mathcal{H}}(T)$ for a tilting object T in a hereditary abelian k-category $\mathcal{H}$. It was shown by Happel [7] (over algebraically closed fields and generalized in [8] to arbitrary base fields) that there are only two types of quasitilted algebras: the tilted ones and the *quasitilted algebras of canonical type*, that is, algebras $A = \operatorname{End}_{\mathcal{H}}(T)$ for a tilting module T over a hereditary category $\mathcal{H}$ whose derived category $D^b(\mathcal{H})$ is equivalent, as a triangulated category, to the derived category $D^b(\operatorname{mod} C)$ of modules over a canonical algebra C. It is known [10] that quasitilted algebras of canonical type admit a trisection of their module categories. In Section 3 we shall give a direct proof of the fact that a quasitilted algebra admitting a trisection of its module category is either a tilted algebra or a quasitilted algebra of canonical type.

Recently, it was shown [16] that the class of Artin algebras A whose Auslander–Reiten quiver admits a trisection whose middle part is a separating family of almost cyclic coherent components coincides with the class of generalized multicoil enlargements of concealed-canonical algebras. This result provides the description of many algebras admitting special classes of trisections.

The purpose of this work is to describe a construction of algebras A admitting trisections of their module categories and in special cases, describe also the structure of the components of the Auslander–Reiten quiver lying in $\operatorname{mod}_0 A$.

In Section 1 we study subcategories $\mathcal{H}_0 \subset \operatorname{ind} A$ called *hearts* of $\operatorname{mod} A$, which under certain conditions yield trisections of $\operatorname{mod} A$. We shall construct (and recall) many examples of algebras admitting (or not) trisections of their module categories, in particular we provide examples of algebras which are not quasitilted but admit trisections. In Section 2 we prove that for an algebra $A = kQ/I$ admitting a trisection $(\operatorname{mod}_+ A, \operatorname{mod}_0 A, \operatorname{mod}_- A)$ of its module category, any quotient $\overline{A} = A/Ae_xA$, where e_x is the idempotent corresponding to a sink x in Q such that the indecomposable injective I_x is in $\operatorname{mod}_0 A$, admits a trisection $(\operatorname{mod}_+ \overline{A}, \operatorname{mod}_0 \overline{A}, \operatorname{mod}_- \overline{A})$ such that $\operatorname{mod}_- A = \operatorname{mod}_- \overline{A}$. This observation is fundamental for the proof of the results concerning quasitilted algebras in Section 3.

Section 4 is devoted to the study of oriented cycles in the quiver Q of an algebra of the form $A = kQ/I$ admitting a trisection of $\operatorname{mod} A$. We show that the quotient $\overline{A}$ of A obtained by killing the convex closure of those vertices in Q lying on oriented cycles, still admits a trisection of $\operatorname{mod} \overline{A}$.

In case $A = kQ/I$ is *triangular*, that is, Q has no oriented cycles, we get the following description of the structure of A. For terminology see [13] or Section 3.

THEOREM A. *Let A be a triangular algebra admitting a trisection of* $\operatorname{mod} A$. *Then there are algebras $A_0, A_1, \dots, A_s = A$ with $A_{i+1} = A_i[M_i]$ for $i = 0, \dots, s-1$ such that the following holds*:

(a) *A_i admits a trisection of $\operatorname{mod} A_i$ such that $M_i \in \operatorname{mod}_0 A_i$, $i = 0, \dots, s-1$.*
(b) *A_0 decomposes as a product $B_1 \times \dots \times B_t$ of indecomposable algebras, where each B_i is either a tilted algebra or an almost concealed-canonical algebra.*

In Section 5, we shall give necessary and sufficient conditions for a one-point extension $A[M]$ of an algebra A admitting a trisection $(\operatorname{mod}_+ A, \operatorname{mod}_0 A, \operatorname{mod}_- A)$ with $M \in \operatorname{mod}_0 A$, to admit a trisection $(\operatorname{mod}_+ A[M], \operatorname{mod}_0 A[M], \operatorname{mod}_- A[M])$ with $\operatorname{mod}_+ A[M] = \operatorname{mod}_+ A$. Section 6 provides the proofs of some basic results stated in Section 1. As a consequence of the structure results established in Sections 4 and 5 we get the following fundamental property of algebras admitting trisections.

THEOREM B. *Let A be an algebra admitting a trisection of* $\operatorname{mod} A$. *Then $\operatorname{mod}_0 A$ weakly factorizes maps from $\operatorname{mod}_+ A$ to $\operatorname{mod}_- A$. That is, for any modules $X \in \operatorname{mod}_+ A$ and $Y \in \operatorname{mod}_- A$ and a morphism $f \in \operatorname{Hom}_A(X, Y)$, there exist a module $U \in \operatorname{mod}_0 A$ and morphisms $f' \in \operatorname{Hom}_A(X, U)$ and $f'' \in \operatorname{Hom}_A(U, Y)$ such that $f = f''f'$.*

Factorization properties for trisections have been considered for special classes of algebras (see for example [12, 18]). In Section 1.7 we provide examples to show that stronger factorization properties are not necessarily satisfied by trisections in general.

Concerning the structure of the Auslander–Reiten components corresponding to modules in $\mathrm{mod}_0 A$ we know from [23] that they admit only finitely many non-periodic orbits. A more precise description is given in Section 7 in case the algebra A is strongly simply connected. Following [21], for an algebraically closed field k, we say that $A = kQ/I$ is *strongly simply connected* if it is triangular and for every convex subcategory B of A, the first Hochschild cohomology $H^1(B)$ vanishes. Roughly speaking, a component of the Auslander–Reiten quiver Γ_A is said to be a *multicoil* if it consists of a finite number of coils glued together by some directed parts, a *coil* being a natural generalization of the notion of a tube (see [2, 24] for the precise definitions). The structure theorem in Section 7 is the following.

THEOREM C. *Let A be a strongly simply connected algebra admitting a trisection* $(\mathrm{mod}_+ A, \mathrm{mod}_0 A, \mathrm{mod}_- A)$. *Then every component of Γ_A lying in* $\mathrm{mod}_0 A$ *is a multicoil.*

Obviously, we shall say that $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ is a *trisection* of $\mathrm{ind}\, A$ if $(\mathrm{add}\, \mathcal{H}_+, \mathrm{add}\, \mathcal{H}_0, \mathrm{add}\, \mathcal{H}_-)$ is a trisection of $\mathrm{mod}\, A$. Sometimes we shall say that $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ is a trisection of A.

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1. Hearts of a module category and trisections

1.1. Let A be an Artin algebra and consider the category $\mathrm{mod}\, A$ of finitely generated left A-modules. It is useful to start by investigating a concept more general than trisections, namely hearts of $\mathrm{mod}\, A$, which provide trisections when additional requirements are satisfied.

A set of indecomposable A-modules $\mathcal{H}_0$ is called a *heart* of $\mathrm{mod}\, A$ if there exist sets of indecomposable A-modules $\mathcal{H}_+$ and $\mathcal{H}_-$ satisfying:

- (*Exactness*) For any $X, Y \in \mathcal{H}_+$ (resp. $\mathcal{H}_0, \mathcal{H}_-$) and Z an indecomposable direct summand of an extension of X by Y, we have $Z \in \mathcal{H}_+$ (resp. $\mathcal{H}_0, \mathcal{H}_-$).
- (*Standardness*) The indecomposable modules in $\mathcal{H}_0$ yield a family of pairwise orthogonal generalized standard components of the Auslander–Reiten quiver Γ_A.
- (*Separation*) Every indecomposable A-module belongs to $\mathcal{H}_+, \mathcal{H}_0$ or $\mathcal{H}_-$. If $X \in \mathcal{H}_+$ and $Y \in \mathcal{H}_-$, then $\mathrm{Hom}_A(\mathcal{H}_0, X) = 0 = \mathrm{Hom}_A(Y, \mathcal{H}_0)$.

Recall that, in case the ground field k is algebraically closed, a component $\mathcal{C}$ of Γ_A is *standard* if the mesh category $k(\mathcal{C})$, obtained from the path category $k\mathcal{C}$ by factoring out the sums $\sum_{i=1}^{s} \beta_i \alpha_i$ for every almost split sequence

$$0 \to \tau_A X \overset{(\alpha_i)}{\to} \bigoplus_{i=1}^{s} Y_i \overset{(\beta_i)}{\to} X \to 0$$

with $X \in \mathcal{C}$, is equivalent to the full subcategory of $\operatorname{mod} A$ induced by the modules in $\mathcal{C}$. A standard component $\mathcal{C}$ is *generalized standard*, that is, $\operatorname{rad}^\infty(X,Y) = 0$ for every pair $X, Y \in \mathcal{C}$ (see [15]). Observe that if a family $(\mathcal{C}_i)_i$ of components of Γ_A is generalized standard, then $\operatorname{Hom}_A(\mathcal{C}_i, \mathcal{C}_j) = 0$ for $i \neq j$.

We say that a heart $\mathcal{H}_0$ of $\operatorname{mod} A$ is *sincere* if there exists a sincere module in $\operatorname{add}\mathcal{H}_0$. It is *adjusted* if given $\mathcal{H}_+$ and $\mathcal{H}_-$ as in the definition, for every $X \in \mathcal{H}_+$ and $Y \in \mathcal{H}_-$ we have $\operatorname{Hom}_A(X, \mathcal{H}_0) \neq 0 \neq \operatorname{Hom}_A(\mathcal{H}_0, Y)$.

Lemma. *Let $\mathcal{H}_0$ be a heart of* $\operatorname{mod} A$ *with $\mathcal{H}_+$ and $\mathcal{H}_-$ as in the definition. Then the following statements hold*:

(a) *If $\mathcal{H}_0$ is sincere, then every indecomposable projective (resp. injective) module belongs to $\mathcal{H}_+ \cup \mathcal{H}_0$ (resp. $\mathcal{H}_0 \cup \mathcal{H}_-$).*
(b) *If $\mathcal{H}_0$ is adjusted, then $\mathcal{H}_0$ is sincere.*

Proof. (a) Follows from the separation property.

(b) Consider a simple module S and assume $S \notin \mathcal{H}_0$. Consider the case $S \in \mathcal{H}_-$. As $\mathcal{H}_0$ is adjusted, there is a module $X \in \mathcal{H}_0$ with $\operatorname{Hom}_A(X, S) \neq 0$, hence X has S as a composition factor. The case $S \in \mathcal{H}_+$ is treated dually. ∎

1.2. Lemma. *Let $\mathcal{H}_0$ be a heart of* $\operatorname{mod} A$ *and suppose that $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ satisfies the conditions of the definition. If $\mathcal{H}_0$ is adjusted, then $\mathcal{H}_+$ and $\mathcal{H}_-$ are uniquely determined.*

Proof. Indeed, $\mathcal{H}_+ = \{X \in \Gamma_A \colon \operatorname{Hom}_A(\mathcal{H}_0, X) = 0\}$ and $\mathcal{H}_- = \{Y \in \Gamma_A \colon \operatorname{Hom}_A(Y, \mathcal{H}_0) = 0\}$. ∎

In case $\mathcal{H}_0$ is an adjusted heart, $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ is called a *trisection* of the category $\operatorname{ind} A$ of finitely generated indecomposable A-modules. In this case $\operatorname{mod}_+ A = \operatorname{add}\mathcal{H}_+$, $\operatorname{mod}_0 A = \operatorname{add}\mathcal{H}_0$, $\operatorname{mod}_- A = \operatorname{add}\mathcal{H}_-$ yield a trisection of the module category $\operatorname{mod} A$.

1.3. We present some general properties of trisections essentially proved in [12]. We recall that given a path of non-zero maps between indecomposable A-modules of the form $X = X_0 \to X_1 \to \cdots \to X_s = Y$, the module X is a *predecessor* of Y (and Y a *successor* of X).

Proposition. *Let $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ be a trisection of* $\operatorname{ind} A$. *Then*:

(a) *The categories* $\operatorname{add}\mathcal{H}_+$, $\operatorname{add}\mathcal{H}_0$ *and* $\operatorname{add}\mathcal{H}_-$ *are closed under extensions*; *$\mathcal{H}_+$ is closed under predecessors and $\mathcal{H}_-$ under successors.*
(b) *Each module in $\mathcal{H}_+$ embeds into a module in $\mathcal{H}_0$.*
(c) *If $X \in \mathcal{H}_+$ and $Y \in \mathcal{H}_-$, then* $\operatorname{Hom}_A(Y, X) = 0$.

(d) *Each module* $X \in \mathcal{H}_+$ *has projective dimension* $\operatorname{pd}_A X \le 1$. *Dually, every* $Y \in \mathcal{H}_-$ *has injective dimension* $\operatorname{id}_A Y \le 1$.
(e) *The modules in* $\mathcal{H}_+$ (*resp.* $\mathcal{H}_-$) *form complete components of* Γ_A.
(f) *The components* $\mathcal{C}$ *in* $\mathcal{H}_0$ *are convex* (*that is*, $X \to X_1 \to \cdots \to X_n \to Y$ *being a chain of non-zero maps between indecomposable modules with* $X, Y \in \mathcal{C}$ *implies* $X_i \in \mathcal{C}$ *for all* i).

Proof. (a) is clear.

