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THE MULTIPLICITY PROBLEM FOR INDECOMPOSABLE DECOMPOSITIONS OF MODULES OVER A FINITE-DIMENSIONAL ALGEBRA. ALGORITHMS AND A COMPUTER ALGEBRA APPROACH

BY

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Abstract. Given a module M over an algebra Λ and a complete set $\mathcal{X}$ of pairwise nonisomorphic indecomposable Λ-modules, the problem of determining the vector $m(M) = (m_X)_{X\in\mathcal{X}} \in \mathbb{N}^{\mathcal{X}}$ such that $M \cong \bigoplus_{X\in\mathcal{X}} X^{m_X}$ is studied. A general method of finding the vectors $m(M)$ is presented (Corollary 2.1, Theorem 2.2 and Corollary 2.3). It is discussed and applied in practice for two classes of algebras: string algebras of finite representation type and hereditary algebras of type $\widetilde{\mathbb{A}}_{p,q}$. In the second case detailed algorithms are given (Algorithms 4.5 and 5.5).

Introduction. The main problem of contemporary representation theory of finite-dimensional algebras is to describe in a possibly precise way the structure of the module category for a given algebra; in particular, to determine its representation type. From this point of view results containing classification of all (up to isomorphism) indecomposable modules in terms of some invariant (e.g. dimension vector) have been considered to be quite satisfactory; especially, if they additionally provide extra information on morphisms encoded by the shape of the Auslander–Reiten quiver.

Most of the research methods developed in the last thirty years for studying representations of algebras have been dedicated to study this kind of problems. Nevertheless before they were invented, another rather universal and natural approach to the classification of indecomposables was common. It led via an answer to the more specialized and in fact difficult question: *how to decompose* (*effectively*) *an arbitrary module into a direct sum of indecomposable submodules* (isomorphic to indecomposables from a "candidate list"). Some variants of this method were successfully used in several very important classical classification results [16, 10, 14, 18, 19, 7].

The above question is interesting in its own right, even if a full list $\mathcal{X}$ of pairwise nonisomorphic indecomposables is already known. It seems to be

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particularly important if one thinks of applications (see [16, 13]). The weaker version of this question, asking for a "normal form" of a module M (i.e. the *full multiplicity sequence* of direct summands from the list $\mathcal{X}$ for M) is also of importance (e.g. in an algebraic geometry approach to module categories).

These two problems can be considered as a final step of studies for a given module category $\operatorname{mod} \Lambda$ of finite or tame representation type. They have a rather computational and algorithmic character. The actually available standard tools of representation theory are not particularly useful and well adjusted for successful, comprehensive discussion of these problems, on the level that allows formulating general answers.

The main aim of this paper is to deal with the second of them. More precisely, *given a module M and a complete family $\mathcal{X}$ of pairwise nonisomorphic indecomposable modules in* $\operatorname{mod} \Lambda$ we want to determine the sequence

$$m(M) = (m_X) \in \mathbb{N}^{\mathcal{X}} \quad \text{such that} \quad M \cong \bigoplus_{X \in \mathcal{X}} X^{m_X}.$$

The sequence $(m_X)_{X \in \mathcal{X}}$ is uniquely determined by the Krull–Remak–Schmidt theorem. We provide a general method for handling this problem. It relies (Theorem 2.2) on computing the sequence

$$h(m) = (h_X) \in \mathbb{N}^{\mathcal{X}}$$

of the dimensions $h_X = \dim_k \operatorname{Hom}_\Lambda(M, X)$ and the so-called Auslander–Reiten matrix T_Λ for Λ. (Sometimes instead of T_Λ it is enough to find the Cartan matrix of the Auslander category for Λ.) In principle this method can be effectively applied only in case the canonical forms of all indecomposable modules are known.

We discuss this method in practice for two simple classes of string (special biserial) algebras: string algebras of finite representation type and canonical hereditary algebras of type $\widetilde{\mathbb{A}}_{p,q}$ (later called simply $\widetilde{\mathbb{A}}_{p,q}$-algebras, see also [20]). In the second case we present rather precise algorithmic procedures for solving the problem (Algorithms 4.5 and 5.5). In constructing the algorithms, and to improve their efficiency, we use some general information on the structure of the relevant module categories and basic methods of representation theory (Auslander–Reiten theory, Galois coverings). To decrease the complexity of algorithms computing the coordinates of the vector $h(M)$, we also apply certain simple results obtained by a detailed "numerical analysis" of some computational linear algebra problems, strongly related to specific canonical forms for indecomposables (Lemmas 4.7 and 5.7).

The paper is organized as follows. In Section 1 we recall basic definitions and fix the notation. In Section 2 we introduce the notion of the multiplicity vector $m(M)$, the Cartan matrix $C(\Lambda)$ of the Auslander category and the Auslander–Reiten matrix T_Λ. We prove that $C(\Lambda)$ is always invertible and its inverse is just T_Λ (Theorem 2.2). We show the importance of these two matrices in determining the multiplicity sequence. Also some strange properties of the partial algebra $\mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$, where $\mathcal{Y}$ is an infinite set, are discussed.

Section 3 is devoted to a discussion of how using a combinatorial description of indecomposables, one can effectively construct the matrices $C(\Lambda)$ for string algebras of finite representation type (see Algorithms 3.3 and 3.5). These algorithms are based on Proposition 3.2 (tree case) and Proposition 3.4 (general case), describing how the dimensions of the homomorphism spaces between indecomposables can be computed.

In Section 4 we give a description of the algorithm determining the multiplicity sequence for modules over the Kronecker algebra (Algorithm 4.5). In particular, we show how to restrict a list of candidates for indecomposable direct summands of a given module to a finite list (Proposition 4.4). Also an inductive metod of computing the dimensions of the homomorphism spaces from a given module to indecomposables from individual Auslander–Reiten components is presented (Lemmas 4.6 and 4.7).

Section 5 is devoted to a description of elements responsible for an analogous algorithm for modules over $\widetilde{\mathbb{A}}_{p,q}$-algebras in the general case (Corollary 5.3, Lemma 5.4, Algorithm 5.5). In particular, we construct a nice functor that allows us to reduce some considerations for $\widetilde{\mathbb{A}}_{p,q}$-algebras to the Kronecker algebra case (see Lemma 5.2). Also an inductive method of computing the dimensions of the respective homomorphism spaces in the general $\widetilde{\mathbb{A}}_{p,q}$-algebra case is proposed (Lemmas 5.6 and 5.7), and a pessimistic complexity of the given algorithms is discussed (Lemma 5.8).

1. Preliminaries and notations. We use the definitions and notation which are well known and commonly used. Nevertheless, for the benefit of the reader, we briefly recall the most important of them. For other information concerning representation theory of algebras (respectively, ring theory, linear algebra, algorithm theory) we refer to [2, 4] (respectively, [1], [15], [6]).

1.1. Throughout the paper k always denotes a field, usually algebraically closed. By a *k-algebra* we always mean a finite-dimensional associative connected basic algebra with unit over k. For a k-algebra Λ (respectively, locally bounded category Λ, see [12]) we denote by $\operatorname{mod}\Lambda$ the category of all finite-dimensional right Λ-modules, and by $\mathcal{J}_\Lambda$ the Jacobson radical in the category $\operatorname{mod}\Lambda$. If (Q,I) is a bounded quiver (see [17]) and the algebra (resp. locally bounded category) Λ has the form $\Lambda = kQ/I$, then we always identify $\operatorname{mod}\Lambda$ with the category of all finite-dimensional representations of the quiver Q, satisfying the relations from the ideal I (for the definition of the path algebra kQ we refer to [2]). In this case for any V in $\operatorname{mod}\Lambda$ we denote by $\operatorname{supp} V$ the set of all vertices x of Q such that $V_x \neq 0$, where V_x is the direct summand of the vector space V corresponding to x.

By the *Auslander–Reiten quiver* Γ_Λ of Λ we always mean the translation valued quiver

$$\Gamma = (\Gamma_0, \Gamma_1, \tau, d, d').$$

The translation quiver $(\Gamma_0, \Gamma_1, \tau)$ is defined in the standard way (the set of vertices Γ_0 consists of the isoclasses of indecomposable objects in $\operatorname{mod}\Lambda$, arrows $[Y] \to [X]$ in Γ_1 reflect the existence of irreducible maps from X to Y, and $\tau[X] = [\tau X]$, where τ is the Auslander–Reiten translate). The valuation (d, d') is given by the collections $d = (d_{[X],[Y]})$ and $d' = (d'_{[X],[Y]})$ indexed by Γ_1, where for fixed $[X] \to [Y] \in \Gamma_1$, $d_{[X],[Y]}$ (resp. $d'_{[X],[Y]}$) is the number of indecomposable direct summands isomorphic to Y (resp. X) in a decomposition of the codomain (resp. domain) of a minimal left (resp. right) almost split map for X (resp. for Y).

For any $[X] \in \Gamma_0$ we denote by ${}^-[X]$ (resp. $[X]^+$) the set of all direct predecessors (resp. successors) of $[X]$ in Γ_Λ, i.e. the set of all vertices $[Y] \in \Gamma_0$ such that there exists an arrow $[Y] \to [X]$ (resp. $[X] \to [Y]$) in Γ_Λ.

For any X and Y in $\operatorname{mod}\Lambda$, we denote by (X, Y) the k-space $\operatorname{Hom}_\Lambda(X, Y)$ and by $[X, Y]$ its dimension.

1.2. Let $Q = (Q_0, Q_1)$ be a quiver. For an arrow $\gamma \in Q_1$, we denote by γ^{-1} the formal inverse of γ (we set $(\gamma^{-1})^{-1} = \gamma$). Any unoriented path w in Q can be presented as a sequence $\gamma_1 \cdots \gamma_n$, for some $n \geq 0$, where $\gamma_i \in Q_1$ or $\gamma_i^{-1} \in Q_1$, for $1 \leq i \leq n$. In the paper we consider only *walks*, i.e. unoriented paths w such that if $w = w_1\alpha\beta^{-1}w_2$ or $w = w_1\alpha^{-1}\beta w_2$ for some unoriented paths w_1, w_2 and arrows $\alpha, \beta \in Q_1$, then $\alpha \neq \beta$. For any walk $w = \gamma_1 \cdots \gamma_n$ we denote by w^{-1} the formal inverse of w, i.e. the walk $\gamma_n^{-1} \cdots \gamma_1^{-1}$. The trivial walks of length 0 are simply identified with the vertices of Q_0.

Assume that Q is a tree. Then any walk w defines an indecomposable Λ-module $V(w)$, called a *line module* or simply a *line*, where $\Lambda = kQ$. The module $V(w)$, as a representation of Q, has the one-dimensional space k at each vertex visited by w and zero spaces otherwise; the structure maps are identities for the arrows belonging to w and zero maps otherwise. Note that $V(w) = V(w^{-1})$ and $V(w) \simeq V(v)$ if and only if $w = v$ or $w = v^{-1}$.

1.3. The following notations are used in this paper.

For any set S, we denote by $|S|$ the cardinality of S.

Let R be a ring. For any $m, n \in \mathbb{N}$ we denote by $\mathbb{M}_{m\times n}(R)$ the set of all $m \times n$-matrices with entries in R. More generally, for any sets $\mathcal{X}$ and $\mathcal{Y}$ we denote by $\mathbb{M}_{\mathcal{X}\times\mathcal{Y}}(R)$ the set of all $\mathcal{X} \times \mathcal{Y}$-matrices M with coefficients in R, that is, functions $M : \mathcal{X} \times \mathcal{Y} \to R$. For any $M \in \mathbb{M}_{\mathcal{X}\times\mathcal{Y}}(R)$ we denote by M^t the matrix transposed to M; M^X (resp. M_X) is the column (resp.

row) of M corresponding to $X \in \mathcal{X}$; finally, for $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$, $M_{|\mathcal{X}' \times \mathcal{Y}'} \in \mathbb{M}_{\mathcal{X}' \times \mathcal{Y}'}(R)$ is the restriction of M to $\mathcal{X}' \times \mathcal{Y}'$. For a sequence $r = (r_X) \in \prod_{X \in \mathcal{X}} R$, $\operatorname{diag}(r)$ is the "diagonal" matrix in $\mathbb{M}_{\mathcal{X} \times \mathcal{X}}(R)$ defined by r; we set $\mathrm{I}_{\mathcal{X}} = \operatorname{diag}(r)$ where $r_X = 1$ for every $X \in \mathcal{X}$.

Given a matrix M in $\mathbb{M}_{m \times n}(k)$, we denote by $\mathrm{r}(M)$ the rank of M. By $\widehat{M}$ we mean the generalized upper triangular matrix obtained by deleting all zero rows from the matrix which is the result of the standard Gaussian-row elimination procedure applied to M (see [15]).