(b) Let $0 \neq X \in \mathcal{H}_+$ and choose $0 \neq f\colon X \to U$ with $U \in \mathcal{H}_0$. Let $I = \operatorname{Im} f$ and consider the exact sequence

$$0 \to K \to X \xrightarrow{f} I \to 0.$$

Then $K \in \operatorname{add}\mathcal{H}_+$. Factorize $I = I_+ \oplus I_0$ with $I_+ \in \operatorname{add}\mathcal{H}_+$ and $I_0 \in \operatorname{add}\mathcal{H}_0$. If f is not mono, then $\dim_k I_+ < \dim_k X$. By induction hypothesis, we may suppose $f'\colon I_+ \to U_0$ is mono for some $U_0 \in \operatorname{add}\mathcal{H}_0$ and similarly, $f_0\colon K \to U_0'$ is mono for some $U_0' \in \operatorname{add}\mathcal{H}_0$. Form the commutative and exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d, "f_0"'] & X \arrow[r, "f"] \arrow[d] & I \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U_0' \arrow[r] & Y \arrow[r] & I_+ \oplus I_0 \arrow[r] & 0
\end{tikzcd}$$

As $U_0', I_0 \in \operatorname{add}\mathcal{H}_0$ and $I_+ \in \operatorname{add}\mathcal{H}_+$ by (a) we see that $Y \in \operatorname{add}(\mathcal{H}_0 \cup \mathcal{H}_+)$. Since $\operatorname{Ext}^1_A(I_+, U_0') = 0$, we have $Y \cong I_+ \oplus U_1$ for some $U_1 \in \operatorname{add}\mathcal{H}_0$. Therefore, $X \hookrightarrow Y \hookrightarrow U_0' \oplus U_1$ is the desired inclusion.

(c) Let $X \in \mathcal{H}_+$ and $Y \in \mathcal{H}_-$. Consider $X \hookrightarrow X_0$ with $X_0 \in \operatorname{add}\mathcal{H}_0$, which is possible by (b). Then $0 \to \operatorname{Hom}_A(Y, X) \to \operatorname{Hom}_A(Y, X_0)$ is exact and we know that $\operatorname{Hom}_A(Y, X_0) = 0$.

(d) Let X be an indecomposable module in $\mathcal{H}_+$. Since the Auslander–Reiten translation $\tau_A X$ has no injective predecessor by 1.1 and (a), it follows that $\operatorname{pd}_A X \le 1$ by [20, p. 74].

(e) Assume $X \to Y$ is an irreducible map in Γ_A with $X \in \mathcal{H}_+$. By standardness $Y \notin \mathcal{H}_0$. Assume $Y \in \mathcal{H}_-$ to get a contradiction.

Since Y is not projective, we consider the almost split sequence

$$0 \to X_1 \xrightarrow{\alpha} X \oplus \bigoplus_{i=1}^{s} Z_i \xrightarrow{\beta} Y \to 0.$$

Then $X_1 \in \mathcal{H}_+$. Moreover by adjustment, there is a non-zero map $f\colon Z \to Y$

with $Z \in \mathcal{H}_0$, hence f factorizes through β. Then there is some $j \in \{1, \dots, s\}$ with a non-zero map $Z \to Z_j$. It follows that $Z_j \in \mathcal{H}_-$, because $\mathcal{H}_0$ is standard and there is an irreducible map $X_1 \to Z_j$ with $X_1 \in \mathcal{H}_+$. Set $Y_1 = Z_j$. We have found an irreducible map $X_1 \to Y_1$. In a similar way we construct a family of irreducible maps $X_i \to Y_i$ with $X_i = \tau_A Y_{i-1} \in \mathcal{H}_+$ and $Y_i \in \mathcal{H}_-$ for $i \in \mathbb{N}$. The module $T = \bigoplus_{i \in \mathbb{N}} Y_i$ is a partial tilting module because $\mathrm{Ext}^1_A(Y_i, Y_j) \cong D\overline{\mathrm{Hom}}_A(Y_j, X_i) = 0$ (see [20]). Moreover, the modules $(Y_i)_i$ are pairwise non-isomorphic since they lie on a sectional path. Indeed, there is an infinite path $\cdots \to Y_n \xrightarrow{\beta_n} Y_{n-1} \to \cdots \to Y_1 \xrightarrow{\beta_1} Y$ of irreducible maps which is sectional (since $Y_{n+1} \cong \tau_A Y_{n-1}$ implies that $X_{n-1} \in \mathcal{H}_0$ by (a), which is a contradiction). By [4], sectional paths have no cycles. Therefore T should have at most $\operatorname{rank} K_0(A)$ indecomposable pairwise non-isomorphic direct summands, a contradiction.

(f) Follows from standardness and (a). ∎

1.4. Examples.

(a) *Representation-finite algebras.* In this case $\mathcal{H}_0 = \operatorname{ind} A$ is an adjusted heart of $\operatorname{mod} A$. Observe that when k is algebraically closed, the components in $\mathcal{H}_0$ are not necessarily standard.

(b) *Tilted algebras.* Let H be a hereditary algebra and ${}_HT$ a tilting module. Consider the tilted algebra $A = \mathrm{End}_A(T)$. There are several possibilities:

• A is representation-finite.

• A is representation-infinite and T is a sum of preprojective modules. Then A is a *concealed algebra* and Γ_A has two components $\mathcal{P}$ and $\mathcal{J}$ with complete slices. In this case both $\mathcal{H}_0 = \mathcal{P}$ and $\mathcal{H}_0 = \mathcal{J}$ are adjusted hearts of $\operatorname{mod} A$. Moreover, if H is tame, then the regular modules of A form a generalized standard family $\mathcal{T}$ of stable tubes. Also $\mathcal{H}_0 = \mathcal{T}$ is an adjusted heart of $\operatorname{mod} A$. Similarly when T is a sum of preinjective modules.

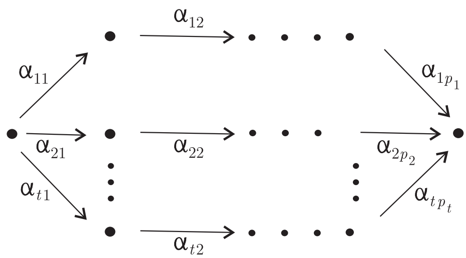
• A is representation-infinite and T has regular summands. Consider the preprojective component $\mathcal{P}$ and the preinjective component $\mathcal{J}$ of Γ_A.

In case H is tame, either $\mathcal{P}$ or $\mathcal{J}$ has a complete slice (but not both). Suppose $\mathcal{P}$ has a complete slice. Then both $\mathcal{H}_0 = \mathcal{P}$ and $\mathcal{H}_0 = \operatorname{ind} A \setminus (\mathcal{P} \cup \mathcal{J})$ are adjusted hearts of $\operatorname{mod} A$. In this case, A is a *domestic tubular algebra* (see [20]).

In case H is wild, let $\mathcal{C}$ be the *connecting component* of Γ_A (that is, the unique component having a complete slice); then $\mathcal{H}_0 = \mathcal{C}$ is the unique choice for an adjusted heart of $\operatorname{mod} A$.

For the next examples (c) to (h) we *assume* that k is algebraically closed.

(c) *Canonical algebras.* Let $C = (C(\underline{p}, \underline{\lambda})$ be the canonical algebra associated to the data $\underline{p} = (p_1, \dots, p_t) \in \mathbb{N}^n$, $\underline{\lambda} = (\lambda_3, \dots, \lambda_t)$, where the λ_i's are pairwise different elements in k^* and $C = kQ/I$, where Q is the quiver



and I is generated by $\alpha_{1p_1}\ldots\alpha_{11}-\lambda_i\alpha_{2p_2}\ldots\alpha_{21}-\alpha_{ip_i}\ldots\alpha_{i1}$ for $i=3,\ldots,t$.

It was shown in [20] that there is a standard family $\mathcal{T}=(T_\lambda)_{\lambda\in\mathbb{P}_1k}$ of stable tubes such that $\mathcal{H}_0=\mathcal{T}$ is an adjusted heart of $\operatorname{mod}A$.

If C is a tubular algebra (that is, $\underline{p}=(2,2,2,2),(3,3,3),(2,4,4),(2,3,6)$), then there are standard families $\mathcal{T}^\gamma=(T^\gamma_\lambda)_{\lambda\in\mathbb{P}_1k}$ of stable tubes in Γ_A ($\gamma\in\mathbb{Q}^+\cup\{0,\infty\}$) such that $\mathcal{H}^\gamma_0=\mathcal{T}^\gamma$ is an adjusted heart of $\operatorname{mod}A$ (see [20]).

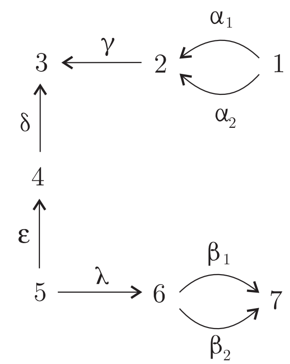
(d) *Quasitilted algebras of canonical type*. We recall that an algebra A is quasitilted if $\operatorname{gl\,dim}A\le 2$ and for every indecomposable A-module X either $\operatorname{pd}_A X\le 1$ or $\operatorname{id}_A X\le 1$ (see [10]). Quasitilted algebras of the form $A=\operatorname{End}_C(T)$, where C is a canonical algebra with a trisection $(\mathcal{H}_+,\mathcal{H}_0,\mathcal{H}_-)$ and T is a tilting module in $\operatorname{add}\mathcal{H}_+$, are said to be *concealed-canonical*. In case A is not a tubular algebra, $\operatorname{mod}A$ has a unique adjusted heart. A general quasitilted algebra of canonical type may be obtained as a semiregular branch enlargement of a concealed-canonical algebra (see [13]).

(e) Consider the path algebra B associated to the quiver Q below. The

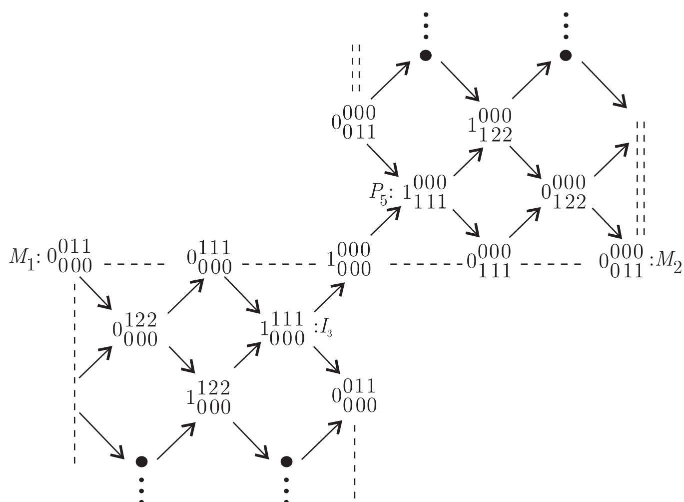
$$Q\colon\ \begin{tikzcd}[row sep=small] & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] & & 4 \\ & 3 \arrow[ur] & \end{tikzcd} \qquad M\colon\ \begin{tikzcd}[row sep=small] & k \arrow[dr, "0"] & \\ k \arrow[ur, "1"] \arrow[dr, "1"'] & & k \\ & k \arrow[ur, "1"'] & \end{tikzcd}$$

quiver of Γ_B contains a standard family $\mathcal{T}=(T_\lambda)_{\lambda\in\mathbb{P}_1k}$ of stable tubes which are all homogeneous exception two tubes T_1 and T_2 of rank 2. Let M be an indecomposable module in T_1 with regular length 2. Consider the one-point extension $A=B[M]$ $\left(=\begin{pmatrix} B & {}_BM_k \\ 0 & k\end{pmatrix}\right.$ with the usual matrix operations, see [20]). Then $\mathcal{T}'=(T_\lambda)_{\lambda\ne 1}$ is a family of components of Γ_A. We find that $\mathcal{H}_0=\mathcal{T}'$ is a heart of $\operatorname{mod}A$ with $\mathcal{H}_+=\mathcal{P}$ where $\mathcal{P}$ is the preprojective component of Γ_B (and also of Γ_A) and $\mathcal{H}_-=\Gamma_A\setminus(\mathcal{P}\cup\mathcal{T}')$. Since M is not simple regular, the component $\mathcal{C}'$ of Γ_A where M lies is not standard (see for example [18]). It follows that $\operatorname{ind}A$ does not have an adjusted heart.

(f) Consider the algebra A given as $A=kQ/I$ where Q is the quiver

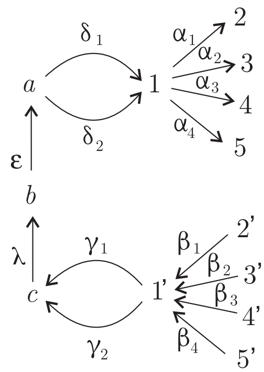


and I is generated by $\gamma\alpha_1$, $\delta\varepsilon$, and $\beta_1\lambda$. As shown in [2], A is a *multicoil* algebra, that is, every component of Γ_A is a multicoil. The components of Γ_A may be described as follows: let $\mathcal{P}$ be the preprojective component of the algebra $A(1,2)$ associated with the full subcategory of A with vertices $1,2$; let $(T^1_\lambda)_\lambda$ be the standard family of tubes of $A(1,2)$, where $M_1\colon\ (k2 \underset{1}{\overset{0}{\rightleftarrows}} k1) \in T^1_1$ (where ki is the vector space k “sitting” at the vertex i); let $\mathcal{J}'$ be the preinjective component of $A(1,2,3,4)$; let $\mathcal{P}'$ be the preprojective component of $A(4,5,6,7)$; let $(T^2_\lambda)_\lambda$ be the standard family of tubes of $A(6,7)$, where $M_2\colon\ (k6 \underset{1}{\overset{0}{\rightleftarrows}} k7) \in T^2_1$; let $\mathcal{J}$ be the preinjective component of $A(6,7)$. Then $\mathcal{P}, \mathcal{P}', (T^1_\lambda)_{\lambda\neq 1}, (T^2_\lambda)_{\lambda\neq 1}, \mathcal{J}, \mathcal{J}'$ are the components of Γ_A with exception of the following multicoil $\mathcal{C}$:



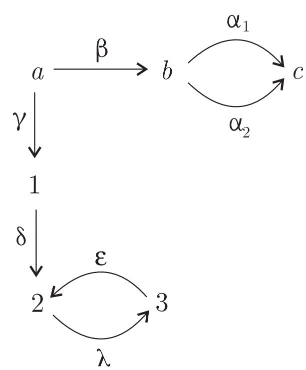
where dotted lines should be identified and indecomposable modules are represented by their dimension vectors. Then $\mathcal{H}_0 = \operatorname{add}((T^1_\lambda)_{\lambda\neq 1} \cup (T^2_\lambda)_{\lambda\neq 1} \cup \mathcal{C})$ is an adjusted heart of $\operatorname{mod} A$.

(g) The algebra $A = kQ/I$ where Q is the quiver



and I is generated by $\alpha_i\delta_1 - \alpha_i\delta_2$, $\gamma_1\beta_i - \gamma_2\beta_i$, $i = 1, 2, 3, 4$, $\delta_1\varepsilon - \delta_2\varepsilon$, $\lambda\gamma_1 - \lambda\gamma_2$ and $\varepsilon\lambda$. The algebra $A(a, b, 1, 2, 3, 4, 5)$ is a tubular extension of the hereditary tame algebra $B = A(1, 2, 3, 4, 5)$ of tubular type $(2, 2, 2, 3)$ and hence is wild. Thus A is wild. Let $(T^1_\lambda)_\lambda$ be the tubular family of Γ_B and $M = \operatorname{rad} P_a \in T^1_1$; similarly, let $(T^2_\lambda)_\lambda$ be the tubular family of $\Gamma_{B'}$ where $B' = A(1', 2', 3', 4', 5')$ and $M' = I_b/\operatorname{soc} I_b \in T^2_1$. Then T^1_λ, T^2_λ, $\lambda \neq 1$, are components of Γ_A and there is a multicoil $\mathcal{C}$ of Γ_A containing both M and M'. The category $\mathcal{H}_0$ given by the modules in $(T^1_\lambda)_{\lambda\neq 1} \cup (T^2_\lambda)_{\lambda\neq 1} \cup \mathcal{C}$ is an adjusted heart of $\operatorname{mod} A$.

(h) Let A be the algebra given by kQ/I where Q is the quiver



with I generated by $\alpha_1\beta$, $\delta\gamma$, $\lambda\delta$, $\varepsilon\lambda$ and $\lambda\varepsilon$. Then Γ_A consists of all components of $\Gamma_{A'}$ where $A' = A(a, b, c, 1)$ except the component T where the projective P_a lies which is substituted by a new component T' obtained by identifying T and $\Gamma_{A''}$, where $A'' = A(1, 2, 3)$, along the simple module S_1.

Then $\operatorname{mod} A$ admits an adjusted heart which contains a cycle of maps between indecomposable projective modules.

1.5. Observe that there are algebras A admitting two trisections $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ and $(\mathcal{H}'_+, \mathcal{H}'_0, \mathcal{H}'_-)$ with $\mathcal{H}_0 \neq \mathcal{H}'_0$. Indeed, the following cases are well known (see [20]):

(a) A is a representation-infinite concealed algebra with preprojective component $\mathcal{P}$ and preinjective component $\mathcal{J}$. We may choose $\mathcal{H}_0$ to be $\mathcal{P}$ or $\mathcal{J}$.
(b) A is domestic tubular with a tubular family $\mathcal{T}$ and a preinjective component $\mathcal{J}$ containing all injectives. Then $\mathcal{H}_0 = \mathcal{T}$ or $\mathcal{H}_0 = \mathcal{J}$.
(c) A is a tubular algebra with separating tubular families $(\mathcal{T}^{\varrho})$, $\varrho \in \{0\} \cup \mathbb{Q}^+ \cup \{\infty\}$. Then we may choose $\mathcal{H} = \mathcal{T}^{\varrho}$ for any ϱ.

Theorem. *Let A be an algebra admitting two trisections $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ and $(\mathcal{H}'_+, \mathcal{H}'_0, \mathcal{H}'_-)$ with $\mathcal{H}_0 \neq \mathcal{H}'_0$. Then there is a quotient $\overline{A}$ of A satisfying the following conditions*:

(a) *there are only finitely many indecomposable A-modules* (*up to isomorphism*) *which are not $\overline{A}$-modules*;
(b) *$\overline{A} = A_1 \times \cdots \times A_s$ is a product of indecomposable algebras, at least one of which is either a concealed algebra, a domestic tubular algebra or a tubular algebra.*

The proof is based upon Theorem A, whose proof will only be given in Section 3, and on the reduction process given in Section 4. We postpone the proof until Section 6.

1.6. We say that a triple $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ is a *weak trisection* of $\operatorname{ind} A$ if $\mathcal{H}_0$ is a heart of $\operatorname{mod} A$ and $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ are sets of indecomposable A-modules satisfying the exactness, standardness and separation conditions and moreover all indecomposable projective (resp. injective) modules lie in $\mathcal{H}_+ \cup \mathcal{H}_0$ (resp. $\mathcal{H}_0 \cup \mathcal{H}_-$).

We introduce this definition to simplify some statements, since in Section 6 we shall show the following result.

Theorem. *Any weak trisection of* $\operatorname{ind} A$ *is a trisection.*

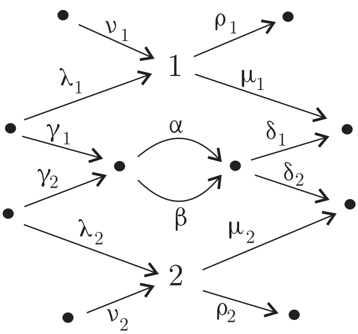
1.7. Let $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ be a trisection of $\operatorname{ind} A$. Consider the following properties of the trisection:

- (*Strong factorization property*) for all $X \in \mathcal{H}_+$, $Y \in \mathcal{H}_-$ and $f \in \operatorname{Hom}_A(X, Y)$ and for every component $\mathcal{T}$ of $\mathcal{H}_0$, there exists $U \in \mathcal{T}$ and maps $f' \in \operatorname{Hom}_A(X, U)$ and $f'' \in \operatorname{Hom}_A(U, Y)$ such that $f = f''f'$.
- (*Weak factorization property*) for all $X \in \mathcal{H}_+$, $Y \in \mathcal{H}_-$ and $f \in \operatorname{Hom}_A(X, Y)$ there exists a module $U \in \operatorname{add} \mathcal{H}_0$ and maps $f' \in \operatorname{Hom}_A(X, U)$ and $f'' \in \operatorname{Hom}_A(U, Y)$ such that $f = f''f'$.

In [12] it is shown that the concealed-canonical algebras have the strong factorization property. On the other hand, Theorem B shows that every trisection has the weak factorization property. We now give an example showing that trisections do not always have the strong factorization property.

Example. Consider the algebra $A = kQ/I$, over an algebraically closed field k, given by the quiver

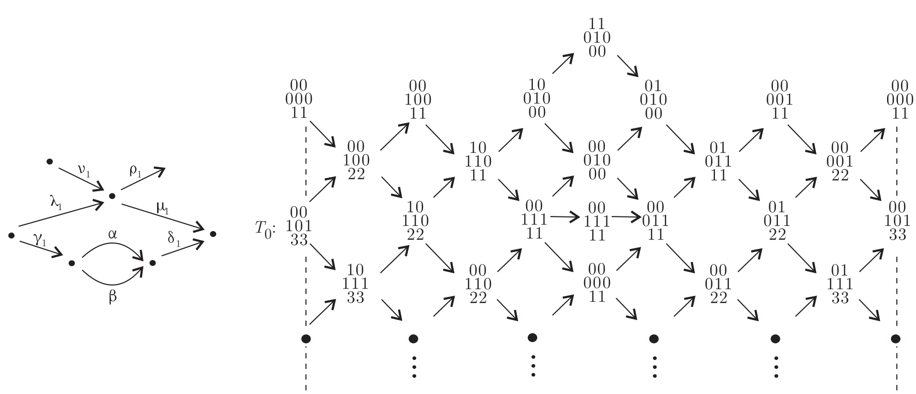
Q:

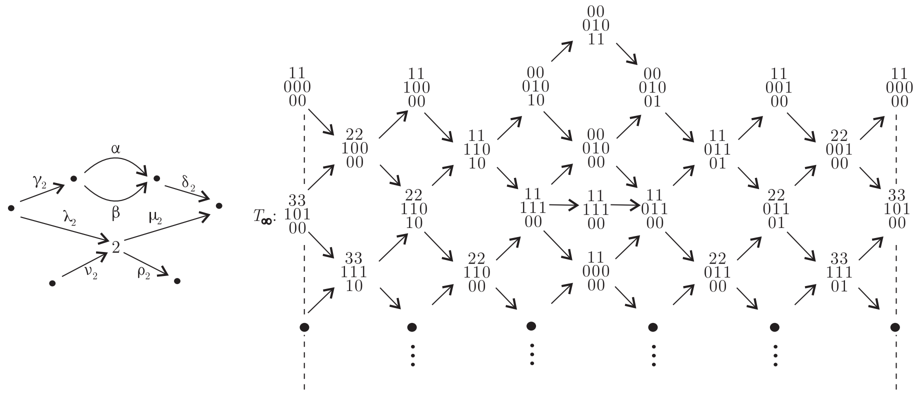


bound by the relations $\mu_i\nu_i = 0$, $\varrho_i\lambda_i = 0$ $(i = 1, 2)$, $\alpha\gamma_1 = 0 = \delta_1\alpha$, $\delta_1\beta\gamma_1 = \mu_1\lambda_1$, $\beta\gamma_2 = 0 = \delta_2\beta$ and $\delta_2\alpha\gamma_2 = \mu_2\lambda_2$. Then Γ_A is formed by a unique preprojective component $\mathcal{P}$, a unique preinjective component $\mathcal{J}$, the homogeneous tubes $(T_s)_s$ having in their mouths the Kronecker modules

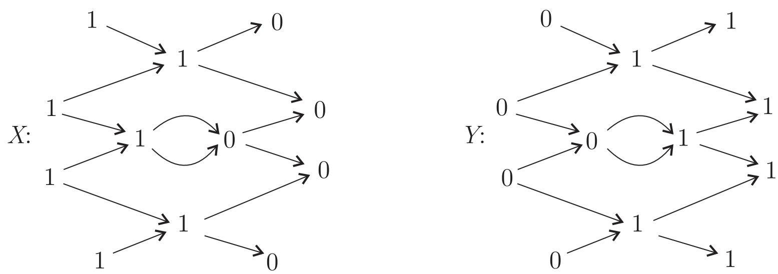
$$M_s\colon\ k \underset{s}{\overset{1}{\rightrightarrows}} k, \qquad s \in k \setminus \{0\},$$

and two additional quasi-tubes T_0 and T_∞ (see [2] for definitions) whose modules are given by dimension vectors as follows:





The family $\mathcal{T} = (T_s)_{s\in\mathbb{P}_1(k)}$ spans a full subcategory $\operatorname{mod}_0 A$ of $\operatorname{mod} A$ such that every map from $\mathcal{P}$ to $\mathcal{J}$ factorizes through $\operatorname{mod}_0 A$, as is not hard to check from the construction. On the other hand, the map $f \in \operatorname{Hom}_A(X, Y)$ between $X \in \mathcal{P}$ and $Y \in \mathcal{J}$ with dimension vectors



and $\operatorname{Im} f = S_1 \oplus S_2$ factorizes through $\operatorname{add}(T_0 \vee T_\infty)$ but not through any tube T_s, $s \in \mathbb{P}_1(k)$.

2. Going down with trisections

2.1. Let $A = kQ/I$ be a finite-dimensional algebra such that $\operatorname{ind} A$ admits a trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$. We shall examine the set Q_0^0 of vertices $x \in Q_0$ such that the corresponding indecomposable projective A-module P_x lies in $\mathcal{H}_0$.

LEMMA. *Q_0^0 is closed under predecessors in Q.*

Proof. Let $x \in Q_0^0$ and $y \to x$ be an arrow in Q. Then there is a non-zero map $P_x \to P_y$ in $\operatorname{mod} A$. Since $P_x \in \mathcal{H}_0$, we have $P_y \in \mathcal{H}_0 \cup \mathcal{H}_-$ but since $\mathcal{H}_-$ does not contain projective modules, we conclude that $y \in Q_0^0$. ∎

2.2. Assume $P_x \in \mathcal{H}_0$. First we consider the case when x is a source in Q and we form the quotient algebra $A^{(x)} = A/Ae_xA$.