For any set $\mathcal{X}$ and $R = \mathbb{Z}, \mathbb{Q}, \mathbb{N}$, we set

$$R^{\mathcal{X}} = \prod_{\mathcal{X}} R, \quad R^{(\mathcal{X})} = \bigoplus_{\mathcal{X}} R.$$

2. A general method of determining multiplicities

2.1. Let Λ be a finite-dimensional algebra and $\mathcal{X}$ a fixed complete list of pairwise nonisomorphic indecomposable objects in $\operatorname{mod} \Lambda$. Denote by $C = C(\Lambda)$ the matrix (usually infinite!) in $\mathbb{M}_{\mathcal{X} \times \mathcal{X}}(\mathbb{Z})$ defined by the formula

$$c_{X,Y} = [Y, X]$$

for $X, Y \in \mathcal{X}$.

Given a module M in $\operatorname{mod} \Lambda$, for any $X \in \mathcal{X}$ we denote by $h_X = h_X(M)$ (resp. $h'_X = h'_X(M)$) the dimension $[M, X]$ (resp. $[X, M]$) and by $m_X = m_X(M)$ the multiplicity of X in the decomposition of M into a direct sum of indecomposables (in particular, $M \cong \bigoplus_{Y \in \mathcal{X}} Y^{m_Y}$). We view the sequences

$$h(M) = (h_X), \quad h'(M) = (h'_X) \quad \text{and} \quad m(M) = (m_X)$$

as column vectors in $\mathbb{N}^{\mathcal{X}}$. Note that in contrast to $m = m(M)$, the vectors $h = h(M)$ and $h' = h'(M)$ can have an infinite number of nonzero coordinates.

Definition.

(a) The matrix $C(\Lambda)$ is called the *Cartan matrix of the Auslander category* $E(\Lambda)$ of the algebra Λ.
(b) The vector $m(M)$ is called the *multiplicity vector of the Λ-module M*.

Lemma. *For any M in* $\operatorname{mod} \Lambda$ *we have $h = C(\Lambda) \cdot m$ and $h' = C(\Lambda)^t \cdot m$* (see also Remark below).

Proof. Since $M \cong \bigoplus_{Y \in \mathcal{X}} Y^{m_Y}$, for any $X \in \mathcal{X}$ we have $(M, X) = \bigoplus_{Y \in \mathcal{X}} (Y, X)^{m_Y}$ and $h_X = \sum_{Y \in \mathcal{X}} c_{X,Y} m_Y$. Notice that since $m_Y = 0$ for almost all Y, the above sum is finite and also $C \cdot m$ is well defined. Consequently, $h = C \cdot m$. Similarly, we obtain $h' = C^t \cdot m$. ∎

COROLLARY.

(a) *If* $T \in \mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Q})$ *is a left inverse of* $C(\Lambda)$, *then* $T \cdot h = m$.
(b) *If* $T \in \mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Q})$ *is a right inverse of* $C(\Lambda)$, *then* $T^t \cdot h' = m$.

Proof. Fix $T \in \mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Q})$ such that $T \cdot C = \mathrm{I}_{\mathcal{X}}$, where $C = C(\Lambda)$. Let $\mathcal{X}_0 \subseteq \mathcal{X}$ be the finite subset consisting of all $X \in \mathcal{X}$ such that $m_X \neq 0$. Then $h = \sum_{X\in\mathcal{X}_0} m_X C^X$ (in $\mathbb{Z}^{\mathcal{X}}$), so the product $T \cdot h$ is well defined as $T \cdot h = \sum_{X\in\mathcal{X}_0} m_X (TC^X)$, and consequently $T \cdot h = \sum_{X\in\mathcal{X}_0} m_X (\mathrm{I}_{\mathcal{X}})^X = m$.
The proof of the second assertion is analogous. ■

REMARK. Let $\mathcal{Y}$ be an infinite set.

(i) The set $\mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$ is a partial ring with respect to multiplication of matrices: for $M, N \in \mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$, the product $M \cdot N$ is defined if and only if for each pair $X, Z \in \mathcal{Y}$ the set $\{Y \in \mathcal{Y} : M(X,Y) \neq 0 \neq M(Y,Z)\}$ is finite. The partial ring admits a unit element $\mathrm{I}_{\mathcal{Y}}$ (the identity matrix). In $\mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$ the following pathologies may appear:

• Multiplication in $\mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$ is not associative in the sense that there may exist $M_1, M_2, M_3 \in \mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$ such that the products $M_1 \cdot M_2$, $M_2 \cdot M_3$, $(M_1 \cdot M_2) \cdot M_3$, $M_1 \cdot (M_2 \cdot M_3)$ are well defined and $(M_1 \cdot M_2) \cdot M_3 \neq M_1 \cdot (M_2 \cdot M_3)$. For example, consider the triple

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 2 & 2 & \cdots \\ 1 & 2 & 3 & 3 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 2 & 2 & 2 & 2 & \cdots \\ 3 & 3 & 3 & 3 & \cdots \\ 4 & 4 & 4 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that $M_2 \cdot M_3 = 0$, $M_1 \cdot M_2 = \mathrm{I}_{\mathcal{Y}}$, $M_3 \neq 0$, $(M_1 \cdot M_2) \cdot M_3 = \mathrm{I}_{\mathcal{Y}} \cdot M_3 = M_3$ and $M_1 \cdot (M_2 \cdot M_3) = M_1 \cdot 0 = 0$.

• A matrix $M \in \mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$ may admit two different two-sided inverses (see Remark 2.2(iv) for an example).

(ii) Any matrix $M \in \mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$ induces a $\mathbb{Z}$-linear map $M\cdot : \mathbb{Z}^{(\mathcal{Y})} \to \mathbb{Z}^{\mathcal{Y}}$. If in each row (resp. column) of $M_1 \in \mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$ (resp. $M_2 \in \mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Z})$) almost all entries are zero then M_1 (resp. M_2) induces a $\mathbb{Z}$-linear map $M_1\cdot : \mathbb{Z}^{\mathcal{Y}} \to \mathbb{Z}^{\mathcal{Y}}$ (resp. $M_2\cdot : \mathbb{Z}^{(\mathcal{Y})} \to \mathbb{Z}^{(\mathcal{Y})}$). Moreover, the composites $(M_1\cdot)\circ(M\cdot)$, $(M\cdot)\circ(M_2\cdot) : \mathbb{Z}^{(\mathcal{Y})} \to \mathbb{Z}^{\mathcal{Y}}$ can be represented as multiplications by $M_1 \cdot M$ and $M \cdot M_2$, respectively.

(iii) Statements analogous to (i) and (ii) also hold for $\mathbb{M}_{\mathcal{Y}\times\mathcal{Y}}(\mathbb{Q})$.

The result below was suggested to us by G. Zwara.

PROPOSITION. *Let $C = C(\Lambda)$ be the Cartan matrix.*

(a) *For any nonzero $D \in \mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Q})$ such that $D \cdot C$ is defined, $D \cdot C$ is a nonzero matrix. Moreover, C has at most one left inverse in $\mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Q})$.*

(b) *For any nonzero $D \in \mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Q})$ such that $C \cdot D$ is defined, $C \cdot D$ is a nonzero matrix. Moreover, C has at most one right inverse in $\mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Q})$.*

Proof. First we claim that for any nonzero $d = (d_X) \in \mathbb{Q}^{\mathcal{X}}$ such that $d^t \cdot C$ is defined, i.e. the set $\mathcal{X}_Y = \{X \in \mathcal{X} : d_X \neq 0 \neq c_{X,Y}\}$ is finite for all $Y \in \mathcal{X}$, the product $d^t \cdot C$ is nonzero.

We start by observing that if $d^t \cdot C$ is defined then d is in $\mathbb{Q}^{(\mathcal{X})}$; more precisely, $d_X = 0$ for every $X \in \mathcal{X}' = \mathcal{X} \setminus (\mathcal{X}_{P_1} \cup \cdots \cup \mathcal{X}_{P_n})$, where $P_1, \ldots, P_n$ are all the projective modules in $\mathcal{X}$. Suppose that $d_X \neq 0$ for some $X \in \mathcal{X}'$. Since $P_1 \oplus \cdots \oplus P_n$ is a projective generator in $\operatorname{mod} \Lambda$, there exists i such that $c_{X,P_i} = [P_i, X] \neq 0$, and X belongs to $\mathcal{X}_{P_i}$, a contradiction.

To prove our claim it suffices to show that $d = 0$ whenever $d \in \mathbb{Z}^{(\mathcal{X})}$ and $d^t \cdot C = 0$. Given such a d, we consider two finite sets: $\mathcal{X}^+ = \{X \in \mathcal{X} : d_X > 0\}$ and $\mathcal{X}^- = \{X \in \mathcal{X} : d_X < 0\}$. For any $Y \in \mathcal{X}$ we have

$$0 = (d^t \cdot C)_Y = \sum_{X\in\mathcal{X}} d_X c_{X,Y} = \sum_{X\in\mathcal{X}^+} d_X[Y,X] - \sum_{X\in\mathcal{X}^-} (-d_X)[Y,X]$$
$$= \Big[Y, \bigoplus_{X\in\mathcal{X}^+} X^{d_X}\Big] - \Big[Y, \bigoplus_{X\in\mathcal{X}^-} X^{(-d_X)}\Big] =: [Y, X^+] - [Y, X^-].$$

Consequently, by the result of Auslander [3, 5], we obtain $X^+ \cong X^-$, and $d = 0$.

Now the first assertion of (a) follows immediately from the above claim. To show the second, note that if $T \cdot C = \mathrm{I}_{\mathcal{X}} = T' \cdot C$ then the product $(T - T') \cdot C$ is defined and $(T - T') \cdot C = 0$. Then by the first assertion we have $T = T'$ and the proof of (a) is complete.

The proof of (b) is similar. ∎

2.2. Observe that if Λ is a directed algebra of finite representation type then $C(\Lambda)$ is invertible (Γ_Λ is finite and contains no oriented cycle, so $C(\Lambda)$ is triangular for a suitable ordering of $\mathcal{X}$). Now we show that $C(\Lambda)$ is always invertible in $\mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Q})$. We give a direct description of its inverse. The construction is effective if one knows the Auslander–Reiten quiver Γ_Λ of Λ.

From now on, we identify modules from $\mathcal{X}$ with vertices of Γ_Λ ($\mathcal{X} \ni X \leftrightarrow [X] \in \Gamma_0$). We can assume that for any nonprojective (resp. noninjective) $X \in \mathcal{X}$ the module τX (resp. $\tau^- X$) belongs to $\mathcal{X}$.

Definition. Let $T = T_\Lambda = [t_{X,Y}] \in \mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Z})$ be defined as follows: if $Y, X \in \mathcal{X}$ and X is nonprojective, then

$$t_{X,Y} = \begin{cases} 1 & \text{if } Y = X \neq \tau X \text{ or } Y = \tau X \neq X, \\ 2 & \text{if } Y = X = \tau X, \\ -d'_{Y,X} & \text{if } Y \in {}^-X, \\ 0 & \text{otherwise,} \end{cases}$$

while if X is projective, then

$$t_{X,Y} = \begin{cases} 1 & \text{if } Y = X, \\ -d'_{Y,X} & \text{if } Y \in {}^-X, \\ 0 & \text{otherwise.} \end{cases}$$

We call T_Λ the *Auslander–Reiten matrix* of Λ.

For any $X \in \mathcal{X}$, we set $f_X = \dim_k \operatorname{End}_\Lambda(X)/J(\operatorname{End}_\Lambda(X))$.

Theorem. *Let Λ be a finite-dimensional k-algebra, $C = C(\Lambda)$ the corresponding Cartan matrix, $T = T_\Lambda$ the Auslander–Reiten matrix of Λ and $F = \operatorname{diag}((f_X)_{X\in\mathcal{X}})$.*

(a) *C is invertible in $\mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Q})$ and $F^{-1}T$ is the unique two-sided inverse of C.*
(b) *If $\operatorname{End}_\Lambda(X)/J(\operatorname{End}_\Lambda(X)) \cong k$ for every $X \in \mathcal{X}$ (which holds automatically if k is algebraically closed), then T is an inverse of C, so C is invertible in $\mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Z})$, and $m(M) = T \cdot h(M)$ (resp. $T^t \cdot h'(M) = m(M)$) for all M in $\operatorname{mod}\Lambda$.*

Proof. (a) We show first that $TC = F$.

For any $X \in \mathcal{X}$ we have either the almost split sequence

$$0 \to \tau X \to \bigoplus_{Z\in{}^-X} Z^{d'_{Z,X}} \to X \to 0$$

if X is nonprojective, or the right minimal almost split sequence

$$\bigoplus_{Z\in{}^-X} Z^{d'_{Z,X}} \cong JX \hookrightarrow X$$

if X is projective. They induce respectively exact sequences

$$(*)\qquad 0\to(-,\tau X)\to\bigoplus_{Z\in{}^-X}(-,Z)^{d'_{Z,X}}\to(-,X)\to\overline{(-,X)}\to 0$$

and

$$(**)\qquad 0\to\bigoplus_{Z\in{}^-X}(-,Z)^{d'_{Z,X}}\to(-,X)\to\overline{(-,X)}\to 0$$

of contravariant k-functors from the category $\operatorname{mod}\Lambda$ to $\operatorname{mod}k$, where $\overline{(-,X)}$ denotes the simple functor $(-,X)/\mathcal{J}(-,X)$ (see [11]). Fix an arbitrary $Y\in\mathcal{X}$. Then applying the exact sequences $(*)(Y)$ and $(**)(Y)$, and computing dimensions, we obtain the formulas

$$c_{\tau X,Y}+c_{X,Y}-\sum_{Z\in{}^-X}d'_{Z,X}\,c_{Z,Y}=\begin{cases}f_X & \text{if } Y=X,\\ 0 & \text{if } Y\neq X,\end{cases}$$

and

$$c_{X,Y}-\sum_{Z\in{}^-X}d'_{Z,X}\,c_{Z,Y}=\begin{cases}f_X & \text{if } Y=X,\\ 0 & \text{if } Y\neq X,\end{cases}$$

respectively. Consequently, the required equality $TC=F$ holds.

To prove that C is invertible with inverse $F^{-1}T$, consider the matrix $T^-=(t^-_{X,Y})\in\mathbb{M}_{\mathcal{X}\times\mathcal{X}}(\mathbb{Z})$, dual to T in some sense, which is defined as follows:

$$t^-_{X,Y}=\begin{cases}1 & \text{if } X=Y\neq\tau^-Y \text{ or } X=\tau^-Y\neq Y,\\ 2 & \text{if } X=Y=\tau^-Y,\\ -d_{Y,X} & \text{if } X\in Y^+,\\ 0 & \text{otherwise,}\end{cases}$$

if Y is noninjective, and

$$t^-_{X,Y}=\begin{cases}1 & \text{if } X=Y,\\ -d_{Y,X} & \text{if } X\in Y^+,\\ 0 & \text{otherwise,}\end{cases}$$

if X is injective, for $Y,X\in\mathcal{X}$. We show that $CT^-=F$.

As before, for $Y\in\mathcal{X}$ we have either the almost split sequence

$$0\to Y\to\bigoplus_{Z\in Y^+}Z^{d_{Y,Z}}\to\tau^-Y\to 0$$

if X is noninjective, or the left minimal almost split sequence

$$Y\twoheadrightarrow Y/\operatorname{Soc}Y\cong\bigoplus_{Z\in Y^+}Z^{d_{Y,Z}}$$

if Y is injective. Again we obtain exact sequences

$$(*)' \qquad 0 \to (\tau^- Y, -) \to \bigoplus_{Z \in Y^+} (Z, -)^{d_{Y,Z}} \to (Y, -) \to \overline{(Y, -)} \to 0$$

and respectively

$$(**)' \qquad 0 \to \bigoplus_{Z \in Y^+} (Z, -)^{d_{Y,Z}} \to (Y, -) \to \overline{(Y, -)} \to 0$$

of covariant k-functors from $\operatorname{mod} \Lambda$ to $\operatorname{mod} \Lambda$, where $\overline{(Y, -)}$ denotes the simple functor $(Y, -)/\mathcal{J}(Y, -)$ (see [11]). Then for any $X \in \mathcal{X}$ the exact sequences $(*)'(X)$ and $(**)'(X)$ yield the formulas

$$c_{X,\tau^- Y} + c_{X,Y} - \sum_{Z \in Y^+} c_{X,Z}\, d_{Y,Z} = \begin{cases} f_X & \text{if } X = Y, \\ 0 & \text{if } X \neq Y, \end{cases}$$

and

$$c_{X,Y} - \sum_{Z \in Y^+} c_{X,Z}\, d_{Y,Z} = \begin{cases} f_X & \text{if } X = Y, \\ 0 & \text{if } X \neq Y, \end{cases}$$

respectively. Consequently, we have $CT^- = F$.

To complete the proof of (a), we observe that the matrices $\overline{T}^- = T^- F^{-1}$, C and $\overline{T} = F^{-1} T$ induce $\mathbb{Q}$-linear maps: $\overline{T}^- \cdot : \mathbb{Q}^{(\mathcal{X})} \to \mathbb{Q}^{(\mathcal{X})}$, $C \cdot : \mathbb{Q}^{(\mathcal{X})} \to \mathbb{Q}^{\mathcal{X}}$ and $\overline{T} \cdot : \mathbb{Q}^{\mathcal{X}} \to \mathbb{Q}^{\mathcal{X}}$, respectively. Now, by applying the associativity of composition (for $(\overline{T}\cdot) \circ (C\cdot) \circ (\overline{T}^-)$), the two equalities proved above and the rules from Remark 2.1(ii), we infer that the maps $\overline{T}^- \cdot, \overline{T} \cdot : \mathbb{Q}^{(\mathcal{X})} \to \mathbb{Q}^{\mathcal{X}}$ are equal, hence $T^- F^{-1} = F^{-1} T$, and $F^{-1} T$ is the unique (by Proposition 2.1) inverse of C in $\mathbb{M}_{\mathcal{X} \times \mathcal{X}}(\mathbb{Q})$.

Since (b) is a consequence of (a) and of Corollary 2.1, the theorem is proved. ∎

COROLLARY. *If Λ is of finite representation type then the Cartan matrix $C = C(\Lambda)$ is uniquely determined by the formula $C = T^{-1} F$, where $T = T_\Lambda$ and F are as above. In particular, $C = T^{-1}$ if k is algebraically closed (cf. Remark 2.1(i)).*

REMARK. (i) The equality $T^- F^{-1} = F^{-1} T$ implies immediately that in each row and each column of T (resp. T^-) almost all entries are zero, and also that $f_Y d_{X,Y} = f_X d'_{X,Y}$ for all $X, Y \in \mathcal{X}$, and $f_X = f_{\tau X}$ for every nonprojective $X \in \mathcal{X}$.

(ii) Let $\mathcal{X}' \subseteq \mathcal{X}$ be the subclass of all vertices of a fixed connected component in Γ_Λ. Then $m(M)_{|\mathcal{X}'} = T_{|\mathcal{X}' \times \mathcal{X}'} \cdot h(M)_{|\mathcal{X}'}$ and $m(M)_{|\mathcal{X}'} = (T^-{}_{|\mathcal{X}' \times \mathcal{X}'})^t \cdot h'(M)_{|\mathcal{X}'}$ for any M in $\operatorname{mod} \Lambda$, provided $f_X = 1$ for all $X \in \mathcal{X}'$, where $m(M)_{|\mathcal{X}'} = (m_X(M))_{X \in \mathcal{X}'}$, $h(M)_{|\mathcal{X}'} = (h_X(M))_{X \in \mathcal{X}'}$ and $h'(M)_{|\mathcal{X}'} = (h'_X(M))_{X \in \mathcal{X}'}$. In particular, for any $X \in \mathcal{X}'$ we have $m(M)_X = (T_{|\mathcal{X}' \times \mathcal{X}'})_X \cdot h(M)_{|\mathcal{X}'}$ (resp. $m(M)_X = (h'(M)_{|\mathcal{X}'})^t \cdot (T^-{}_{|\mathcal{X}' \times \mathcal{X}'})^X$).

(iii) If $\mathcal{X} = \bigcup_{i\in I} \mathcal{X}_i$ is a splitting of $\mathcal{X}$ corresponding to the decomposition of Γ_Λ into a disjoint union of connected components then $T = \operatorname{diag}(\{T_{|\mathcal{X}_i\times\mathcal{X}_i}\}_{i\in I})$ and $T^- = \operatorname{diag}(\{T^-{}_{|\mathcal{X}_i\times\mathcal{X}_i}\}_{i\in I})$ (here "diag" denotes a block diagonal matrix). In particular, $T_{|\mathcal{X}_i\times\mathcal{X}_i} = T^-{}_{|\mathcal{X}_i\times\mathcal{X}_i}$ if $f_X = 1$ for all $X \in \mathcal{X}_i$.

(iv) Assume that $f_X = 1$ for all $X \in \mathcal{X}$, and $|I| \geq 2$, where I is as above. Then the matrix C, in spite of the equalities $CT = TC = \mathrm{I}_{\mathcal{X}}$ and $T = \operatorname{diag}(\{T_{|\mathcal{X}_i\times\mathcal{X}_i}\}_{i\in I})$, always differs from $\operatorname{diag}(\{C_{|\mathcal{X}_i\times\mathcal{X}_i}\}_{i\in I})$, since Λ is of infinite representation type and $\mathcal{J}_\Lambda^\infty(\mathcal{X}_i, \mathcal{X}_j) \neq 0$ for some $i \neq j$, so $C_{|\mathcal{X}_j\times\mathcal{X}_i} \neq 0$. Nevertheless, we clearly have $C_{|\mathcal{X}_i\times\mathcal{X}_i} \cdot T_{|\mathcal{X}_i\times\mathcal{X}_i} = T_{|\mathcal{X}_i\times\mathcal{X}_i} \cdot C_{|\mathcal{X}_i\times\mathcal{X}_i} = \mathrm{I}_{\mathcal{X}_i}$ for every $i \in I$, so $\operatorname{diag}(\{C_{|\mathcal{X}_i\times\mathcal{X}_i}\}_{i\in I})$ forms another, different from C, inverse for T. As a simplest concrete example of this strange behaviour one can consider the matrices T and C for a Kronecker algebra (see Section 4 for a detailed description of T and C in block form).

(v) In case Λ is of finite representation type and k is algebraically closed, the formula from the corollary provides a method of finding the matrix C if one knows the shape of Γ_Λ, and conversely, the matrix T if one knows C.

From now on we assume that k is an algebraically closed field.

2.3. As a conclusion from Theorem 2.2 and Corollary 2.1 it follows that to determine the multiplicity vector $m(M) = (m_X) \in \mathbb{Z}^{(\mathcal{X})}$ of an arbitrary module M in $\operatorname{mod}\Lambda$, one has to construct the Auslander–Reiten matrix T_Λ for Λ and compute the infinite vector $h(M) = (h_X) \in \mathbb{Z}^{\mathcal{X}}$ (resp. $h'(M) = (h'_X) \in \mathbb{Z}^{\mathcal{X}}$). In fact, once we know these two data the coordinates of the vector $m(M)$ can be computed as follows.

Corollary. *For any $X \in \mathcal{X}$, we have*

$$m_X = \begin{cases} h_X + h_{\tau X} - \sum\limits_{Y\in {}^-X} d'_{Y,X}\, h_Y & \text{if } X \text{ is nonprojective},\\ h_X - \sum\limits_{Y\in {}^-X} d'_{Y,X}\, h_Y & \text{if } X \text{ is projective}, \end{cases}$$

and

$$m_X = \begin{cases} h'_X + h'_{\tau^- X} - \sum\limits_{Y\in X^+} d_{Y,X}\, h'_Y & \text{if } X \text{ is noninjective},\\ h'_X - \sum\limits_{Y\in X^+} d_{Y,X}\, h'_Y & \text{if } X \text{ is injective.} \end{cases}$$

Remark. To compute $m(M)$ we do not need to know the entire, usually infinite, vector $h(M)$ ($m(M) \in \mathbb{Z}^{(\mathcal{X})}$). If we are able to find a finite subset $\mathcal{X}_0 \subset \mathcal{X}$ containing $\{X \in \mathcal{X} : m_X \neq 0\}$ then we need to compute the coordinates h_X for X from some finite subset $\mathcal{X}_1 \subseteq \mathcal{X}$ containing $\mathcal{X}_0$, which can be effectively constructed from $\mathcal{X}_0$. If Λ is representation finite then we can always set $\mathcal{X}_0 = \mathcal{X}$.

3. String algebras of finite representation type. If Λ is an algebra of finite representation type then in some situations it may be easier to determine the matrix $C = C(\Lambda)$ (than T_Λ itself), and afterwards compute T_Λ as its inverse (in case Λ is representation infinite this may be difficult, since C is then an infinite matrix). This is the case for string algebras of finite representation type, though combinatorial formulas for Auslander–Reiten sequences in $\operatorname{mod}\Lambda$ (expressed in terms of V-sequences) are known ([22]). In this section we briefly describe how to compute the matrices C in this case. First we consider the tree case and next the general case.