THEOREM. *Let $A = kQ/I$ be an algebra with a trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ of* $\operatorname{ind} A$. *Let x be a source of Q such that $P_x \in \mathcal{H}_0$. Then $A^{(x)}$ admits a trisection $(\mathcal{H}'_+, \mathcal{H}'_0, \mathcal{H}'_-)$ with $\mathcal{H}'_+ = \mathcal{H}_+$.*

Proof. First observe that any module $X \in \mathcal{H}_+$ is an $A^{(x)}$-module. Then define $\mathcal{H}'_+ = \mathcal{H}_+$.

Define $\mathcal{H}'_0$ as the set of indecomposable $A^{(x)}$-modules $\{\tau^{-i}_{A^{(x)}} X\colon\ i \geq 0$ and X is an indecomposable $A^{(x)}$-module in $\mathcal{H}_0\}$ and $\mathcal{H}'_- = \Gamma_{A^{(x)}} \setminus (\mathcal{H}'_+ \cup \mathcal{H}'_0)$. Observe that $\mathcal{H}'_- \subset \mathcal{H}_-$.

We claim that $\mathcal{H}'_0$ is an adjusted heart of $\operatorname{mod} A^{(x)}$. We divide the proof into several steps.

(1) By definition, $\mathcal{H}'_+$ is closed under predecessors. Moreover, $\mathcal{H}'_-$ is closed under successors. Indeed, assume that Y is a successor of $X \in \mathcal{H}'_-$. If $Y \notin \mathcal{H}'_-$, then $Y \in \mathcal{H}_+ \cup \mathcal{H}_0$ and also $X \in (\mathcal{H}_+ \cup \mathcal{H}_0) \cap \mathcal{H}_- = \emptyset$.

(2) We check that $\mathcal{H}'_0$ is formed by complete components of $\Gamma_{A^{(x)}}$. Let $X \in \mathcal{H}_0$ be an indecomposable $A^{(x)}$-module and let $i \geq 0$. Let Y be a neighbour of $Z = \tau^{-i}_{A^{(x)}} X$ in $\Gamma_{A^{(x)}}$.

Assume $Y \to Z$. If some $P = \tau^{j}_{A^{(x)}} Y$ is projective for $0 \leq j < i$, then X is a predecessor of P and hence $P \in \mathcal{H}_0$. Since $\tau^{-j}_{A^{(x)}} P = Y$, it follows that $Y \in \mathcal{H}'_0$. If no $\tau^{j}_{A^{(x)}} Y$ is projective for $0 \leq j \leq i$, then $\tau^{i}_{A^{(x)}} Y \to X$ is an arrow in $\Gamma_{A^{(x)}}$. Therefore $\tau^{i}_{A^{(x)}} Y$ is a predecessor of $X \in \mathcal{H}_0$ but $\tau^{i}_{A^{(x)}} Y$ is not in $\mathcal{H}_+ = \mathcal{H}'_+$, since otherwise $X \in \mathcal{H}_+$. Hence $\tau^{i}_{A^{(x)}} Y \in \mathcal{H}_0$ and $Y = \tau^{-i}_{A^{(x)}}(\tau^{i}_{A^{(x)}} Y) \in \mathcal{H}'_0$.

Assume $Z \to Y$. If Y is projective, then, since Y is a successor of X, we get $Y \in \mathcal{H}_0$. Hence $Y \in \mathcal{H}'_0$. Otherwise, there is an arrow $\tau_{A^{(x)}} Y \to Z$ in $\Gamma_{A^{(x)}}$ and the case above implies that $\tau_{A^{(x)}} Y \in \mathcal{H}_0$, that is, $\tau_{A^{(x)}} Y = \tau^{-i}_{A^{(x)}} N$ for some indecomposable $N \in \mathcal{H}_0$. Therefore, $Y = \tau^{-i-1}_{A^{(x)}} N \in \mathcal{H}'_0$.

(2′) We shall prove that $\mathcal{H}'_0$ is formed by a generalized standard family of components. For this purpose, consider indecomposable $A^{(x)}$-modules $X_1, X_2 \in \mathcal{H}_0$ and $Y_1 = \tau^{-a}_{A^{(x)}} X_1$, $Y_2 = \tau^{-b}_{A^{(x)}} X_2$ for some $a, b \geq 0$. We shall prove that $\operatorname{rad}^{\infty}_{A^{(x)}}(Y_1, Y_2) = 0$ by induction on $a + b$.

Assume $\operatorname{rad}^{\infty}_{A^{(x)}}(Y_1, Y_2) \neq 0$. Since $\underline{\operatorname{Hom}}_{A^{(x)}}(Y_1, Y_2) \cong \overline{\operatorname{Hom}}_{A^{(x)}}(\tau_{A^{(x)}} Y_1, \tau_{A^{(x)}} Y_2)$ and by induction hypothesis $\operatorname{rad}^{\infty}_{A^{(x)}}(\tau_{A^{(x)}} Y_1, \tau_{A^{(x)}} Y_2) = 0$, we get an indecomposable projective $A^{(x)}$-module P_y (which is therefore projective as an A-module) with $0 \neq f = gh \in \operatorname{rad}^{\infty}_{A^{(x)}}(Y_1, Y_2)$ and $h \in \operatorname{rad}_{A^{(x)}}(Y_1, P_y)$, $g \in \operatorname{rad}_{A^{(x)}}(P_y, Y_2)$. Since $P_y \in \mathcal{H}_+ \cup \mathcal{H}_0$, both Y_1 and P_y belong to $\mathcal{H}_0$. Standardness implies that $Y_2 \in \mathcal{H}_-$ and $g \in \operatorname{rad}^{\infty}_{A}(P_y, Y_2)$.

Consider the left almost split map $P_y \xrightarrow{r} E$ in $\operatorname{mod} A$. Then there is an indecomposable direct summand E_1 of E and $g_1 \in \operatorname{rad}^{\infty}_{A}(E_1, Y_2)$ with $P_y \xrightarrow{r_1} E_1 \xrightarrow{g_1} Y_2$ non-zero. Observe that $E_1 \in \mathcal{H}_0$ and $E_1 \in \operatorname{mod} A^{(x)}$. If E_1 is non-projective, then $\underline{\operatorname{Hom}}_{A^{(x)}}(E_1, Y_2) \cong \overline{\operatorname{Hom}}_{A^{(x)}}(\tau_{A^{(x)}} E_1, \tau_{A^{(x)}} Y_2)$ and by induction hypothesis $\operatorname{rad}^{\infty}_{A^{(x)}}(\tau_{A^{(x)}} E_1, \tau_{A^{(x)}} Y_2) = 0$. Hence there exists a projective $A^{(x)}$-module $P_{y'} \in \mathcal{H}_0$ and maps $h' \in \operatorname{rad}_{A^{(x)}}(E_1, P_{y'})$,

$g' \in \operatorname{rad}^{\infty}_{A^{(x)}}(P_{y'}, Y_2)$ with the composition $P_y \xrightarrow{r_1} E_1 \xrightarrow{h'} P_{y'} \xrightarrow{g'} Y_2$ non-zero. Iteration of this process yields $\operatorname{rad}^{\infty}_A(P_s, P_t) \neq 0$ for some projective modules $P_s, P_t \in \mathcal{H}_0$. This contradiction proves the claim.

(3) By definition every indecomposable $A^{(x)}$-module lies in $\mathcal{H}'_t \cup \mathcal{H}'_0 \cup \mathcal{H}'_-$. Moreover, $\operatorname{Hom}_{A^{(x)}}(\mathcal{H}'_0, \mathcal{H}'_+) = 0$ since $\mathcal{H}'_+ = \mathcal{H}_+$ and $\mathcal{H}'_0 \subset \operatorname{add}(\mathcal{H}_0 \cup \mathcal{H}_-)$. We show that $\operatorname{Hom}_{A^{(x)}}(Y, \tau^{-i}_{A^{(x)}} X) = 0$ for $X \in \mathcal{H}_0 \cap \operatorname{mod} A^{(x)}$, $i \geq 0$ and $Y \in \mathcal{H}'_-$. Assume otherwise, that is, $\operatorname{Hom}_{A^{(x)}}(Y, \tau^{-i}_{A^{(x)}} X) \neq 0$, implying that $\tau^{-i}_{A^{(x)}} X \in \mathcal{H}_-$. Since there are no projective modules in $\mathcal{H}_-$, we get

$$0 \neq \operatorname{Hom}_{A^{(x)}}(Y, \tau^{-i}_{A^{(x)}} X) \cong \overline{\operatorname{Hom}}_{A^{(x)}}(\tau_{A^{(x)}} Y, \tau^{-i+1}_{A^{(x)}} X)$$

and therefore $\operatorname{Hom}_{A^{(x)}}(\tau_{A^{(x)}} Y, \tau^{-i+1}_{A^{(x)}} X) \neq 0$. Repeating the procedure we infer that $\operatorname{Hom}_{A^{(x)}}(\tau^{i}_{A^{(x)}} Y, X) \neq 0$, where $\tau^{i}_{A^{(x)}} Y \in \mathcal{H}_-$ and $X \in \mathcal{H}_0$, a contradiction. Hence $\operatorname{Hom}_{A^{(x)}}(Y, \tau^{-i}_{A^{(x)}} X) = 0$ as desired.

(4) We prove the adjustment condition. Let $X \in \mathcal{H}'_+ = \mathcal{H}_+$ and $Y \in \mathcal{H}'_-$ be indecomposable $A^{(x)}$-modules. There are modules $Z, Z' \in \mathcal{H}_0$ such that $\operatorname{Hom}_A(X, Z) \neq 0 \neq \operatorname{Hom}_A(Z', Y)$.

Consider first the decomposition of the restriction of Z to $A^{(x)}$ into indecomposable modules, $\bigoplus_{i=1}^{s} Z_i$, such that $\operatorname{Hom}_{A^{(x)}}(R, Z_i) \neq 0$ for all $i = 1, \ldots, s$, where $R = \operatorname{rad} P_x$. Since each Z_i is a submodule of Z and both $R, Z \in \mathcal{H}_0$, we have $Z_i \in \mathcal{H}'_0$ $(i = 1, \ldots, s)$. Moreover, $\operatorname{Hom}_A(X, Z) \neq 0$ implies that $\operatorname{Hom}_{A^{(x)}}(X, Z_j) \neq 0$ for some j. Similarly we get an indecomposable direct summand Z'_j of the restriction of Z' to $A^{(x)}$ such that $Z'_j \in \mathcal{H}'_0$ and $\operatorname{Hom}_{A^{(x)}}(Z'_j, Y) \neq 0$. ∎

2.3. Let $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ be a trisection of $\operatorname{ind} A$. Let P^+ be the direct sum of all indecomposable projective A-modules $P_x \in \mathcal{H}_+$ and define $A^+ = \operatorname{End}_A(P^+)^{\mathrm{op}}$. Similarly, define $A^- = \operatorname{End}_A(I^-)^{\mathrm{op}}$, where I^- is the direct sum of all indecomposable projective A-modules P_x such that the corresponding injective I_x is in $\mathcal{H}_-$. We call A^+ (resp. A^-) the *left* (resp. *right*) *extremal quotient* of A.

For the following, recall that a component $\mathcal{C}$ of Γ_A is *semiregular* if either $\tau^n_A X$ or $\tau^{-n}_A X$ is well defined for every $X \in \mathcal{C}$ and $n \geq 0$.

Theorem.

(a) *The module category* $\operatorname{mod} A^+$ *admits a trisection* $(\mathcal{H}^+_+, \mathcal{H}^+_0, \mathcal{H}^+_-)$ *with* $\mathcal{H}^+_+ = \mathcal{H}_+$ *and such that the indecomposable modules in* $\mathcal{H}^+_0$ *form semiregular generalized standard components of the Auslander–Reiten quiver* Γ_{A^+} *without projective modules.*

(b) *The module category* $\operatorname{mod} A^-$ *admits a trisection* $(\mathcal{H}^-_+, \mathcal{H}^-_0, \mathcal{H}^-_-)$ *with* $\mathcal{H}^-_- = \mathcal{H}_-$ *and such that the indecomposable modules in* $\mathcal{H}^-_0$ *form* *semiregular generalized standard components of Γ_{A^-} without injective modules.*

(c) *The algebras A^+ and A^- are quasitilted.*

Proof. (a) Let $A = kQ/I$ and observe by 2.1 that A^+ is *convex* (= path closed) in A. The proof follows verbatim 2.2.

(b) Dual to (a).