3.1. Recall that an algebra Λ is called a *string special biserial algebra* (or simply a *string algebra*) if it has the form kQ/I, where I is an admissible ideal in kQ, and the bound quiver (Q, I) satisfies the following conditions:

- (S1) The numbers of arrows starting from, respectively ending in, any fixed vertex of Q are bounded by 2.
- (S2) For any arrow α of Q, there is at most one arrow β and at most one arrow γ such that $\beta\alpha$ and $\alpha\gamma$ are not in I.
- (S3) The ideal I is generated by zero-relations.

To describe indecomposable modules over a string algebra $\Lambda = kQ/I$ one uses some special walks. Following [22], a walk w in the quiver Q is called a *V-sequence* in (Q, I) if for any oriented path u such that u or u^{-1} is a subpath of w, u does not belong to I. A V-sequence w is *primitive* if for any $n \in \mathbb{N}$ the composite walk w^n is defined and w^n is again a V-sequence in (Q, I). Denote by $\mathcal{V} = \mathcal{V}(Q, I)$ the set of all V-sequences in (Q, I) and by $\mathcal{V}_0 = \mathcal{V}_0(Q, I)$ a fixed selection of representatives of sets $\{w, w^{-1}\}$ (these sets define a splitting of $\mathcal{V}$). Note that if Q is a tree then the indecomposable kQ-module $V(w)$ defined by any walk $w \in \mathcal{V}$ (see 1.2) belongs to $\operatorname{mod}\Lambda$, since $V(w)$ is annihilated by I.

Let $(\widetilde{Q}, \widetilde{I})$ be a universal cover of the bound quiver (Q, I) in the sense of [17] (in our situation $\widetilde{Q}$ is a universal cover of Q, so a tree). We denote by $\mathrm{F} : (\widetilde{Q}, \widetilde{I}) \to (Q, I)$ the canonical Galois covering of bound quivers, defined by passing to orbits under the identification $Q = \widetilde{Q}/G$, where G is the fundamental group of Q acting on $\widetilde{Q}$ in the usual way. Denote by $\widetilde{\Lambda}$ the locally bounded k-category $k\widetilde{Q}/\widetilde{I}$ which is a factor of the path category $k\widetilde{Q}$ of $\widetilde{Q}$ modulo the ideal $\widetilde{I}$. We have at our disposal the induced pull-up functor $\mathrm{F}_\bullet : \operatorname{mod}\Lambda \to \operatorname{Mod}\widetilde{\Lambda}$ and its left (and right) adjoint, the pull-down functor F_λ: $\operatorname{mod}\widetilde{\Lambda} \to \operatorname{mod}\Lambda$, which, due to the fact that G acts freely on indecomposables, have nice properties (see [12]). (Here $\operatorname{Mod}\widetilde{\Lambda}$ denotes the category of locally finite-dimensional $\widetilde{\Lambda}$-modules, see [9].) It is clear that the notions of walk, V-sequence and line module, introduced earlier, can be extended to the situation of $\widetilde{Q}$, $(\widetilde{Q}, \widetilde{I})$ and $\widetilde{\Lambda}$, respectively.

For any V-sequence $w \in \mathcal{V} = \mathcal{V}(Q, I)$ we denote by $X(w)$ the indecomposable Λ-module of the form $\mathrm{F}_\lambda(V(\widetilde{w}))$, where $\widetilde{w}$ is a fixed lifting of w to $\widetilde{Q}$ (clearly $\widetilde{w} \in \widetilde{\mathcal{V}} = \mathcal{V}(\widetilde{Q}, \widetilde{I})$). By the properties of F_λ the definition of $X(w)$ does not depend on the choice of the lifting $\widetilde{w}$ (for two liftings $\widetilde{w}$, $\widetilde{w}'$ of w we have $\widetilde{w} = g\widetilde{w}'$, so $V(\widetilde{w}) = {}^gV(\widetilde{w}')$ for some $g \in G$, hence $\mathrm{F}_\lambda(V(\widetilde{w})) = \mathrm{F}_\lambda(V(\widetilde{w}')))$. Notice that $X(w)$ is indecomposable since so is every line. Clearly, $X(w) = V(w)$ if Q is a tree, since then $\widetilde{Q} = Q$ and $\mathcal{V} = \widetilde{\mathcal{V}}$.

Following [22], we have the following characterization of representation finite string algebras.

Proposition. *Let $\Lambda = kQ/I$ be an arbitrary string algebra. The algebra Λ is of finite representation type if and only if (Q, I) admits no primitive V-sequence. In this case, the set*

$$\mathcal{X} = \{X(w) : w \in \mathcal{V}_0\}$$

is a complete family of pairwise nonisomorphic indecomposable Λ-modules.

Corollary. *If $\Lambda = kQ/I$ is a string algebra such that Q is a tree* (*we call such algebras string tree algebras*), *then Λ is of finite representation type.*

3.2. Now we show how to construct the Cartan matrix

$$C(\Lambda) = C = [c_{X,Y}]_{X,Y \in \mathcal{X}} = [c_{v,w}]_{v,w \in \mathcal{V}_0}$$

for a string tree algebra $\Lambda = kQ/I$. First we state some technical facts. We set $S_{w,v} = \operatorname{supp} X(w) \cap \operatorname{supp} X(v)$. Note that for any $w \in \mathcal{V}$, the support $\operatorname{supp} X(w)$ coincides with the set S_w consisting of all vertices belonging to the walk w.

Lemma. *For any $v, w \in \mathcal{V}_0$ the following hold true*:

(i) $c_{v,w} \leq 1$,
(ii) $c_{v,v} = 1$,
(iii) *if* $S_{w,v} = \emptyset$ *then* $c_{w,v} = c_{v,w} = 0$,
(iv) *if* $S_{w,v} \neq \emptyset$ *then any Λ-homomorphism* $f = \{f_c\}_{c \in Q_0} : X(w) \to X(v)$ *such that $f_s = 0$ for some $s \in S_{w,v}$ is zero*,
(v) *if* $c_{w,v} = 1$ *for some* $v \neq w$, *then* $c_{v,w} = 0$.

Proof. Assertions (i)–(iv) follow easily from the definition of homomorphism and the facts that for any $w \in \mathcal{V}$ the k-spaces corresponding to vertices in the representation $X(w) = V(w)$ are k or 0 and that the full subquiver of Q formed by the support of S_w is connected.

To prove (v), note that Λ is of finite representation type and k is algebraically closed, so there exist oriented paths from $X(v)$ to $X(w)$ and from $X(w)$ to $X(v)$ in the Auslander–Reiten quiver Γ_Λ, provided $c_{w,v} = c_{v,w} = 1$.

Consequently, Γ_Λ contains oriented cycles, a contradiction. Note that Γ_Λ is acyclic, since Q is a tree (see [2]). ∎

The result below furnishes a necessary and sufficient condition for the dimension $c_{v,w}$ to be 1.

PROPOSITION. *For $v, w \in \mathcal{V}_0$ such that $S_{v,w} \neq \emptyset$, $c_{v,w} = 1$ if and only if for any arrow $\alpha : a \to b \in Q_1$ the following two conditions are satisfied*:

(i) *If $a \in S_{v,w}$ and $b \in S_v$ then $b \in S_w$.*
(ii) *If $b \in S_{v,w}$ and $a \in S_w$ then $a \in S_v$.*

Proof. Observe first that if for some $\alpha \in Q_1$, (i) (or (ii)) is not satisfied then $c_{v,w} = 0$ from Lemma 3.2(iv). Assume now that (i) and (ii) are satisfied. Then it is easy to show that the collection $\{f_c\}_{c \in Q_0}$ given by the formula

$$f_c = \begin{cases} \mathrm{id}_k & \text{if } c \in S_{v,w}, \\ 0 & \text{if } c \in Q_0 \setminus S_{v,w}, \end{cases}$$

yields a (nonzero) Λ-homomorphism from $X(w)$ to $X(v)$. Consequently, by Lemma 3.2(i) we have $c_{v,w} = 1$. ∎

3.3. The proposition above yields an easy, purely combinatorial, method of computing the dimensions $c_{v,w}$. As a consequence, we can construct the Cartan matrix $C = C(\Lambda)$ using only the shape of the quiver (Q, I). We summarize our considerations by an algorithm.

ALGORITHM (computing $C(\Lambda)$ for string tree algebras).

Input: The set $\mathcal{V}_0 = \mathcal{V}_0(Q, I)$ of V-sequences in (Q, I) for a given string tree algebra $\Lambda = kQ/I$.

Output: The Cartan matrix $C = C(\Lambda)$.

```
for any w ∈ V_0 set c_{w,w} := 1;
for any distinct v, w ∈ V_0 do
  if c_{v,w} is not computed then do {
    if S_{v,w} = ∅ then set c_{v,w} := 0, c_{w,v} := 0;
    else do {
      for any a ∈ S_{v,w}, a → b ∈ Q_1, b ∈ S_v do
        if b ∉ S_w then {set c_{v,w} := 0; break;}
      if c_{v,w} is not computed then do
        for any b ∈ S_{v,w}, a → b ∈ Q_1, a ∈ S_w do
          if a ∉ S_v then {set c_{v,w} = 0; break;}
    }
    if c_{v,w} is not computed then set c_{v,w} := 1, c_{w,v} := 0;
  }
```

Observe that the matrix T_Λ can be easily computed as an inverse of C since all elements of the set $\mathcal{V}_0$ (see 3.1) can be viewed as a sequence $(w_1, \ldots, w_n)$ such that $[c_{w_i,w_j}]_{1 \le i,j \le n} \in \mathbb{M}_{n\times n}(k)$ is a triangular 0-1 matrix (cf. 2.1).

3.4. Now we consider general string algebras. Let $\Lambda = kQ/I$ be any string algebra of finite representation type and $\widetilde{\mathcal{V}}'_0 = \{\widetilde{w} : w \in \mathcal{V}_0\}$ a subset of $\widetilde{\mathcal{V}}_0$ formed by a fixed complete selection of liftings (see 3.1). For any $v, w \in \mathcal{V}_0$ we set $G_{v,w} = \{g \in G : S_{\widetilde{v}} \cap gS_{\widetilde{w}} \neq \emptyset\}$. Note that each $G_{v,w}$ is finite since the standard action of G on $\widetilde{Q}_0$ is free. Denote by $Q' = (Q'_0, Q'_1)$ the smallest connected (finite) subquiver of $\widetilde{Q}$ containing the finite set of all $x \in \widetilde{Q}_0$ that are visited by the walks $g\widetilde{w}$ for some $\widetilde{w} \in \widetilde{\mathcal{V}}'_0$ and $g \in \bigcup_{v \in \mathcal{V}_0} G_{v,w}$.

Denote by Λ' the algebra corresponding to the full subcategory of $\widetilde{\Lambda}$ formed by Q'_0. As Λ' is a string tree algebra, it is of finite representation type, and it has the form $\Lambda' = kQ'/I'$, where I' corresponds to the appropriate restriction of the ideal $\widetilde{I}$ in $k\widetilde{Q}$ to kQ'. We can assume that $\widetilde{\mathcal{V}}'_0 \subseteq \mathcal{V}'_0 = \mathcal{V}_0(Q', I')$. Clearly, $\mathcal{V}' = \mathcal{V}(Q', I') \subset \widetilde{\mathcal{V}}$. Set $C' = C(\Lambda') = [c'_{v',w'}]_{v',w' \in \mathcal{V}'_0}$. The following result yields a method for computing the matrix $C = C(\Lambda)$ in the form $C = [c_{v,w}]_{v,w \in \mathcal{V}_0}$, once we know C' (the latter can be computed by applying the algorithm above).