(c) Observe that $X \in \mathcal{H}_+^+$ has $\mathrm{pd}_{A^+} X \le 1$ and $Y \in \mathcal{H}_0^+ \cup \mathcal{H}_-^+$ has $\mathrm{id}_{A^+} Y \le 1$. Moreover, every indecomposable direct summand R of $\mathrm{rad}\, P_x$ in $\mathrm{mod}\, A^+$ lies in $\mathcal{H}_+^+$, hence $\mathrm{pd}_{A^+} R \le 1$ and $\mathrm{gl\,dim}\, A^+ \le 2$. Therefore A^+ is quasitilted. For A^- the proof is dual. ∎

Corollary. *Assume moreover that $A = kQ/I$ is a triangular algebra. Then*:

(a) *The algebra A may be reconstructed from A^+ by a sequence of one-point extensions $A_0 = A^+$, $A_1 = A_0[M_0], \ldots, A_{n+1} = A_n[M_n]$ and $A_{n+1} = A$ in such a way that for each $i = 0, 1, \ldots, n+1$, $\mathrm{ind}\, A_i$ admits a trisection $(\mathcal{H}_+^i, \mathcal{H}_0^i, \mathcal{H}_-^i)$ with $\mathcal{H}_+^i = \mathcal{H}_+$, $M_i \in \mathrm{add}\, \mathcal{H}_0^i$ and $\mathcal{H}_0^{n+1} = \mathcal{H}_0$, $\mathcal{H}_-^{n+1} = \mathcal{H}_-$.*

(b) *The algebra A may be reconstructed from A^- by a sequence of one-point coextensions $A_0 = A^-$, $A_1 = [N_1]A_0, \ldots, A_{n+1} = [N_n]A_n = A$, in such a way that for each $i = 0, 1, \ldots, n+1$, $\mathrm{ind}\, A_i$ admits a trisection $(\mathcal{H}_+^i, \mathcal{H}_0^i, \mathcal{H}_-^i)$ with $\mathcal{H}_-^i = \mathcal{H}_-$, $N_i \in \mathrm{add}\, \mathcal{H}_0^i = \mathcal{H}_0$ and $\mathcal{H}_+^{n+1} = \mathcal{H}_+$.* ∎

2.4. Let $A = kQ/I$ be an algebra. We consider the process of "killing" projectives in $\mathcal{H}_0$ when $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ is a weak trisection of $\mathrm{ind}\, A$.

Proposition. *Let $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ be a weak trisection of* $\mathrm{ind}\, A$. *Then*:

(a) *Let x be a source of Q. Then $A^{(x)}$ admits a weak trisection $(\mathcal{H}'_+, \mathcal{H}'_0, \mathcal{H}'_-)$ of* $\mathrm{ind}\, A^{(x)}$ *with $\mathcal{H}'_+ = \mathcal{H}_+$ and* $\mathrm{rad}\, P_x \in \mathrm{add}\, \mathcal{H}'_0$.

(b) *Let $A^+ = \mathrm{End}_A(P^+)^{\mathrm{op}}$ where P^+ is the direct sum of all indecomposable projectives $P_x \in \mathcal{H}_+$. Then A^+ admits a weak trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ of* $\mathrm{ind}\, A^+$ *with $\mathcal{H}_+^+ = \mathcal{H}_+$ and $\mathcal{H}_0^+$ without projective modules. In particular, A^+ is a quasitilted algebra.*

Proof. The proof given in 2.2 may be repeated verbatim. Additionally, we have to check that all indecomposable projective (resp. injective) $A^{(x)}$-modules lie in $\mathcal{H}'_+ \cup \mathcal{H}'_0$ (resp. $\mathcal{H}'_0 \cup \mathcal{H}'_-$). Since $\mathcal{H}'_+ = \mathcal{H}_+$ and $\mathcal{H}_0 \cap \mathrm{ind}\, A^{(x)} \subset \mathcal{H}'_0$, this is clear. ∎

3. Directing components in the heart of a module category

3.1. Recall that an indecomposable module X in $\mathrm{mod}\, A$ is said to be *directing* if there are no cycles $X = X_1 \to X_2 \to \cdots \to X_n = X$ of non-zero non-isomorphisms between indecomposable A-modules. A component $\mathcal{C}$ of Γ_A is *directing* if all modules $X \in \mathcal{C}$ are directing.

The following result is fundamental for the understanding of the structure of trisections in module categories. It follows from Theorem 3.2 in [23], but we present a different proof below.

PROPOSITION. *Let A be an indecomposable triangular algebra admitting a trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ of* $\operatorname{ind} A$. *Suppose that $\mathcal{H}_0$ contains a semiregular directing component $\mathcal{C}$. Then A is a tilted algebra and $\mathcal{C}$ is a connecting component.*

Proof. Assume $A = kQ/I$, where the quiver Q has no oriented cycles. Since $\mathcal{C}$ is semiregular, we may assume that $\mathcal{C}$ has no injective modules. Let B be the full subcategory of A formed by the vertices x in Q such that $\operatorname{Hom}_A(P_x, M) \neq 0$ for some $M \in \mathcal{C}$.

We first show that B is convex in A: assume there is a path

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} x_2 \to \cdots \to x_{s-1} \xrightarrow{\alpha_{s-1}} x_x = y$$

in Q such that $\operatorname{Hom}_A(P_x, M) \neq 0$ and $\operatorname{Hom}_A(P_y, N) \neq 0$ for $M, N \in \mathcal{C}$ and that $x_1, \ldots, x_{s-1}$ are not in B. Consider the algebra $C = A/J$, where J is the ideal generated by the paths $x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\gamma} t$ and $t' \xrightarrow{\delta} x_{s-1} \xrightarrow{\alpha_{s-1}} y$ for any vertices t, t' in Q. Denote by I_0 the indecomposable injective C-module associated to the vertex y and by P_0 the indecomposable projective C-module associated to x. Then clearly, $S_{x_{s-1}}$ is a summand of $I_0/\operatorname{soc} I_0$ and S_{x_1} is a summand of $\operatorname{rad} P_0$. Since moreover $M, N \in \operatorname{mod} C$ and $\operatorname{Hom}_C(P_0, M) \neq 0$ and $\operatorname{Hom}_C(N, I_0) \neq 0$, we obtain a chain of non-zero maps

$$N \to I_0 \to S_{x_{s-1}} \to \begin{pmatrix} S_{x_{s-2}} \\ S_{x_{s-1}} \end{pmatrix} \to S_{x_{s-2}} \to \cdots \to \begin{pmatrix} S_{x_1} \\ S_{x_2} \end{pmatrix} \to S_{x_1} \to P_0 \to M,$$

where $\left(\begin{smallmatrix} S_a \\ S_b \end{smallmatrix}\right)$ denotes the indecomposable two-dimensional module with socle S_b and top S_a. Since $N, M \in \mathcal{H}_0$, all the modules of the chain lie in $\mathcal{H}_0$. Since $N \in \mathcal{C}$ and $I_0 \notin \mathcal{C}$, we have $\operatorname{rad}_A^{\infty}(N, I_0) \neq 0$, contradicting $\mathcal{H}_0$ having only generalized standard components (cf. [5]).

In this way we get an algebra B admitting a semiregular directing and sincere component $\mathcal{C}$ in Γ_B, hence B is a tilted algebra [20]. Therefore we get a trisection $(\mathcal{H}_+^B, \mathcal{H}_0^B, \mathcal{H}_-^B)$ of $\operatorname{ind} B$ with $\mathcal{H}_0^B = \mathcal{C}$.

We now show that $A = B$. Assume to the contrary that $B \neq A$. By duality we may suppose that there is an arrow $x_0 \to y$ in Q with $x_0 \notin B$ and $y \in B$; we may moreover assume that x_0 has no successors in the path order of Q with the same property. Let C be the full subcategory of A formed by B and x_0, that is, $C = B[M]$ is a one-point extension with $M = \operatorname{rad} P_{x_0}^C$ and $P_{x_0}^C$ the projective C-module associated with the vertex x_0. Write $M = \bigoplus_{i=1}^s M_i$ for the indecomposable decomposition of M.

We want to show that each M_i is in $\mathcal{H}_-^B$. Otherwise let $M_i \in \mathcal{H}_+^B \cup \mathcal{C}$ and let $X \in \mathcal{C} = \mathcal{H}_0^B$ be such that $\mathrm{Hom}_B(M_i, X) \neq 0$. Since $P_{x_0} \notin \mathcal{C}$, we get $M_i \in \mathcal{H}_+^B$. Since X is not injective, we consider the almost split sequence in $\mathrm{mod}\, B$

$$0 \to X \to E \to \tau_B^- X = Y \to 0.$$

Hence in $\mathrm{mod}\, C$, $\tau_C Y = (\mathrm{Hom}_B(M, X), X, \mathrm{id}) \ncong X$, which is impossible since in the component $\mathcal{C}$ of Γ_A, we have $\tau_A Y = X$. Therefore all M_i are in $\mathcal{H}_-^B$.

Observe that since $\mathcal{H}_0^B \subset \mathcal{H}_0$, we have $\mathcal{H}_-^B \subset \mathcal{H}_-$. Since $P_{x_0} \notin \mathcal{H}_-$ and $P_{x_0}^C$ lies in the same component where the $M_i \in \mathcal{H}_-$ lie, this implies that no M_i is a direct summand of $\mathrm{rad}\, P_{x_0}$, a contradiction proving that $A = B$. ∎

3.2. We get a better insight into the above result in the context of quasitilted algebras. Indeed, let $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ be a trisection of $\mathrm{ind}\, A$ with a directing semiregular component $\mathcal{C}$ in $\mathcal{H}_0$. Suppose $\mathcal{C}$ has no injective modules. Consider the algebra A^- as in 2.3. Then A^- is quasitilted and $\mathcal{C}$ is a component Γ_{A^-}. By the classification theorem for quasitilted algebras [16], A^- is a tilted algebra and $\mathcal{C}$ is the connecting component, that is, $\mathcal{H}_0^- = \mathcal{C}$. Coinserting back the injectives which were killed when forming A^- gives rise to A. But these injectives should lie in $\mathcal{H}_0^-$ and hence in $\mathcal{C}$, which did not have injective modules. Therefore $A^- = A$ is a tilted algebra.

Due to the importance of the classification of quasitilted algebras given in [7], we think it is convenient to present an alternative proof of 3.1 which is closer in spirit to the homological definition of quasitilted algebras. Independent proofs of this statement have already appeared in print (Corollary B in [25]; see also [1, 19] for related results).

Recall that for a quasitilted algebra A, the class $\mathcal{L}_A$ is formed by those indecomposable A-modules X such that any predecessor $Y \rightsquigarrow X$ has $\mathrm{pd}_A Y \leq 1$; dually, the class $\mathcal{R}_A$ is formed by those indecomposable A-modules Y such that any successor $Y \rightsquigarrow X$ has $\mathrm{id}_A X \leq 1$. By [10], $\mathrm{ind}\, A = \mathcal{L}_A \cup \mathcal{R}_A$.

Proposition. *Let A be a quasitilted algebra with a directing module $X \in \mathcal{R}_A \cap \mathcal{L}_A$. Then A is tilted.*

Proof. Let $\mathcal{T} = \mathrm{add}\, \mathcal{R}_A$ and $\mathcal{F} = \mathrm{add}(\mathcal{L}_A \setminus \mathcal{R}_A)$ be classes of A-modules. As shown in [10, II.2.3] the pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathrm{mod}\, A$. Consider the construction $\phi(\mathrm{mod}\, A; (\mathcal{T}, \mathcal{F}))$ as defined in [10, I.2]. We find that $\mathcal{H}$ is a full subcategory of the *derived category* $D^b(\mathrm{mod}\, A)$ whose objects $Z_\bullet$ have homology $Z_{-1} \in \mathcal{Y} = \mathrm{add}(\mathcal{L}_A \setminus \mathcal{R}_A)[1]$ and $Z_0 \in \mathrm{add}\, \mathcal{R}_A$. Then $\phi(\mathrm{mod}\, A; (\mathcal{T}, \mathcal{F})) = (\mathcal{H}; (\mathcal{X}, \mathcal{Y}))$ and $\mathcal{H}$ is a hereditary category derived equivalent to $\mathrm{mod}\, A$.

We shall prove that the object $X \in \mathcal{L}_A \cap \mathcal{R}_A \subset \mathcal{X}$ is still a directing object in $\mathcal{H}$. Indeed, letting $0 \neq f\colon Z_\bullet \to X'$ be a map in $\mathcal{H}$ with $X' \in \mathcal{L}_A \cap \mathcal{R}_A$, we show that $Z_\bullet \in \mathcal{L}_A \cap \mathcal{R}_A$. This shows that X stays directing in $\mathcal{H}$.

Suppose that $Z_\bullet$ is equivalent to the projective complex $P_\bullet\colon \cdots \to P_2 \to P_1 \xrightarrow{g} P_0 \to 0$ with homologies $Z_{-1} \to 0$ and $Z_0 \xrightarrow{0\neq f} X'_0 = X' \in \mathcal{L}_A \cap \mathcal{R}_A$. Hence $Z_0 \in \mathcal{L}_A$ and $\mathrm{pd}_A Z_0 \le 1$. The exactness of $P_1 \xrightarrow{g} P_0 \to Z_0 \to 0$ implies that g is mono and $Z_{-1} = 0$. Thus $Z_\bullet \in \mathcal{R}_A \cap \mathcal{L}_A$.

Observe that $\mathrm{Hom}_k(A,k)$ is a tilting object in $\mathcal{H}$. Then $\mathcal{H}$ is a hereditary category with a tilting object and a directing object. By [9], $\mathcal{H}$ is derived equivalent to $\mathrm{mod}\, H$ for a hereditary algebra H. There is a directing component $\mathcal{C}$ in $\mathcal{H}$ where X lies. Therefore A is a tilted algebra. ∎

3.3. Recall from [13] that an algebra $A = \mathrm{End}_C(T)$, where T is a tilting module over a canonical algebra C, is called *almost concealed-canonical* if T is the sum of indecomposable modules of non-negative rank.

Corollary. *Let A be a triangular algebra admitting a trisection. Let A^+ and A^- be the extremal quotients of A as defined in* 2.3. *Let $A^+ = A_1^+ \times \cdots \times A_s^+$ and $A^- = A_1^- \times \cdots \times A_t^-$ be indecomposable decompositions of algebras. Then for each $1 \le i \le s$ and $1 \le j \le t$, the factors A_i^+ and A_j^- are either tilted or almost concealed-canonical algebras.*

Proof. Follows directly from [16] or by 3.1 combined with the characterization of quasitilted algebras of canonical type in [13]. ∎

Theorem A stated in the introduction follows from the Corollary above and 2.3.

4. Cycles of projectives

4.1. We shall look more attentively at the cycles in Q for algebras $A = kQ/I$ admitting a trisection of $\mathrm{mod}\, A$.

Proposition. *Let $A = kQ/I$ be an algebra admitting a trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ of* $\mathrm{ind}\, A$. *Let x be a vertex on an oriented cycle of Q. Then*:

(a) *There is a component $\mathcal{C}$ of $\mathcal{H}_0$ such that for every vertex $y \in Q_0$ lying on a cycle containing both x and y, the modules P_y and I_y are in $\mathcal{C}$.*

(b) *There are only finitely many indecomposable modules Y with $Y(x) \neq 0$. They all belong to $\mathcal{C}$.*

Proof. (a) Let $x = x_0 \to x_1 \to \cdots \to x_s = x$ be a cycle in Q. If $I_{x_i} \notin \mathcal{H}_0$, then $I_{x_i} \in \mathcal{H}_-$ and then $I_{x_j} \in \mathcal{H}_-$ for all $1 \le j \le s$. Hence $A^- = kQ^-/I^-$ where Q^- contains the above cycle, which contradicts the fact that A^- is quasitilted. Hence all I_{x_i} are in $\mathcal{H}_0$.

Since $\mathcal{H}_0$ is generalized standard, all I_{x_i} belong to the same component $\mathcal{C}$ in $\mathcal{H}_0$. Similarly all P_{x_i} belong to $\mathcal{C}$ (since P_x, $I_x \in \mathcal{C}$ and $\mathrm{Hom}_A(P_x, I_x) \neq 0$).

(b) By standardness, there is $m \in \mathbb{N}$ with $\mathrm{rad}_A^m(P_x, I_y) = 0$. Let Y be an indecomposable A-module with $Y(x) \neq 0$. Then there are $g \in \mathrm{Hom}_A(P_x, Y)$ and $h \in \mathrm{Hom}_A(Y, I_x)$ with $hg \neq 0$. Therefore, there exists a path $P_x = Y_0 \to Y_1 \to \cdots \to Y_s = Y \to Y_{s+1} \to \cdots \to Y_t = I_x$ of irreducible maps between indecomposable A-modules, of length $t \leq m+1$. Clearly, there are only finitely many modules Y (up to isomorphism) that may appear in such a path. All those modules belong to $\mathcal{C}$. ∎

4.2. Let $A = kQ/I$ admit a trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ of $\mathrm{ind}\, A$. Let Q^{c} be the convex closure in Q of points lying on cycles in Q. Consider the quotient $\overline{A} = A/(x\colon\ x \in Q^{\mathrm{c}})$.

Proposition. *The algebra $\overline{A}$ admits a trisection $(\overline{\mathcal{H}}_+, \overline{\mathcal{H}}_0, \overline{\mathcal{H}}_-)$ of* $\mathrm{ind}\, \overline{A}$ *such that $\overline{\mathcal{H}}_+ = \mathcal{H}_+$, $\overline{\mathcal{H}}_- = \mathcal{H}_-$ and there are only finitely many indecomposable A-modules $Y \in \mathcal{H}_0$ such that $Y \notin \overline{\mathcal{H}}_0$.*

Proof. Let $Q^{\mathrm{c}} = Q^1 \amalg \cdots \amalg Q^s$ be written as the union of connected quivers. By 4.1, it is clear that there is a component $\mathcal{C}^i$ of $\mathcal{H}_0$ such that any indecomposable A-module Y with $Y(x) \neq 0$ for some $x \in Q^i$ lies in $\mathcal{C}^i$. Moreover, there are only finitely many such modules. Hence for all $Y \in \mathcal{H}_+ \cup \mathcal{H}_-$, $Y(x) = 0$ for any $x \in Q^{\mathrm{c}}$.

Set $\overline{\mathcal{H}}_+ = \mathcal{H}_+$, $\overline{\mathcal{H}}_- = \mathcal{H}_-$ and $\overline{\mathcal{H}}_0 = \{Y \in \mathcal{H}_0\colon\ Y(x) = 0 \text{ for all } x \in Q^{\mathrm{c}}\}$. Clearly $(\overline{\mathcal{H}}_+, \overline{\mathcal{H}}_0, \overline{\mathcal{H}}_-)$ is a trisection of $\mathrm{mod}\, \overline{A}$. ∎

5. Going up with trisections

5.1. As seen in Section 2 (and using 4.2), the problem of finding trisections can be reduced to the question of deciding when a one-point extension $B[M]$ of an algebra B admitting a trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ and with $M \in \mathrm{add}\, \mathcal{H}_0$ has again a trisection. We shall prove the following characterization for extensions of (weak) trisections.

Theorem. *Let B be an algebra admitting a weak trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ of* $\mathrm{ind}\, B$. *Let $M \in \mathrm{add}\, \mathcal{H}_0$ and let $\mathcal{C}$ be the finite union of components in $\mathcal{H}_0$ containing indecomposable summands of M. Let $A = B[M]$ and denote by $\mathcal{C}'$ the component of Γ_A containing a new projective P_ω (and all indecomposable direct summands of M). Then* $\mathrm{ind}\, A$ *has a weak trisection $(\widehat{\mathcal{H}}_+, \widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_-)$ with $P_\omega \in \widehat{\mathcal{H}}_0$ if and only if the following conditions hold*:

(a) *$\mathcal{C}'$ is generalized standard.*
(b) *If $P_x \in \mathcal{C}$, then $P_x \in \mathcal{C}'$ and for every arrow $X \to Y$ in $\mathcal{C}$ with $Y \in \mathcal{C}'$, also $X \in \mathcal{C}'$.*

We give the proof in 5.4 after some technical preparation.

5.2. LEMMA. *Let $A = B[M]$, $\mathcal{C}$, $\mathcal{C}'$ and $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ be as above. Let X be an indecomposable B-module with $X \in \mathcal{C}$ but $X \notin \mathcal{C}'$. Then X is in the same component in $\varGamma_A$ as a successor Y of M, that is, there is a chain $M' = Y_0 \to Y_1 \to \cdots \to Y_m = Y$ of non-zero maps between indecomposable B-modules, where M' is an indecomposable direct summand of M and X, Y lie in the same component of $\varGamma_A$.*

Proof. Take a walk $M' \overset{\alpha_1}{—} X_1 \overset{\alpha_2}{—} X_2 \overset{\alpha_n}{—} X_n = X$ of irreducible maps in $\mathcal{C}$, where M' is an indecomposable direct summand of M. If $\operatorname{Hom}_B(M, X_i) = 0$ for all $i = 1, \ldots, n$, then α_i remains irreducible in $\operatorname{mod} A$, which would imply $X \in \mathcal{C}'$.

Choose $j \geq 1$ maximal with $\operatorname{Hom}_B(M, X_j) \neq 0$. Hence $X_{j+1} — X_{j+2} — \cdots — X_{n-1} — X_n = X$ is a walk of irreducible maps in $\varGamma_A$, that is, X_{j+1} and X lie in the same component of $\varGamma_A$. If $\alpha_j \colon X_{j+1} \to X_j$, then by [20, 2.2.5], also X_j is in the same component as X in $\varGamma_A$ and $\operatorname{Hom}_A(M, X_j) \neq 0$. If $\alpha_j \colon X_j \to X_{j+1}$, we have a path of non-zero maps $M \to X_j \to X_{j+1}$. Hence the claim follows. ∎

5.3. LEMMA. *Keep the notations as above. Let*

$$X = \Big(V, \bigoplus_{i=1}^{s} Y_i,\ f \colon\ V \to \operatorname{Hom}_B\Big(M, \bigoplus_{i=1}^{s} Y_i\Big)\Big)$$

be an indecomposable $A(= B[M])$-module lying in $\mathcal{C}'$ and suppose all Y_i are indecomposable B-modules. Then $Y_i \in \mathcal{C}$ for $1 \leq i \leq s$.

Proof. Let

$$M' = X_0 — X_1 — \cdots — X_{t-1} \overset{\alpha}{—} X_t = X$$

be a walk of irreducible maps in $\mathcal{C}'$, where M' is an indecomposable direct summand of M and t is smallest possible. Let $X' = X_{t-1} = (V', \bigoplus_{i=1}^{r} Y_i', f')$ with all Y_i' indecomposable B-modules. By the minimality of t, $Y_i' \in \mathcal{C}$ for $1 \leq i \leq r$.

If $X \xrightarrow{\alpha} X'$ is irreducible, then the restriction $Y_i \xrightarrow{g} \bigoplus_{i=1}^{r} Y_i'$ is either irreducible or split mono. The latter implies $Y_i \simeq Y_j' \in \mathcal{C}$ for some j. Otherwise, g is irreducible in $\operatorname{mod} B$, which implies again $Y_i \in \mathcal{C}$. The case $X' \xrightarrow{\alpha} X$ is similar. ∎

5.4. *Proof of Theorem 5.1.* Assume first that $\operatorname{ind} A$ has a weak trisection $(\widehat{\mathcal{H}}_+, \widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_-)$ such that $M \in \operatorname{add} \widehat{\mathcal{H}}_0$. It follows that $\mathcal{C}'$ is in $\widehat{\mathcal{H}}_0$ and therefore $\mathcal{C}'$ is generalized standard.

Let $P_x \in \mathcal{C}$. If $P_x \notin \mathcal{C}'$, then by 5.2, P_x lies in the same component in $\varGamma_A$ as an indecomposable $X \in \operatorname{mod} A$ such that there is a chain of non-zero maps

$$M' \to X_1 \to X_2 \to \cdots \to X_n = X$$

between indecomposable B-modules and M' an indecomposable direct summand of M. Since $(\widehat{\mathcal{H}}_+, \widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_-)$ is a weak trisection, we have $P_x \in \widehat{\mathcal{H}}_+ \cup \widehat{\mathcal{H}}_0$. Since $M' \in \widehat{\mathcal{H}}_0$, it follows that $X \in \widehat{\mathcal{H}}_0 \cup \widehat{\mathcal{H}}_-$. Therefore $X, P_x \in \widehat{\mathcal{H}}_0$ and $\widehat{\mathcal{H}}_0$ being generalized standard, we infer that $X_i \in \mathcal{C}'$ for $i = 1, \ldots, n$. Thus $P_x \in \mathcal{C}'$, which is a contradiction.

Let $\alpha\colon X \to Y$ be an irreducible map in $\mathcal{C}$ with $Y \in \mathcal{C}'$. Assume $X \notin \mathcal{C}'$. Then α is not irreducible in $\operatorname{mod} A$ and necessarily $\operatorname{Hom}_B(M, X) \neq 0$. We get a path $M \to X \to Y$ with M and Y in $\mathcal{C}'$ which is generalized standard. This yields again $P_x \in \mathcal{C}'$, a contradiction. Hence conditions (a) and (b) hold.

For the converse, assume that (a) and (b) are satisfied. We want to define a weak trisection as follows:

$$\widehat{\mathcal{H}}_+ = \mathcal{H}_+, \quad \widehat{\mathcal{H}}_0 = (\mathcal{H}_0 \setminus \mathcal{C}) \cup \mathcal{C}', \quad \widehat{\mathcal{H}}_- = \operatorname{ind} A \setminus (\widehat{\mathcal{H}}_+ \cup \widehat{\mathcal{H}}_0).$$

It is clear that $\widehat{\mathcal{H}}_0$ is formed by components of Γ_A, the new component $\mathcal{C}'$ is generalized standard and by 5.3, $\mathcal{H}_0 \setminus \mathcal{C}$ and $\mathcal{C}'$ are orthogonal. Hence the standardness condition is clearly satisfied by the triple $(\widehat{\mathcal{H}}_+, \widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_-)$.

(Separation) Clearly $\operatorname{Hom}_A(\mathcal{C}', \mathcal{H}_+) = 0$. Assume $0 \neq f\colon X = (V, \bigoplus X_i, g) \to Y = (U, \bigoplus Y_i, g')$ is a map with $X \in \widehat{\mathcal{H}}_-$ and $Y \in \widehat{\mathcal{H}}_0$. Hence each $X_i \in \mathcal{H}_- \cup \mathcal{C}$ and $Y_j \in \mathcal{H}_0$ by 5.3.

If $Y \in \mathcal{C}'$, then each $Y_j \in \mathcal{C}$. Since $g'\colon U \to \operatorname{Hom}_B(M, \bigoplus Y_j)$ is injective, there are i and j with $\operatorname{Hom}_B(X_i, Y_j) \neq 0$. Therefore $X_i \in \mathcal{C}$. The chain of maps $X_i \hookrightarrow X \xrightarrow{f} Y$ with $X \notin \mathcal{C}'$ would contradict the convexity of $\mathcal{C}'$ if $X_i \in \mathcal{C}'$. Therefore $X_i \in \widehat{\mathcal{H}}_-$.