PROPOSITION. *For any $v, w \in \mathcal{V}_0$ the (w, v)th entry of $C = C(\Lambda)$ is*

$$c_{w,v} = \sum_{g \in G_{v,w}} c'_{g\widetilde{w},\widetilde{v}}.$$

Proof. For any $w' \in \mathcal{V}'_0$ let $V'(w')$ denote the line module in $\operatorname{mod} \Lambda'$ defined by w' (as opposed to the line $V(w')$ in $\operatorname{mod} \widetilde{\Lambda}$). It is easily seen that for any $v', w' \in \mathcal{V}'_0$ there exists a canonical k-isomorphism

$$\operatorname{Hom}_{\Lambda'}(V'(v'), V'(w')) \cong \operatorname{Hom}_{\widetilde{\Lambda}}(V(v'), V(w')).$$

Then for fixed $v, w \in \mathcal{V}_0$ we have

$$\begin{aligned}\operatorname{Hom}_{\Lambda}(X(v), X(w)) &\cong \bigoplus_{g \in G_{v,w}} \operatorname{Hom}_{\widetilde{\Lambda}}(V(\widetilde{v}), V(g\widetilde{w})) \\ &\cong \bigoplus_{g \in G_{v,w}} \operatorname{Hom}_{\Lambda'}(V'(\widetilde{v}), V'(g\widetilde{w}))\end{aligned}$$

(apply properties of the functor F_λ, see [12]). Consequently, taking dimensions we obtain the required equality. ∎

3.5. Applying Proposition 3.4 (in fact its proof) and Proposition 3.2 we formulate an analogue of Algorithm 3.3 for all string algebras of finite representation type. We use the notation introduced in 3.4.

ALGORITHM (computing $C(\Lambda)$ for string algebras of finite representation type).

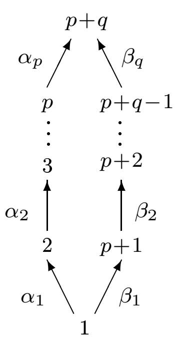
Input: The sets $\mathcal{V}_0 = \mathcal{V}_0(Q, I)$, $\widetilde{\mathcal{V}}'_0 = \{\widetilde{w} \in \widetilde{\mathcal{V}}_0 : w \in \mathcal{V}_0\}$ and $G_{v,w}$, $v, w \in \mathcal{V}_0$, for a given string algebra $\Lambda = kQ/I$ of finite representation type.

Output: The Cartan matrix $C = C(\Lambda)$.

```
for any $w, v \in \mathcal{V}_0$ do {
  set $c_{w,v} := 0$;
  for any $g \in G_{v,w}$ do {
    if $g = 1$, $\widetilde{w} = \widetilde{v}$ then set $c'_{g\widetilde{w},\widetilde{v}} := 1$;
    else if $c'_{g\widetilde{w},\widetilde{v}}$ is not computed then do {
      for any $a \in S_{g\widetilde{w},\widetilde{v}}$, $a \to b \in Q'_1$, $b \in S_{g\widetilde{w}}$ do
        if $b \notin S_{\widetilde{v}}$ then {set $c'_{g\widetilde{w},\widetilde{v}} := 0$; break;}
      if $c'_{g\widetilde{w},\widetilde{v}}$ is not computed then do
        for any $b \in S_{g\widetilde{w},\widetilde{v}}$, $a \to b \in Q'_1$, $a \in S_{\widetilde{v}}$ do
          if $a \notin S_{g\widetilde{w}}$ then {set $c'_{g\widetilde{w},\widetilde{v}} := 0$; break;}
      if $c'_{g\widetilde{w},\widetilde{v}}$ is not computed then set $c'_{g\widetilde{w},\widetilde{v}} := 1$, $c'_{\widetilde{v},g\widetilde{w}} := 0$;
    }
    set $c_{w,v} := c_{w,v} + c'_{g\widetilde{w},\widetilde{v}}$;
  }
}
```

Remark. One can give an algorithmic recursive method of computing the vectors $h(M)$ for M in $\operatorname{mod}\Lambda$, where Λ is a string algebra. It is based on a deep analysis of changes in the systems of linear equations describing the space $(M, X(w))$, $w \in \mathcal{V}_0$, under the process of extending walks w by arrows or their inverses. It has a technical and rather complicated character, and will be presented in a separate publication.

4. $\widetilde{\mathbb{A}}_{p,q}$-algebras: the Kronecker algebra case. In the next two sections we discuss how to apply the general method, outlined in Section 1, to modules over $\widetilde{\mathbb{A}}_{p,q}$-algebras, i.e. the path algebras of the quivers



where $p, q \geq 1$. These string algebras, in contrast to biserial trees, are of tame and so infinite representation type, and require a slightly different approach (see Remark 2.3). For a given M in $\operatorname{mod}\Lambda$, it may be hard to find the infinite vector $h(M)$. Moreover, we cannot sequentially compute all multiplicities $m(M)_X$, $X \in \mathcal{X}$, applying Corollary 2.3. We show how to extract effectively a finite set of potential indecomposable direct summands for M

and compute a finite number of coordinates of the vector $h(M)$, which are necessary to determine $m(M)$. (The description of T_Λ is not a problem in this case.)

First we consider the simplest case of $\widetilde{\mathbb{A}}_{1,1}$ (the Kronecker algebra) and give a precise description of the algorithm. This algorithm differs essentially from the classical one (see [13, 16]). In the next section we show how to deal, in spite of some differences, with the general $\widetilde{\mathbb{A}}_{p,q}$-algebra case, and to reduce it partially to the previous one.

4.1. Let Λ be the Kronecker algebra, i.e. $\Lambda = kQ$, where Q looks as follows:

$$1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$$

We denote by e_1, e_2 the idempotents corresponding to the vertices. We first distinguish three classes of indecomposable Λ-modules:

$$P_i\colon\quad k^{i-1} \overset{V_i}{\underset{W_i}{\rightrightarrows}} k^i, \qquad I_j\colon\quad k^j \overset{V_j^t}{\underset{W_j^t}{\rightrightarrows}} k^{j-1}, \qquad R_{\lambda,l}\colon\quad k^l \overset{J_l(\lambda)}{\underset{\mathrm{I}_l}{\rightrightarrows}} k^l,$$

where $i, j, l \geq 1$, $\lambda \in k$, $W_i, V_i \in \mathbb{M}_{i\times(i-1)}(k)$ are of the form

$$V_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad W_i = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

$J_l(\lambda) \in \mathbb{M}_{l\times l}(k)$ is an upper triangular Jordan block with eigenvalue λ and $\mathrm{I}_l \in \mathbb{M}_{l\times l}(k)$ the unit matrix. The representations from the sets $\mathcal{P} = \{P_i\}_{i\geq 1}$ and $\mathcal{I} = \{I_i\}_{i\geq 1}$ are called respectively postprojective and preinjective (P_1, P_2 are projective, I_1, I_2 injective, and P_1, I_1 simple, see [2]). All modules $\{R_{\lambda,l}\}_{\lambda\in k,\, l\geq 1}$ together with representations of the form

$$R_{\infty,n}\colon\ k^n \overset{\mathrm{I}_n}{\underset{J_n(0)}{\rightrightarrows}} k^n$$

for $n \geq 1$, are called *regular*. We set $\mathcal{R} = \{R_{\lambda,n}\}_{\lambda\in k\cup\{\infty\},\, n\geq 1}$.

We briefly list below those (well known, see e.g. [8, 21]) facts concerning the structure of the module category $\operatorname{mod}\Lambda$ that we use in this paper.

PROPOSITION.

(i) *The disjoint union $\mathcal{X} = \mathcal{P} \cup \mathcal{R} \cup \mathcal{I}$ is a complete family of pairwise nonisomorphic indecomposable Λ-modules.*

(ii) *For any* $P \in \mathcal{P}$, $R \in \mathcal{R}$, $I \in \mathcal{I}$ *we have*

$$(R, P) = (I, P) = (I, R) = 0.$$

(iii) *For any* $n_1, n_2 \geq 1$ *we have*

$$[P_{n_1}, P_{n_2}] = \begin{cases} n_2 - n_1 + 1 & \text{if } n_1 \leq n_2, \\ 0 & \text{if } n_1 > n_2, \end{cases}$$

$$[I_{n_1}, I_{n_2}] = \begin{cases} n_1 - n_2 + 1 & \text{if } n_1 \geq n_2, \\ 0 & \text{if } n_1 < n_2. \end{cases}$$

(iv) *For any* $n_1, n_2 \geq 1$ *and* $\lambda, \lambda_1, \lambda_2 \in k \cup \{\infty\}$, $\lambda_1 \neq \lambda_2$, *we have*

$$[R_{\lambda,n_1}, R_{\lambda,n_2}] = \min\{n_1, n_2\}, \qquad [R_{\lambda_1,n_1}, R_{\lambda_2,n_2}] = 0.$$

(v) *The classes* $\mathcal{P}$ *and* $\mathcal{I}$ *are connected components in* Γ_Λ. *More precisely, a minimal right* (*respectively, left*) *almost split map for a projective* (*respectively, injective*) *nonsimple module has the form* $0 \to P_1^2 \to P_2$ (*respectively,* $I_2 \to I_1^2 \to 0$); *moreover, the almost split sequences are of the form*

$$0 \to P_i \to P_{i+1}^2 \to P_{i+2} \to 0$$

and respectively

$$0 \to I_{j+2} \to I_{j+1}^2 \to I_j \to 0$$

for $i, j \geq 1$. *In particular,* $P_{2s-1} = \tau^{-(s-1)} P_1$ *and* $P_{2s} = \tau^{-(s-1)} P_2$ (*respectively,* $I_{2s-1} = \tau^{(s-1)} I_1$ *and* $I_{2s} = \tau^{(s-1)} I_2$) *for every* $s \geq 1$, *where* τ *denotes the Auslander–Reiten translate.*

(vi) *The class* $\mathcal{R}$ *is a* 1*-parameter family* $\{\mathcal{T}_\lambda\}_{\lambda \in k \cup \{\infty\}}$ *of connected components in* Γ_Λ, *where each* $\mathcal{T}_\lambda$ *is a rank* 1 *stable tube with vertices represented by* $\{R_{\lambda,n}\}_{n \geq 1}$. *More precisely, the almost split sequences in the tube* $\mathcal{T}_\lambda$, $\lambda \in k \cup \{\infty\}$, *have the form*

$$0 \to R_{\lambda,1} \to R_{\lambda,2} \to R_{\lambda,1} \to 0$$

and

$$0 \to R_{\lambda,i} \to R_{\lambda,i-1} \oplus R_{\lambda,i+1} \to R_{\lambda,i} \to 0$$

for all $i \geq 2$. *In particular,* $\tau^i R_{\lambda,n} = R_{\lambda,n}$ *for all* $i \in \mathbb{Z}$, $n \geq 1$ *and* $\lambda \in k \cup \{\infty\}$.

COROLLARY. *Consider the matrices* $T = T_\Lambda$ *and* $C = C(\Lambda)$ *as block matrices with respect to the splitting* $\mathcal{X} = \mathcal{P} \cup (\bigcup_{\lambda \in k \cup \{\infty\}} \mathcal{T}_\lambda) \cup \mathcal{I}$.

(a) *The nonzero block-coordinates for T look as follows*:

$$T_{|\mathcal{P}\times\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ 0 & 0 & 1 & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

$$T_{|\mathcal{T}_\lambda\times\mathcal{T}_\lambda} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & 0 & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ 0 & 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}, \quad \lambda \in k \cup \{\infty\},$$

$$T_{|\mathcal{I}\times\mathcal{I}} = (T_{|\mathcal{P}\times\mathcal{P}})^t.$$

(b) *The diagonal coordinate-blocks for the lower block-triangular matrix C look as follows*:

$$C_{|\mathcal{P}\times\mathcal{P}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & \cdots \\ 4 & 3 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$C_{|\mathcal{T}_\lambda\times\mathcal{T}_\lambda} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 2 & 2 & \cdots \\ 1 & 2 & 3 & 3 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \quad \lambda \in k \cup \{\infty\},$$

$$C_{|\mathcal{I}\times\mathcal{I}} = (C_{|\mathcal{P}\times\mathcal{P}})^t.$$

Moreover, $C_{|\mathcal{P}\times\mathcal{T}_\lambda} = C_{|\mathcal{P}\times\mathcal{I}} = C_{|\mathcal{T}_\lambda\times\mathcal{I}} = 0$ *for all* $\lambda \in k \cup \{\infty\}$.

4.2. To compute the multiplicity sequences $m(M) = (m_X)_{X\in\mathcal{X}}$, $\mathcal{X} = \mathcal{P} \cup \mathcal{R} \cup \mathcal{I}$, for modules M in $\operatorname{mod} \Lambda$, we can apply the following rules.

LEMMA. *For any* M *as above the following equalities hold*:

(i) $m_{P_1} = [M, P_1]$, $m_{P_2} = [M, P_2] - 2[M, P_1]$ *and* $m_{P_i} = [M, P_i] - 2[M, P_{i-1}] + [M, P_{i-2}]$ *for all* $i \geq 3$;

(ii) $m_{I_1} = [I_1, M]$, $m_{I_2} = [I_2, M] - 2[I_1, M]$ *and* $m_{I_j} = [I_j, M] - 2[I_{j-1}, M] + [I_{j-2}, M]$ *for all* $j \geq 3$;

(iii) $m_{R_{\lambda,1}} = 2[M, R_{\lambda,1}] - [M, R_{\lambda,2}]$ *and* $M_{R_{\lambda,l}} = [M, R_{\lambda,l}] - [M, R_{\lambda,l-1}] - [M, R_{\lambda,l+1}]$ *for all* $l \geq 2$, $\lambda \in k \cup \{\infty\}$.