Since $X_i, Y_j \in \mathcal{C}$ and $\mathcal{C}$ is generalized standard, there is a chain of irreducible maps $X_i \to Z_1 \to Z_2 \to \cdots \to Z_s = Y_j$. Since Y_j is a submodule of Y, it follows that $Y_j \in \widehat{\mathcal{H}}_0$. Moreover, $\widehat{\mathcal{H}}_0$ being generalized standard shows that $Y_j \in \mathcal{C}'$. Again hypothesis (b) implies that $X_i \in \mathcal{C}'$, a contradiction.

If $Y \in \widehat{\mathcal{H}}_0 \setminus \mathcal{C}'$, then $Y = (0, Y_1, 0)$. Then for some i, $\operatorname{rad}_B^\infty(X_i, Y_1) \neq 0$, which contradicts $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ being a weak trisection of $\operatorname{ind} B$. This shows that $\operatorname{Hom}_A(\widehat{\mathcal{H}}_-, \widehat{\mathcal{H}}_0) = 0$.

Finally, by condition (b), all indecomposable projective A-modules lie in $\widehat{\mathcal{H}}_+ \cup \widehat{\mathcal{H}}_0$. Clearly, injective modules I_y with $y \neq \omega$ lie in $\widehat{\mathcal{H}}_0 \cup \widehat{\mathcal{H}}_-$. Since the separation condition is already proved, it is not possible that $I_\omega \in \widehat{\mathcal{H}}_+$.

Therefore $(\widehat{\mathcal{H}}_+, \widehat{\mathcal{H}}_0, \widehat{\mathcal{H}}_-)$ is a weak trisection of $\operatorname{ind} A$. ∎

5.5. We present the following structure theorem for components in the middle part $\mathcal{H}_0$ of a trisection.

THEOREM. *Let $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ be a trisection of A. Then*:

(a) *Any component $\mathcal{C}''$ of $\mathcal{H}_0$ admits only finitely many non-periodic orbits.*

(b) *Assume $A = B[M]$ where $M \in \operatorname{add}\mathcal{C} \cap \mathcal{H}_0$ and $\mathcal{C}$ is a finite union of generalized standard components in Γ_B. Let $\mathcal{C}'$ be the component in $\mathcal{H}_0$ containing the summands of M. Then any $X \in \mathcal{C}$ with $X \notin \mathcal{C}'$ is a non-periodic B-module.*

Proof. (a) is known to hold for any generalized standard component [23].

(b) Suppose $X \in \mathcal{C} \setminus \mathcal{C}'$ and X is a τ_B-periodic module. By 5.1, $X^{\tau_B} \cap \mathcal{C}' = \emptyset$. Let Y^{τ_B} be a neighbour periodic orbit to X^{τ_B}. We may therefore assume $X \xrightarrow{\alpha} Y$ is an arrow in $\mathcal{C}$. Since $X \notin \mathcal{C}'$, by 5.1 also $Y \notin \mathcal{C}'$.

Proceeding by induction on the connected component $\mathcal{C}'$ we conclude that only non-periodic τ_B-orbits of $\mathcal{C}$ may not lie in $\mathcal{C}'$. ■

5.6. *Proof of Theorem B.* Let $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ be a trisection of $\operatorname{ind} A$. Let $X \in \mathcal{H}_+$ and $Y \in \mathcal{H}_-$ and choose a morphism $f \in \operatorname{Hom}_A(X, Y)$.

By 4.2 we may suppose that A is a triangular algebra. By 2.3, A may be reconstructed from A^+ by a sequence of one-point extensions $A_0 = A^+$, $A_1 = A_0[M_0], \ldots, A_{n+1} = A_n[M_n] = A$ in such a way that for each $i = 0, 1, \ldots, n+1$, $\operatorname{ind} A_i$ admits a trisection $(\mathcal{H}^i_+, \mathcal{H}^i_0, \mathcal{H}^i_-)$ with $\mathcal{H}^i_+ = \mathcal{H}_+$, $M_i \in \operatorname{add}\mathcal{H}^i_0$ and $\mathcal{H}^{n+1}_0 = \mathcal{H}_0$, $\mathcal{H}^{n+1}_- = \mathcal{H}_-$. Moreover, by 3.3, $A^+ = A^+_1 \times \cdots \times A^+_s$, where each A^+_i is either tilted or an almost concealed-canonical algebra.

Hence $X \in \mathcal{H}^n_+$ and $Y = (Y_0, V, \gamma\colon V \to \operatorname{Hom}_{A_n}(M_n, Y_0))$ for some $Y_0 = Y^0_0 \oplus Y^-_0$ with $Y^0_0 \in \operatorname{add}(\mathcal{H}^n_0)$ and $Y^-_0 \in \operatorname{add}(\mathcal{H}^n_-)$. If $Y^0_0 \neq 0$, then for every direct summand Y' of Y_0 we have $\operatorname{Hom}_{A_n}(M_n, Y') \neq 0$. Since A^+ has the strong separation property (see 1.7), we may proceed by induction and suppose that A_n has the weak separation property.

Let $f = (f_0, f_1)\colon X \to Y^0_0 \oplus Y^-_0$. By hypothesis, there exist $U_1 \in \operatorname{add}\mathcal{H}^n_0$ and maps $f'_1 \in \operatorname{Hom}_{A_n}(X, U_1)$ and $f''_1 \in \operatorname{Hom}_{A_n}(U_1, Y^-_0)$ such that $f_1 = f''_1 f'_1$. Consider the A-module $U = Y^0_0 \oplus U_1 \in \operatorname{add}\mathcal{H}^n_0$ and the indecomposable decompositions $Y^0_0 = \bigoplus_{i=1}^s V_i$ and $U_1 = \bigoplus_{i=s+1}^t V_i$ which determine maps $g_i = f_0|\colon X \to V_i$ $(1 \le i \le s)$ and $g_i = f'_1|\colon X \to V_i$ $(s+1 \le i \le t)$. Then f factorizes as the composition of $(g_i)\colon X \to U$ and $((\mathbf{1}, f''_1), 0) : (U, 0, 0) \to (Y_0, V, \gamma) = Y$. There remains the problem that not necessarily $V_i \in \mathcal{H}^{n+1}_0 = \mathcal{H}_0$.

Let $\mathcal{C}$ be the union of the components of $\mathcal{H}^n_0$ containing summands of M_n. Let $\mathcal{C}'$ be the component of $\mathcal{H}^{n+1}_0 = \mathcal{H}_0$ containing the summands of M_n. By the proof of 5.4 we know that $\mathcal{H}_0 = (\mathcal{H}^n_0 \setminus \mathcal{C}) \cup \mathcal{C}'$. We shall prove that for each $1 \le i \le t$, there exist $Z_i \in \mathcal{C}'$ and maps $g'_i\colon X \to Z_i$ and $g''_i\colon Z_i \to V_i$ such that $g_i = g''_i g'_i$. This yields the desired factorization of f.

Consider a map $g\colon X \to V$ such that $V \in \mathcal{C}$. If V is projective, then $V \in \mathcal{C}'$ by 5.1 and the result follows. Assume $V \notin \mathcal{C}'$ and V is not projective.

Consider the almost split sequence in $\operatorname{mod} A_n$

$$0 \to \tau_{A_n} V \xrightarrow{\nu} \bigoplus_{j=1}^{m} L_j \xrightarrow{\mu} V \to 0,$$

where the L_j are indecomposable modules in $\mathcal{C}$. Hence there exists a map $h\colon X \to \bigoplus_{j=1}^{m} L_j$ such that $\mu h = g$. For each $1 \le j \le m$, either $L_j \in \mathcal{C}'$ or we proceed as before. The procedure should stop after finitely many steps.

Indeed, otherwise we get a chain of maps between indecomposable modules in $\mathcal{C}$

$$\cdots \to N_{r+1} \xrightarrow{\mu_{r+1}} N_r \cdots \to N_2 \xrightarrow{\mu_2} N_1 \xrightarrow{\mu_1} V$$

with $N_i \notin \mathcal{C}'$ and morphisms $h_i \in \operatorname{Hom}_{A_n}(X, N_i)$ such that $0 \neq \mu_1 \cdots \mu_{i-1} h_i\colon X \to V$, $i \ge 1$. By 5.5, all N_i are non-periodic A_n-modules and there exists a sequence $(s_m)_m$ of natural numbers such that the τ_{A_n}-orbit of all the N_{s_m} is the same, say $N_{s_m}^{\tau_{A_n}} = \mathcal{O}$ $(m \ge 1)$. Let L be a module in $\mathcal{C}$ whose orbit is a neighbour of $\mathcal{O}$. By 5.1, $L \notin \mathcal{C}'$ and hence L is a non-periodic A_n-module. We infer that the connected component of $\mathcal{C}$ where V lies is formed by non-periodic modules.

By [14], $\mathcal{C}$ is contained in a translation quiver of the form $\mathbb{Z}\Delta$ for Δ a finite quiver without oriented cycles (in fact, by [22], A_n is a tilted algebra). Hence for some $m_0 \in \mathbb{N}$, any predecessor L of $N_{s_{m_0}}$ in $\mathcal{C}$ has $\operatorname{Hom}_{A_n}(M_n, L) = 0$. In particular, for all $m \ge m_0$, $N_{s_m} \in \mathcal{C}'$, a contradiction which completes the proof of the theorem. ∎

6. Proofs of Theorems 1.5 and 1.6

6.1. With the tools developed in the previous sections we now return to the problems left open in the first section.

Proof of Theorem 1.5. Assume $A = kQ/I$ is an algebra admitting two trisections $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ and $(\mathcal{H}_+^*, \mathcal{H}_0^*, \mathcal{H}_-^*)$ of $\operatorname{ind} A$ with $\mathcal{H}_0 \neq \mathcal{H}_0^*$. By Section 4, the quotient $\overline{A} = A/(x\colon x \in Q^{\mathrm{c}})$, where Q^{c} is the convex closure in Q of vertices lying on cycles in Q, satisfies the following conditions:

(a) there are only finitely many indecomposable A-modules which are not $\overline{A}$-modules;
(b) $\operatorname{mod} \overline{A}$ admits two trisections $(\overline{\mathcal{H}}_+, \overline{\mathcal{H}}_0, \overline{\mathcal{H}}_-)$ and $(\overline{\mathcal{H}}_+^*, \overline{\mathcal{H}}_0^*, \overline{\mathcal{H}}_-^*)$ with $\overline{\mathcal{H}}_0 \neq \overline{\mathcal{H}}_0^*$.

Therefore, $\overline{A} = A_1 \times \cdots \times A_s$ is a product of indecomposable triangular algebras, where at least one of the factors A_i admits two trisections $(\mathcal{H}_+(i), \mathcal{H}_0(i), \mathcal{H}_-(i))$ and $(\mathcal{H}_+^*(i), \mathcal{H}_0^*(i), \mathcal{H}_-^*(i))$ with $\mathcal{H}_0(i) \neq \mathcal{H}_0^*(i)$.

To prove the claim, it is enough to assume that A is an indecomposable triangular algebra and show that A is of one of the following well known types of algebras: concealed, domestic tubular or tubular algebras. We distinguish several situations.

(a) If A is a tilted algebra, then in order that A admits two different trisections, A should be either concealed or a domestic tubular algebra.

(b) Suppose that every component in $\mathcal{H}_0$ is semiregular. First observe that by 3.1, the components in $\mathcal{H}_0$ may be assumed to be non-directing. Then by [14], $\mathcal{H}_0$ is a sincere separating family of standard semiregular tubes. Therefore [13] implies that A is a quasitilted algebra of canonical type. In order that A admits two different trisections, A should be either domestic tubular or a tubular algebra.

(c) Assume $\mathcal{C}$ is a component in $\mathcal{H}_0$ with projective and injective modules. Then $\mathcal{C}$ cannot lie in $\mathcal{H}^*_+$ or $\mathcal{H}^*_-$, hence $\mathcal{C}$ is in $\mathcal{H}^*_0$. Let x be a source in the quiver Q such that $P_x \in \mathcal{C}$, hence $P_x \in \mathcal{H}_0 \cap \mathcal{H}^*_0$. Consider the algebra $A^{(x)}$ and the construction 2.2 of going down trisections $(\mathcal{H}'_+, \mathcal{H}'_0, \mathcal{H}'_-)$ and $(\mathcal{H}^{*\prime}_+, \mathcal{H}^{*\prime}_0, \mathcal{H}^{*\prime}_-)$ of $\operatorname{ind} A^{(x)}$. By 2.2, $\mathcal{H}'_0 \neq \mathcal{H}^{*\prime}_0$ since otherwise $\mathcal{H}_0 = \mathcal{H}^*_0$. By induction hypothesis $A^{(x)}$ is either concealed, domestic tubular or tubular.

According to 5.1, the only situations in which $A = A^{(x)}[R]$ may have two different trisections are the following:

- $A^{(x)}$ is concealed and R is preprojective. Then A is concealed.
- $A^{(x)}$ is domestic tubular and R is indecomposable such that A is domestic tubular or tubular, since otherwise A does not admit a trisection (see for instance [3] or [17]).