Proof. The assertions follow easily from Corollary 2.3, Remark 2.2 and Proposition 4.1. ∎

By the lemma above, to determine the multiplicity vector $m(M)_X$ for all X from one Auslander–Reiten component, it suffices to compute "consecutively" the dimensions $[M, X]$ (resp. $[X, M]$), referring to the natural linear order in that component (note that in this case all components have such an order). We use this general observation to give an algorithmic procedure yielding the sequence $m(M)$. We show how to reduce the considerations to a finite number of components and potential direct summands contained in them. We also discuss the stop problem for the constructed algorithm. Finally, we propose an inductive method of effective computation of the consecutive dimensions for individual components.

4.3. Given a module, we apply the following technical fact to restrict the list of candidates for its indecomposable direct summands to a finite list.

LEMMA. *For any* $j, n \geq 1$ *and* $\lambda \in k \cup \{\infty\}$ *we have* $[P_j, R_{\lambda,n}] = n$. *In particular*,

$$C_{|\mathcal{T}_\lambda \times \mathcal{P}} = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 2 & 2 & 2 & \cdots \\ 3 & 3 & 3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Proof. Recall that the functor $\tau^- : \overline{\operatorname{mod}}\, \Lambda \to \underline{\operatorname{mod}}\, \Lambda$ establishes an equivalence between the full subcategories of Λ-modules without injective and respectively projective direct summands. Moreover, $R_{\lambda,n}$ is τ-invariant and $\operatorname{Hom}_\Lambda(e_i\Lambda, R) \simeq Re_i$. Therefore, for any $s, n \geq 1$ and $\lambda \in k \cup \{\infty\}$,

$$\begin{aligned}[P_{2s-1}, R_{\lambda,n}] &= [\tau^{-(s-1)} P_1, R_{\lambda,n}] = [\tau^{-(s-1)} P_1, \tau^{-(s-1)} R_{\lambda,n}] \\ &= [P_1, R_{\lambda,n}] = [e_2\Lambda, R_{\lambda,n}] = \dim R_{\lambda,n} e_2 = n,\end{aligned}$$

and analogously

$$[P_{2s}, R_{\lambda,n}] = [P_2, R_{\lambda,n}] = [e_1\Lambda, R_{\lambda,n}] = \dim R_{\lambda,n} e_1 = n. \ ∎$$

COROLLARY. *Let* P *be a* Λ*-module such that* $P \simeq \bigoplus_{i=1}^{n_P} P_i^{s_i}$ *for some* $n_P, s_1, \ldots, s_{n_P} \in \mathbb{N}$. *Then* $[P, R_{\lambda,n}] = \sum_{i=1}^{n_P} s_i n$ *for any* $\lambda \in k$ *and* $n \geq 1$.

REMARK. The remaining nonzero coordinate-blocks of the lower-triangular matrix $C = C(\Lambda)$ look as follows:

$$C_{|\mathcal{I}\times\mathcal{T}_\lambda} = (C_{|\mathcal{T}_\lambda\times\mathcal{P}})^t, \qquad C_{|\mathcal{I}\times\mathcal{P}} = \begin{bmatrix} 0 & 1 & 2 & \cdots \\ 1 & 2 & 3 & \cdots \\ 2 & 3 & 4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(cf. Corollary 4.1 and Lemma 4.3). The first formula is straightforward. To prove the second, one shows that $[P_i, I_j] = i + j - 2$ for all $i, j \geq 1$ (apply arguments similar to those in the proof above).

4.4. Now we formulate a necessary and sufficient condition for a module from the tube $\mathcal{T}_\lambda$, $\lambda \in k$, to be a direct summand of a given Λ-module.

PROPOSITION. *Let $M : k^{n_1} \overset{A}{\underset{B}{\rightrightarrows}} k^{n_2}$ be a finite-dimensional Λ-module, where $A, B \in \mathbb{M}_{n_2\times n_1}(k)$, $n_1, n_2 \geq 1$. The module $R_{\lambda_0,n}$, $\lambda_0 \in k$, is a direct summand of M, for some $n \geq 1$, if λ_0 is a common root of all $(n_2 - \sum_{i=1}^{n_P} s_i)$-minors of the matrix $A - \lambda B$, regarded as polynomials from $k[\lambda]$, where $P_1^{s_1} \oplus \cdots \oplus P_{n_P}^{s_{n_P}}$ is a maximal postprojective direct summand of M.*

Proof. We can assume that M has the form $M = P \oplus R \oplus I$, where $P \in \operatorname{add}\mathcal{P}$, $R \in \operatorname{add}\mathcal{R}$, $I \in \operatorname{add}\mathcal{I}$ and $P = \bigoplus_{i=1}^{n_P} P_i^{s_i}$. Fix $\lambda_0 \in k$ and assume that for some $n \geq 1$, $R \simeq R_{\lambda_0,n} \oplus R'$ for some R'. Then by Corollary 4.3 and Proposition 4.1 we have

$$[M, R_{\lambda_0,1}] = [P, R_{\lambda_0,1}] + [R_{\lambda_0,n}, R_{\lambda_0,1}] + [R', R_{\lambda_0,1}] + [I, R_{\lambda_0,1}] = \sum_{i=1}^{n_P} s_i + 1 + x$$

for some $x \in \mathbb{N}$ (note that x is strictly positive exactly when R' contains a direct summand from $\mathcal{T}_{\lambda_0}$). Conversely, if $R_{\lambda_0,n}$ is not a direct summand of M for any $n \geq 1$, then clearly $[M, R_{\lambda_0,1}] = \sum_{i=1}^{n_P} s_i$. Consequently, the inequality $[M, R_{\lambda_0,1}] \geq \sum_{i=1}^{n_P} s_i$ always holds and it is strict if and only if $R_{\lambda_0,n}$ is a direct summand of M for some $n \geq 1$.

Now we estimate the dimension of $\operatorname{Hom}_\Lambda(M, R_{\lambda_0,1})$. Note that any $f \in (M, R_{\lambda_0,1})$ is a pair $(x, y) \in \mathbb{M}_{1\times n_1}(k) \times \mathbb{M}_{1\times n_2}(k)$, satisfying the system

$$\begin{cases} yA = \lambda_0 x, \\ yB = x, \end{cases}$$

of linear equations, or the equivalent one

$$\begin{cases} yB = x, \\ y(A - \lambda_0 B) = 0. \end{cases}$$

Therefore we have

$$(*) \qquad [M, R_{\lambda_0,1}] = n_2 - \mathrm{r}(A - \lambda_0 B)$$

(x is determined by y and $[M, R_{\lambda_0,1}]$ is equal to the dimension of the solution space of $y(A-\lambda_0 B) = 0$). Since $[M, R_{\lambda_0,1}] \geq \sum_{i=1}^{n_P} s_i$, we have $\mathrm{r}(A-\lambda_0 B) \leq n_2 - \sum_{i=1}^{n_P} s_i$. The last inequality is strict if and only if all $(n_2 - \sum_{i=1}^{n_P} s_i)$-minors of $A - \lambda_0 B$ are zero, and the proof is finished. ∎

Let $\{\lambda_1, \ldots, \lambda_t\} \subset k$, $t \geq 0$, be the set of all $\lambda \in k$ such that M contains a direct summand from the tube $\mathcal{T}_\lambda$ (we can determine this set by applying the proposition). The fact below shows how to compute the number of summands of M in $\mathcal{T}_{\lambda_i}$, knowing the rank of the matrix $A - \lambda_i B$.

Corollary. *Given $i \in \{1, \ldots, t\}$ denote by j_i the number of indecomposable direct summands of M from the tube $\mathcal{T}_{\lambda_i}$. Then $j_i = n_2 - \mathrm{r}(A-\lambda_i B) - \sum_{l=1}^{n_P} s_l$.*

Proof. Fix $i \in \{1, \ldots, t\}$. Let $R_{\lambda_i,m_1}, \ldots, R_{\lambda_i,m_{j_i}}$, $m_1, \ldots, m_{j_i} \geq 1$, be a complete list of indecomposable direct summands of M in $\mathcal{T}_{\lambda_i}$, i.e. $R \simeq R_{\lambda_i,m_1} \oplus \cdots \oplus R_{\lambda_i,m_{j_i}} \oplus R'$, where R' contains no direct summands from $\mathcal{T}_{\lambda_i}$. Then by Corollary 4.3 and Proposition 4.1 we have

$$\begin{aligned}[M, R_{\lambda_i,1}] &= [P, R_{\lambda_i,1}] + [R_{\lambda_i,m_1}, R_{\lambda_i,1}] + \cdots + [R_{\lambda_i,m_{j_i}}, R_{\lambda_i,1}] \\ &\quad + [R', R_{\lambda_i,1}] + [I, R_{\lambda_i,1}] \\ &= \sum_{l=1}^{n_P} s_l + 1 + \cdots + 1 + 0 + 0 = \sum_{l=1}^{n_P} s_l + j_i.\end{aligned}$$

Now applying the equality $(*)$ from the previous proof, we immediately obtain $j_i = n_2 - \mathrm{r}(A - \lambda_i B) - \sum_{l=1}^{n_P} s_l$. ∎

4.5. Now we summarize our previous considerations and present consecutive steps of algorithm whose task is to determine the full mutiplicity sequence $m(M)$, for a given Λ-module M, if we know "sufficiently many" coordinates of the vector $h(M)$ (resp. $h'(M)$).

Algorithm (the Kronecker algebra case).

Input: A Λ-module M in the form

$$M: \qquad k^{n_1} \underset{B}{\overset{A}{\rightrightarrows}} k^{n_2}$$

Output: The integers

$$n_P, n_I, s_1, \ldots, s_{n_P}, t_1, \ldots, t_{n_I} \geq 0;$$
$$s \geq 0, \qquad m_1, \ldots, m_s \geq 1;$$
$$a_1^1, \ldots, a_{m_1}^1, a_1^2, \ldots, a_{m_2}^2, \ldots, a_1^s, \ldots a_{m_s}^s \geq 0;$$
$$m_\infty, b_1, \ldots, b_{m_\infty} \geq 0,$$

and elements $\lambda_1, \dots, \lambda_s \in k$ such that $M \simeq P \oplus R \oplus I$, where $P = \bigoplus_{i=1}^{n_P} P_i^{s_i}$, $I = \bigoplus_{i=1}^{n_I} I_i^{t_i}$ and $R = (\bigoplus_{i=1}^{s} \bigoplus_{j=1}^{m_i} R_{\lambda_i,j}^{a_j^i}) \oplus (\bigoplus_{l=1}^{m_\infty} R_{\infty,l}^{b_l})$.

(1) *Determining the multiplicity vector for a postprojective component*:

$\texttt{set}$ $s_1 := m_{P_1} = [M, P_1]$
$\quad s_2 := m_{P_2} = [M, P_2] - 2[M, P_1]$
$\quad n := 3$
$\texttt{while}$ $\sum_{i=1}^{n-1} s_i(i-1) + (n-1) \le n_1$ $\texttt{and}$ $\sum_{i=1}^{n-1} s_i i + n \le n_2$ $\texttt{do}$ $\{$
$\quad s_n := m_{P_n} = [M, P_n] - 2[M, P_{n-1}] + [M, P_{n-2}]$
$\quad n := n + 1$
$\}$
$\texttt{set}$
$\quad n_P := \texttt{max}\{i : i = 1, \dots, n-1,\ s_i \neq 0\}$
$\quad n_1' := n_1 - \sum_{i=1}^{n_P} s_i(i-1)$
$\quad n_2' := n_2 - \sum_{i=1}^{n_P} s_i i$

(2) *Determining the multiplicity vector for a preinjective component*:

$\texttt{set}$
$\quad t_1 := m_{I_1} = [I_1, M]$
$\quad t_2 := m_{I_2} = [I_2, M] - 2[I_1, M]$
$\quad n := 3$
$\texttt{while}$ $\sum_{i=1}^{n-1} t_i i + n \le n_1'$ $\texttt{and}$ $\sum_{i=1}^{n-1} t_i(i-1) + (n-1) \le n_2'$ $\texttt{do}$ $\{$
$\quad t_n := m_{I_n} = [I_n, M] + 2[I_{n-1}, M] + [I_{n-2}, M]$
$\quad n := n + 1$
$\}$
$\texttt{set}$
$\quad n_I := \texttt{max}\{i : i = 1, \dots, n-1,\ t_i \neq 0\}$

(3) *Determining the multiplicity vector for a regular component $\mathcal{T}_\lambda$ with $\lambda \neq \infty$*: Let $\{\lambda_1, \dots, \lambda_s\} \subset k$ be all common roots of $(n_2 - \sum_{i=1}^{n_P} s_i)$-minors of the matrix $A - \lambda B$ treated as polynomials from $k[\lambda]$ (see Corollary 4.4).

$\texttt{for}$ $i = 1, \dots, s$ $\texttt{do}$
$\quad j_i := n_2 - \mathrm{r}(A - \lambda_i B) - \sum_{l=1}^{n_P} s_l$
$\texttt{for}$ $i = 1, \dots, s$ $\texttt{do}$ $\{$
$\quad a_1^i := m_{R_{\lambda_i,1}} = 2[M, R_{\lambda_i,1}] - [M, R_{\lambda_i,2}]$
$\quad n := 2$
$\quad \texttt{while}$ $\sum_{l=1}^{n-1} a_l^i < j_i$ $\texttt{do}$ $\{$
$\qquad a_n^i := m_{R_{\lambda_i,n}} = 2[M, R_{\lambda_i,n}] - [M, R_{\lambda_1,n-1}] - [M, R_{\lambda_i,n+1}]$
$\qquad n := n + 1$
$\quad \}$
$\quad m_i := n - 1$
$\}$

(4) *Determining the multiplicity vector for a regular component* $\mathcal{T}_\lambda$ *with* $\lambda = \infty$:

```
set
```
$b_1 := m_{R_{\infty,1}} = 2[M, R_{\infty,1}] - [M, R_{\infty,2}]$

$n := 2$

$n_1' := n_1 - \sum_{i=1}^{n_P} s_i(i-1) - \sum_{i=1}^{n_I} t_i i - \sum_{i=1}^{s} \sum_{j=1}^{m_i} a_j^i j$

`while` $\sum_{i=1}^{n-1} b_i i < n_1'$

`do` $\{$

$b_n := m_{R_{\infty,n}} = 2[M, R_{\infty,n}] - [M, R_{\infty,n-1}] - [M, R_{\infty,n+1}]$

$n := n+1$

$\}$

$m_\infty := \mathtt{max}\{i : i = 1, \ldots, n-1,\ m_i > 0\}$ ■

Remark. (i) It is easily seen that the algorithm stops in each of the four steps. In steps (1) and (2) the index n increases in each execution of the loop. In step (3) the loop must stop when the sum of the multiplicities of the direct summands from $\mathcal{T}_{\lambda_i}$ reaches j_i (see Corollary 4.4), i.e. if $a_1^i + \cdots + a_{m_i}^i = j_i$ for some $m_i < \infty$. Finally, in step (4) the loop stops since $2n_1' = \dim_k M'$, where M' is the maximal direct summand of M formed by modules from $\mathcal{T}_\infty$, i.e. $n_1' = \sum_{i=1}^{m_\infty} b_i i$, for some $m_\infty < \infty$ (clearly under the assumption that the algorithm is correct).

(ii) The correctness of the algorithm follows from Lemma 4.2, Proposition 4.4 and Corollary 4.4. After stop of loops in steps (1) and (2) we obtain the multiplicities of all postprojective (respectively, preinjective) direct summands of M. A possible next run of any of these loops would test an indecomposable direct summand whose dimension is already greater than the codimension of the direct sum of all indecomposable summands detected up to that stage. This estimation is very imprecise in case $\dim_k M$ is much greater than $\dim_k P$. In steps (3) and (4) the situation is much better, the execution of loops stops immediately after detecting all the summands searched for.

The algorithm requires consecutive computations of the dimensions $[M, P_i]$ for $i \geq 1$ (and analogously for the remaining connected components). Generally, the complexity of the computations grows fast with increasing i. We show how to avoid full computations of $[M, P_i]$ in each step, reducing them to the already known result of computations for $[M, P_{i-1}]$ and some simple computational problem, depending only on the dimension vector of M and such that its complexity in each step is the same.

4.6. Now we describe an inductive method of computing the dimensions $[M, X]$ (resp. $[X, M]$) where X is of the form P_i, $R_{\lambda,i}$, $R_{\infty,i}$ (resp. I_i) for $i \in \mathbb{N}$. First, one has to translate this problem into the language of systems of linear equations.

LEMMA. *Let* $M : k^{n_1} \overset{A}{\underset{B}{\rightrightarrows}} k^{n_2}$ *be a finite-dimensional representation, where* $A, B \in \mathbb{M}_{n_2\times n_1}(k)$, $n_1, n_2 \geq 1$. *Then*

(i) $$[M, P_i] = in_2 - \mathrm{r}(M_P^i),$$

(ii) $$[I_i, M] = in_1 - \mathrm{r}(M_I^i),$$

(iii) $$[M, R_{\lambda,i}] = in_2 - \mathrm{r}(M_\lambda^i),$$

(iv) $$[M, R_{\infty,i}] = in_2 - \mathrm{r}(M_\infty^i),$$

for all $i \geq 1$ *and* $\lambda \in k$, *where* $M_P^i \in \mathbb{M}_{in_2\times(i+1)n_1}(k)$, $M_I^i \in \mathbb{M}_{in_1\times(i+1)n_2}(k)$, $M_\lambda^i \in \mathbb{M}_{in_2\times in_1}(k)$ *and* $M_\infty^i \in \mathbb{M}_{in_2\times in_1}(k)$ *are the following block matrices*:

$$M_P^i = \begin{bmatrix} -A & B & 0 & 0 & \cdots & 0 \\ 0 & -A & B & 0 & \cdots & 0 \\ 0 & 0 & -A & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -A & B \end{bmatrix},$$

$$M_I^i = \begin{bmatrix} -A^t & B^t & 0 & 0 & \cdots & 0 \\ 0 & -A^t & B^t & 0 & \cdots & 0 \\ 0 & 0 & -A^t & B^t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -A^t & B^t \end{bmatrix},$$

$$M_\lambda^i = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ -B & C & 0 & \cdots & 0 \\ 0 & -B & C & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -B & C \end{bmatrix}, \quad M_\infty^i = \begin{bmatrix} B & 0 & 0 & \cdots & 0 \\ -A & B & 0 & \cdots & 0 \\ 0 & -A & B & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -A & B \end{bmatrix},$$

where $C = C(\lambda) = A - \lambda B$.

Proof. We consider the case of postprojective indecomposables P_i. Formula (i) is clear for $i = 1$, since each $f : M \to P_1$ is given by $x \in \mathbb{M}_{1\times n_2}(k)$ satisfying the system $xA = 0 = xB$, or equivalently $x[A|B] = 0$. In the general case $i \geq 2$ the morphism $f : M \to P_i$ is a pair $(X, Y) \in \mathbb{M}_{(i-1)\times n_1}(k) \times \mathbb{M}_{i\times n_2}(k)$ of matrices, satisfying the system

(∗) $$\begin{cases} YA = V_i X, \\ YB = W_i X, \end{cases}$$

consisting of two subsystems $(*)_\alpha$ and $(*)_\beta$. Denote by $x_1, \ldots, x_{i-1}$ (respectively, $y_1,\dots,y_i$) the rows of X (respectively, Y). Then $(*)$ has the form

$$(*)' \qquad \begin{cases} y_1A &= x_1,\\ \vdots & \quad\vdots\\ y_{i-1}A &= x_{i-1},\\ y_iA &= 0,\\ y_1B &= 0,\\ y_2B &= x_1,\\ \vdots & \quad\vdots\\ y_iB &= x_{i-1},\end{cases}$$

and is equivalent to

$$(*)'' \qquad \begin{cases} y_1A &= x_1,\\ \vdots & \quad\vdots\\ y_{i-1}A &= x_{i-1},\\ y_iA &= 0,\\ y_1B &= 0,\\ y_2B - y_1A &= 0,\\ \vdots & \quad\vdots\\ y_iB - y_{i-1}A &= 0.\end{cases}$$

($(*)''$ is obtained from $(*)'$ by subtracting from the $(j+1)$th matrix equation of $(*)'_\beta$ the jth matrix equation of $(*)'_\alpha$, for every $j = 1,\dots,i-1$.) Let $(**)$ be obtained from $(*)''$ by dropping the first $i-1$ equations; as a block matrix equation it has the form

$$[y_n,\dots,y_1]\cdot M_P^i = 0.$$

Since the vectors $x_1,\dots,x_{i-1}$ are determined in $(*)''$ by $y_1,\dots,y_{i-1}$, the dimensions of the solution spaces for systems $(*)''$ and $(**)$ are the same, and consequently, we have (i).

It is easily seen that applying the standard duality $\mathrm{D} : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{\mathrm{op}}$ and (i) we immediately obtain (ii).

To compute the number $[M, R_{\lambda,i}]$, consider an arbitrary homomorphism $f : M \to R_{\lambda,i}$, given by a pair $(X,Y) \in \mathbb{M}_{i\times n_1}(k) \times \mathbb{M}_{i\times n_2}(k)$ of matrices satisfying the system

$$(***) \qquad \begin{cases} YA = J_i(\lambda)X,\\ YB = X,\end{cases}$$

consisting of two subsystems $(***)_\alpha$ and $(***)_\beta$. Denote by $x_1,\dots,x_i$ and

$y_1, \ldots, y_i$ the rows of X and Y, respectively. Then $(***)$ can be written in the form

$$(***)' \qquad \begin{cases} y_1 A &= \lambda x_1 + x_2, \\ \vdots & \quad \vdots \\ y_{i-1} A &= \lambda x_{i-1} + x_i, \\ y_i A &= \lambda x_i, \\ y_1 B &= x_1, \\ \vdots & \quad \vdots \\ y_i B &= x_i. \end{cases}$$

Now, we proceed as before. For every $j = 1, \ldots, i$, we subtract from the jth equation of $(***)'_\alpha$ the jth equation of $(***)'_\beta$ multiplied by λ, then we drop the last i equations and we arrive at the system

$$[y_1, \ldots, y_i] \cdot M^i_\lambda = 0$$

whose solution space has the same dimension as that of $(***)$. In this way we obtain (iii).

The last formula (iv) follows easily from (iii) (one has to exchange matrices A and B, for $\lambda = 0$). ∎

4.7. Finally, we briefly outline an inductive method of rank computation for matrices from Lemma 4.6. This method follows from their very specific form. We use the notation introduced in 1.3.

LEMMA.

(a) *Let* $\mathcal{N} = \mathcal{N}(A, B)$ *be a family of generalized upper triangular matrices* N_i, $i \in \mathbb{N}$, *defined inductively by setting* $N_1 = \widehat{[-A|B]}$ *and*

$$N_{i+1} = \begin{bmatrix} N^{(i)}_{11} & [N^{(i)}_{12}|0] \\ 0 & \widehat{U}_i \end{bmatrix}$$

where $N_i = \begin{bmatrix} N^{(i)}_{11} & N^{(i)}_{12} \\ N^{(i)}_{21} & N^{(i)}_{22} \end{bmatrix}$ *with maximal zero block* $N^{(i)}_{21}$ *containing* in_1 *columns and* $U_i = \begin{bmatrix} N^{(i)}_{22} & 0 \\ -A & B \end{bmatrix}$. *Then* $\mathrm{r}(N_{i+1}) = \mathrm{r}(N_i) + \mathrm{r}(\widehat{U}_i) - \mathrm{r}(N^{(i)}_{22})$ *and* $\mathrm{r}(M^i_P) = \mathrm{r}(N_i)$, *for every* $i \in \mathbb{N}$; *moreover*, $N^{(i)}_{22} = U^{(i-1)}_{22}$, *where* $\widehat{U}_i = \begin{bmatrix} U^{(i)}_{11} & U^{(i)}_{12} \\ U^{(i)}_{21} & U^{(i)}_{22} \end{bmatrix}$ *with maximal zero block* $U^{(i)}_{21}$ *containing* n_1 *columns.*

(b) *Let* $\mathcal{N} = \mathcal{N}(-C, -B)$ *be a family of matrices as above. Then* $\mathrm{r}(M^i_\lambda) = \mathrm{r}(N_{i-1}) - \mathrm{r}(N^{(i-1)}_{22}) + \mathrm{r}(\widehat{Z}_{i-1})$ *for every* $i \geq 2$, *where* $Z_{i-1} = \begin{bmatrix} N^{(i-1)}_{22} \\ C \end{bmatrix}$.