The proof is complete. ∎

6.2. We proceed to discuss the concept of weak trisections.

LEMMA. *Let $A = kQ/I$ be an algebra with a weak trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ of $\operatorname{ind} A$. Suppose that for some vertex $a \in Q$ we have $X(a) = 0$ for every $X \in \mathcal{H}_0$. Let x be a source in Q such that $P_x \in \mathcal{H}_0$ and consider the quotient $A^{(x)}$. Then the weak trisection $(\mathcal{H}'_+, \mathcal{H}'_0, \mathcal{H}'_-)$ of $\operatorname{ind} A^{(x)}$ as given in 2.2 satisfies $Y(a) = 0$ for every $Y \in \mathcal{H}'_0$.*

Proof. Let $A = B[M]$ for $M = \operatorname{rad} P_x$. Consider the finite union of components $\mathcal{C}'$ in $\mathcal{H}'_0$ containing indecomposable direct summands of M and the component $\mathcal{C}$ in $\mathcal{H}_0$ containing the projective P_x (and hence all indecomposable direct summands of M). Then by 5.1 the following hold:

- if $P_y \in \mathcal{C}'$, then $P_y \in \mathcal{C}$;
- if $X \to Y$ is an arrow in $\mathcal{C}'$ with $Y \in \mathcal{C}$, then also $X \in \mathcal{C}$.

By definition of $\mathcal{H}_0 = (\mathcal{H}'_0 \setminus \mathcal{C}') \cup \mathcal{C}$, we shall check that $Y(a) = 0$ for all $Y \in \mathcal{C}$. Observe that every $Y \in \mathcal{C}$ is of the form $Y = \tau_A^{-i} X$ for some $X \in \mathcal{C}$, $i \geq 0$.

Assume, in order to get a contradiction, that $Y = \tau_A^{-i}X$ with $X \in \mathcal{C}$ ($\subset \mathcal{H}_0$), $Y(a) \neq 0$ and $i > 0$ is minimal. Hence $Y = \tau_A^- X_0$ with $X_0(a) = 0$. Consider the almost split sequence $0 \to X_0 \to E \to Y \to 0$ in $\operatorname{mod} A$. Hence there exists an indecomposable direct summand Y_1 of E with $Y_1(a) \neq 0$. Moreover, observe that for $X_1 = \tau_A Y_1$ we have $X_1(a) = 0$ [indeed, either $\tau_A^{-j} X_1$ is injective for some $0 \le j < i-1$ or $\tau_A^{-i+1} X_1$ is a predecessor of X, in any case in $\mathcal{H}_0$. The minimality of i implies that $X_1(a) = 0$].

By repeating this procedure we get a sequence of irreducible maps

$$\cdots \to Y_n \to Y_{n-1} \to \cdots \to Y_1 \to Y_0 = Y$$

and the corresponding sequence

$$\cdots \to \tau_A Y_n = X_n \to \tau_A Y_{n-1} = X_{n-1} \to \cdots \to \tau_A Y_0 = X_0$$

with $Y_i(a) \neq 0$ and $X_i(a) = 0$. By [4], all modules Y_n are pairwise non-isomorphic. We shall show that $\bigoplus_{n=0}^{\infty} Y_n$ is a partial tilting module, which yields the required contradiction. Indeed, for $n, m \ge 0$, assume that $0 \neq \operatorname{Ext}_A^1(Y_n, Y_m) \cong D\overline{\operatorname{Hom}}_A(Y_m, X_n)$, hence $\operatorname{Hom}_A(Y_m, X_n) \neq 0$. Then either there is some $0 \le j < i-1$ such that $\tau_A^j Y_m$ is a predecessor of a projective in $\mathcal{H}_0$, or $\operatorname{Hom}_A(\tau_A^{i-1} Y_m, \tau_A^i Y_n) \neq 0$ and hence $\tau_A^{i-1} Y_m$ is a predecessor of $\tau_A^i Y_0 = X \in \mathcal{H}_0$. In any case there is some $0 \le j \le i-1$ such that $\tau_A^j Y_m \in \mathcal{H}_0$, which contradicts the choice of i. Therefore $\operatorname{Ext}_A^1(Y_n, Y_m) = 0$ for any $n, m \ge 0$, which means that $\bigoplus_{n=0}^{\infty} Y_n$ is a partial tilting module. Our claim is proved. ■

6.3. *Proof of Theorem 1.6.* Assume $(\mathcal{H}_+, \mathcal{H}_-, \mathcal{H}_-)$ is a weak trisection of $\operatorname{ind} A$ and it is not adjusted.

Killing those vertices on cycles of Q as in Section 4, we may assume that A is a triangular algebra. Since $\mathcal{H}_0$ is not adjusted, we may assume that X is an indecomposable A-module in $\mathcal{H}_+$ with $\operatorname{Hom}_A(X, \mathcal{H}_0) = 0$.

By 2.3, there is a sequence of algebras $A = A_n[M_n], A_n = A_{n-1}[M_{n-1}], \ldots, A_1 = A_0[M_0]$ such that $\operatorname{ind} A_i$ admits a trisection $(\mathcal{H}_+^i, \mathcal{H}_0^i, \mathcal{H}_-^i)$ with $M_i \in \operatorname{add} \mathcal{H}_0^i$ and A_0 is a product of quasitilted algebras. Then by construction 2.2, $X \in \mathcal{H}_+^n$ and $\operatorname{Hom}_{A_n}(X, \mathcal{H}_0^n) = 0$. By induction $X \in \mathcal{H}_+^0$ and $\operatorname{Hom}_{A_0}(X, \mathcal{H}_0^0) = 0$, which is impossible in the class of quasitilted algebras (see 3.3). We conclude that $\operatorname{Hom}_A(X, \mathcal{H}_0) \neq 0$ and $\mathcal{H}_0$ is adjusted. ■

7. Strongly simply connected algebras admitting a trisection

7.1. We recall from [2] that a *coil* is a translation quiver constructed inductively from a stable tube by a sequence of procedures called *admissible operations*. Let Σ be a standard component of Γ_A and $X \in \Sigma$. Consider the following situations:

(ad 1) The modules $Y \in \Sigma$ with $\mathrm{Hom}_A(X,Y) \neq 0$ lie on a path

$$X = X_0 \to X_1 \to X_2 \to \cdots .$$

Let D be the algebra of $m \times m$ lower triangular matrices and Z be the unique indecomposable D-module which is projective-injective.

Let $A' = (A \times D)[X \oplus Z]$.

(ad 2) The modules $Y \in \Sigma$ with $\mathrm{Hom}_A(X,Y) \neq 0$ lie on two sectional paths, the first infinite and the second finite:

$$Y_t \leftarrow \cdots \leftarrow Y_2 \leftarrow Y_1 \leftarrow X = X_0 \to X_1 \to X_2 \to \cdots$$

with $t \geq 1$.

Let $A' = A[X]$.

(ad 3) The modules $Y \in \Sigma$ with $\mathrm{Hom}_A(X,Y) \neq 0$ are of the form Y_i or X_j in:

$$\begin{array}{ccccccccccc} & & Y_1 & \to & Y_2 & \to \cdots \to & Y_t & & & & \\ & & \uparrow & & \uparrow & & \uparrow & & & & \\ X = & & X_0 & \to & X_1 & \to \cdots \to & X_{t-1} & \to & X_t & \to & X_{t+1} \to \cdots \end{array}$$

with X_{t-1} injective.

Let $A' = A[X]$.

Then by [2], the component $\mathcal{C}'$ of $\Gamma_{A'}$ where X lies is a standard component called a *coil*.

A component Σ of Γ_A is called a *multicoil* if it contains a full translation subquiver Σ' such that:

(a) Σ' is a disjoint union of coils,
(b) the modules in $\Sigma \setminus \Sigma'$ are directing.

Examples of multicoils may be found in [2].

7.2. A *strongly simply connected* algebra $A = kQ/I$ over an algebraically closed field k is a triangular algebra such that for every convex subcategory $B = kQ'/I'$ of A and every source x of Q' (resp. sink y of Q') such that $B^{(x)}$ (resp. $B^{(y)}$) is connected, the radical of the projective B-module P_x (resp. $I_y/\operatorname{soc} I_y$ for the injective B-module I_y) is indecomposable. See [21].

The main result in [24] may be (partially) rephrased in the following way (observe the statement does not depend on the representation type of the algebra).

THEOREM ([24]). *Let A be a strongly simply connected algebra. Let Σ be a multicoil in Γ_A. Let $R \in \Sigma$ be such that the component Σ' of $\Gamma_{A[R]}$ where R lies is generalized standard. Then Σ' is a multicoil in $\Gamma_{A[R]}$.* ∎

7.3. We show Theorem C of the introduction.

Proof of Theorem C. Let A be a strongly simply connected algebra admitting a trisection $(\mathcal{H}_+, \mathcal{H}_0, \mathcal{H}_-)$ of $\operatorname{ind} A$. Then by 3.3, A^+ is either a tilted algebra or an almost concealed-canonical algebra and $\mathcal{H}_0$ is formed, respectively, by a directing component or a standard family of semiregular tubes. In any case $\mathcal{H}_0$ is formed by a generalized standard family of multicoils.

Observe that there are indecomposable modules $M_0, M_1, \ldots, M_s$ such that $A_0 = A^+$, $A_1 = A_0[M_0], \ldots, A_{i+1} = A_i[M_i]$ with $A_{s+1} = A$ and there is a trisection $(\mathcal{H}_+^i, \mathcal{H}_0^i, \mathcal{H}_-^i)$ of $\operatorname{ind} A_i$ such that $M_i \in \mathcal{H}_0^i$ for $i = 0, \ldots, s$ by 2.3. The result follows from 7.2 by induction. ∎

7.4. In independent work Malicki and Skowroński [16] generalized the procedures (admissible operations) allowing one to build standard components, called *generalized multicoil components*, in the Auslander–Reiten quiver of certain extensions of algebras with a separating family of generalized standard almost cyclic coherent components. Theorem C above could be generalized to arbitrary base fields using these generalized admissible operations.

REFERENCES

[1] I. Assem and F. Coelho, *Two-sided gluings of tilted algebras*, J. Algebra 269 (2003), 456–479.

[2] I. Assem and A. Skowroński, *Coils and multicoil algebras*, in: CMS Conf. Proc. 19, Amer. Math. Soc., 1996, 1–24.

[3] M. Barot and J. A. de la Peña, *Derived tubular strongly simply connected algebras*, Proc. Amer. Math. Soc. 127 (1999), 647–655.

[4] R. Bautista and S. Smalø, *Non-existing cycles*, Comm. Algebra 11 (1983), 1755–1767.

[5] K. Bongartz, *Algebras and quadratic forms*, J. London Math. Soc. (2) 28 (1983), 461–469.

[6] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebras*, London Math. Soc. Lecture Note Ser. 119, Cambridge Univ. Press, 1988.

[7] D. Happel, *A characterization of hereditary categories with tilting object*, Invent. Math. 144 (2004), 381–398.

[8] D. Happel and I. Reiten, *Hereditary abelian categories with tilting object over arbitrary base fields*, J. Algebra 256 (2002), 414–432.

[9] —, —, *Directing objects in hereditary categories*, in: Contemp. Math. 229, Amer. Math. Soc., 1998, 169–179.

[10] D. Happel, I. Reiten and S. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575 (1996).

[11] H. Lenzing and J. A. de la Peña, *Wild canonical algebras*, Math. Z. 224 (1997), 403–425.

[12] —, —, *Concealed-canonical algebras and separating tubular families*, Proc. London Math. Soc. (3) 78 (1999), 513–540.

[13] H. Lenzing and A. Skowroński, *Quasi-tilted algebras of canonical type*, Colloq. Math. 71 (1996), 161–181.
[14] S. Liu, *Semi-stable components of an Auslander–Reiten quiver*, J. London Math. Soc. 47 (1993), 405–416.
[15] —, *Infinite radicals in standard Auslander–Reiten components*, J. Algebra 166 (1994), 245–254.
[16] P. Malicki and A. Skowroński, *Algebras with separating almost cyclic coherent Auslander–Reiten components*, ibid. 291 (2005), 208–237.
[17] J. A. de la Peña, *On the representation type of one-point extensions of tame concealed algebras*, Manuscripta Math. 61 (1988), 183–194.
[18] J. A. de la Peña and B. Tomé, *Tame algebras with a weakly separating family of coils*, in: CMS Conf. Proc. 18, Amer. Math. Soc., 1996, 555–569.
[19] I. Reiten and A. Skowroński, *Generalized double tilted algebras*, J. Math. Soc. Japan 56 (2004), 269–288.
[20] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, Berlin, 1984.
[21] A. Skowroński, *Simply connected algebras and Hochschild cohomologies*, in: CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 431–447.
[22] —, *Generalized standard Auslander–Reiten components without oriented cycles*, Osaka J. Math. 30 (1993), 515–527.
[23] —, *Generalized standard Auslander–Reiten components*, J. Math. Soc. Japan 46 (1994), 517–543.
[24] —, *Simply connected algebras of polynomial growth*, Compos. Math. 109 (1997), 99–133.
[25] —, *Directing modules and double tilted algebras*, Bull. Polish Acad. Sci. Math. 50 (2002), 77–87.

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