Proof. Assertion (a) follows from the detailed analysis of Gauss elimination. To show (b), note that $\mathrm{r}(M_\lambda^i) = \mathrm{r}\left(\begin{bmatrix} N_{i-1} \\ [0\,|C] \end{bmatrix}\right)$ and apply the arguments used in (a). ■

REMARK. (i) The matrices M_I^i and M_∞^i have respectively the same form as M_P^i and M_λ^i so their ranks can be computed analogously.

(ii) In Algorithm 4.5, for a given number d, we have to compute the dimensions $[M, P_i]$, or equivalently the ranks $\mathrm{r}(M_P^i)$ (see Lemma 4.6), for $i = 1, \dots, d$. The last lemma allows us to reduce the complexity of the above rank computations. They are realized in practice as a sequence of d Gauss eliminations for the matrices M_P^i of linearly increasing sizes $in_2 \times (i+1)n_1$ and now can be replaced by a sequence of d Gauss eliminations for matrices of size at most $2n_2 \times 2n_1$.

5. $\widetilde{\mathbb{A}}_{p,q}$-algebras: the general case. In this section we discuss the difference between the problem of determining the multiplicity vectors of general $\widetilde{\mathbb{A}}_{p,q}$-algebras and of the Kronecker algebra. In the general case we do not present the algorithm in detail, but rather indicate how to modify Algorithm 1.5 and how to reduce partially the problem to the previous one. To deal with indecomposables and handle certain homomorphism spaces in a more convenient way, we use some elements of the covering technique for string algebras (briefly outlined in 3.1).

5.1. Let $\Lambda = kQ$ be the path algebra of the quiver Q of type $\widetilde{\mathbb{A}}_{p,q}$ (see Section 4). The universal cover $\widetilde{Q}$ of Q is then an infinite quiver of the form

$$\cdots \xrightarrow{\alpha_1^n} \cdots \xrightarrow{\alpha_p^n} \xleftarrow{\beta_q^n} \cdots \xleftarrow{\beta_1^n} \xrightarrow{\alpha_1^{n+1}} \cdots \xrightarrow{\alpha_p^{n+1}} \xleftarrow{\beta_q^{n+1}} \cdots \xleftarrow{\beta_1^{n+1}} \cdots$$

for $n \in \mathbb{Z}$. The canonical Galois covering of bound quivers is in fact just an ordinary quiver morphism $F : \widetilde{Q} \to Q$ ($I = 0$!), given by the natural formulas $F(\alpha_i^n) = \alpha_i$ and $F(\beta_i^n) = \beta_i$ for $n \in \mathbb{Z}$, $i = 1, \dots, p$, $j = 1, \dots, q$. The fundamental group G of Q can be identified with $\mathbb{Z}$; under this identification the action of G on $\widetilde{Q}$ is given by $m \cdot \alpha_i^n = \alpha_i^{n+m}$ and $m \cdot \beta_i^n = \beta_i^{n+m}$ for $n, m \in \mathbb{Z}$. Clearly, we have $\Lambda = kQ$ and $\widetilde{\Lambda} = k\widetilde{Q}$.

Note that any walk in Q (resp. $\widetilde{Q}$) is a V-sequence in $(Q, 0)$ (resp. $(\widetilde{Q}, 0)$). Therefore to list all indecomposable Λ-modules of the form $X(w) = \mathrm{F}_\lambda(V(\widetilde{w}))$, $w \in \mathcal{V}_0$ (see 3.1), it suffices to write down all walks in Q consistent with a fixed, arbitrarily selected cycle orientation of the underlying unoriented graph for Q. We do this in some quite ordered and strictly prescribed way.

For this purpose, we fix some notation. For any $1 \le i < j \le p$ (resp. $1 \le s < t \le q$), we distinguish the walk $\alpha_{i,j} = \alpha_i \alpha_{i+1} \cdots \alpha_{j-1} \alpha_j$ (resp.

$\beta_{s,t}^{-1} = \beta_t^{-1}\beta_{t-1}^{-1}\cdots\beta_{s+1}^{-1}\beta_s^{-1}$) in the quiver Q. We also set $\alpha = \alpha_{1,p}$ and $\beta^{-1} = \beta_{1,q}^{-1}$.

First, consider the indecomposable modules $P_{0,p+q} = X(p+q)$, $P_{0,p+q-1} = X(\beta_q^{-1})$, $P_{0,p+q-2} = X(\beta_{q-1,q}^{-1})$, $\ldots$, $P_{0,p+1} = X(\beta_{2,q}^{-1})$, $P_{0,p} = X(\alpha_p)$, $P_{0,p-1} = X(\alpha_{p-1,p})$, $\ldots$, $P_{0,2} = X(\alpha_{2,p})$, and $P_{0,1} = X(\beta_{1,q}^{-1}\alpha_{1,p})$. It is easy to see that they are all projective. Note that $P_{0,i} = P(i)$ for every $i = 1,\ldots,p+q$. Given $i \in \{1,\ldots,p+q\}$, we define by induction the modules $P_{-n,i}$ of the form $X(w)$ for all $n > 0$. Suppose that $P_{-(n-1),i} = X(w)$ for some walk w already constructed. Then we set $P_{-n,i} := X(v)$, where v is a walk obtained from w as follows. We extend w with one arrow α_s on the left-hand side and one inverse β_t^{-1} on the right-hand side (notice that they always exist!); in case $s = 1$ (resp. $t = 1$) we also add the walk β^{-1} (resp. α).

For example, if $p = 2$, $q = 3$ we have $P_{0,3} = X(\beta_{2,3}^{-1})$, $P_{-1,3} = X(\alpha_2\beta^{-1}\alpha)$, $P_{-2,3} = X(\beta^{-1}\alpha\beta^{-1}\alpha\beta_3^{-1})$, $P_{-3,3} = X(\alpha_2\beta^{-1}\alpha\beta^{-1}\alpha\beta_{2,3}^{-1})$ and so on.

Λ-modules from the class

$$\mathcal{P} = \{P_{-n,i}\}_{n\geq 0,\, 1\geq i\geq p+q}$$

are called *postprojective* [2].

We can construct dually the class of preinjective Λ-modules. We set $I_{0,1} = X(1)$, $I_{0,2} = X(\alpha_1)$, $I_{0,3} = X(\alpha_{1,2}),\ldots, I_{0,p} = X(\alpha_{1,p-1})$, $I_{0,p+1} = X(\beta_1^{-1})$, $I_{0,p+2} = X(\beta_{1,2}^{-1}),\ldots, I_{0,p+q-1} = X(\beta_{1,q-1}^{-1})$ and $I_{0,p+q} = X(\alpha\beta^{-1})$ (these modules are injective and $I_{0,i} = I(i)$ for every $i = 1,\ldots,p+q$). If $I_{n-1,i} = X(w)$ for some walk w already constructed, then we set $I_{n,i} := X(v)$, where v is a walk obtained by extending w with one inverse β_s^{-1} on the left-hand side and one arrow α_t on the right-hand side; in case $s = q$ (resp. $t = p$) we also add the walk α (resp. β^{-1}). Λ-modules from the class

$$\mathcal{I} = \{I_{n,i}\}_{n\geq 0,\, 1\geq i\geq p+q}$$

are called *preinjective*.

To define the next two classes of indecomposables in $\operatorname{mod}\Lambda$, we introduce inductively two families of walks defining them. A walk w is called a *walk of type alpha with quasi-length n starting at $a \in \{1,\ldots,p\}$* (denoted by $\overline{\alpha}_{a,n}$) if $w = a$ for $2 \leq a \leq p$, or $w = \beta^{-1}$ for $a = 1$, in the case $n = 1$; and $w = \alpha_a\overline{\alpha}_{t(\alpha_a),n-1}$ for $2 \leq a \leq p$, or $w = \beta^{-1}\alpha_1\overline{\alpha}_{t(\alpha_1),n-1}$ for $a = 1$, in the case $n > 1$ (we identify vertices $p+q$ and 1, if necessary). For example, if $p = 2$, $q = 3$, we have $\overline{\alpha}_{1,2} = \beta^{-1}\alpha_1$, $\overline{\alpha}_{2,2} = \alpha_2\beta^{-1}$, $\overline{\alpha}_{1,3} = \beta^{-1}\alpha\beta^{-1}$, $\overline{\alpha}_{2,3} = \alpha_2\beta^{-1}\alpha_1$, $\overline{\alpha}_{1,4} = \beta^{-1}\alpha\beta^{-1}\alpha_1$, $\overline{\alpha}_{2,4} = \alpha_2\beta^{-1}\alpha\beta^{-1}$. Dually, we say that w is a *walk of type beta with quasi-length n starting at $a \in \{p+1,\ldots,p+q\}$* (denoted by $\overline{\beta}_{a,n}$) if $w = a$ for $p+1 \leq a \leq p+q-1$, or $w = \alpha$ for $a = p+q$, in the case $n = 1$; and $w = \beta_{a-p}^{-1}\overline{\beta}_{s(\beta_{a-p}),n-1}$ for $p+1 \leq a \leq p+q-1$, or

$w = \alpha\beta_q^{-1}\overline{\beta}_{s(\beta_q),n-1}$ for $a = p + q$, in the case $n > 1$ (we identify vertices 1 and $1 + q$ if necessary).

For simplicity, we denote by $A_{a,n}$ and $B_{a,n}$ the indecomposable Λ-modules of the form $X(\overline{\alpha}_{a,n})$ and $X(\overline{\beta}_{a,n})$, respectively, which are defined by walks from the two newly constructed families of walks. We set

$$\mathcal{A} = \{A_{a,n}\}_{1\le a\le p,\, n\ge 1}, \qquad \mathcal{B} = \{B_{a,n}\}_{p+1\le a\le p+q,\, n\ge 1}.$$

Observe that by the construction each of the families $\mathcal{P}$, $\mathcal{I}$, $\mathcal{A}$, $\mathcal{B}$ consists of pairwise nonisomorphic Λ-modules, these families are pairwise disjoint and $\mathcal{P}\cup\mathcal{I}\cup\mathcal{A}\cup\mathcal{B}$ exhausts all indecomposables of the form $X(w)$, for all walks w in Q.

5.2. For any $\lambda \in k \setminus \{0\}$ and $n \ge 1$ we denote by $R_{\lambda,n}$ the Λ-module given by the representation of the quiver Q that has the k-space k^n at each vertex, the k-linear map corresponding to the arrow α_1 is defined by the Jordan block $J_n(\lambda)$ and all remaining structure maps are identities. We set

$$\mathcal{R} = \{R_{\lambda,n}\}_{\lambda\in k\setminus\{0\},\, n\ge 1} \cup \mathcal{A} \cup \mathcal{B}.$$

Λ-modules from this family are called *regular*.

Now we construct a restriction functor $\Psi : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda'$, which allows us to reduce partially computations of homomorphism spaces over Λ to those over the Kronecker algebra $\Lambda' = kQ'$ (here Q' denotes the quiver $1' \rightrightarrows 2'$). In view of applications we define Ψ only on a dense full subcategory consisting of matrix representations.

Let M be a finite-dimensional Λ-module, which as a representation of Q is defined by the k-spaces $k^{a_1}, \dots, k^{a_p}, k^{b_1}, \dots, k^{b_q}$, corresponding to vertices $1, \dots, p, p + 1, \dots, p + q$, and the k-linear maps given by matrices $A_1, \dots, A_p, B_1, \dots, B_q$ of suitable dimensions, corresponding to the arrows $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$, respectively, where $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{N}$. Then we set

$$\Psi(M) = (k^{a_1} \overset{\bar{A}}{\underset{\bar{B}}{\rightrightarrows}} k^{b_q})$$

where $\overline{A} = A_p \cdots A_2A_1$, $\overline{B} = B_q \cdots B_2B_1$. For a homomorphism $f = \{f_i\}_{1\le i\le p+q} : M \to N$ between Λ-modules M and N given by matrix representations of Q, we set

$$\Psi(f) = \{f_1, f_{p+q}\}$$

where the maps f_1, f_{p+q} correspond to vertices $1', 2'$ of Q', respectively. It is easily seen that the above mappings yield a functor between the relevant categories. Notice that $\Psi(R_{\lambda,n}) = R'_{\lambda,n}$ for all $\lambda \in k\setminus\{0\}$, $n \ge 1$, where $R'_{\lambda,n}$ denotes the regular indecomposable $R_{\lambda,n}$ in $\operatorname{mod}\Lambda'$.

LEMMA. *For any Λ-module M and $\lambda \in k \setminus \{0\}$, $n \geq 1$, the functor Ψ yields the isomorphism*

$$\mathrm{Hom}_\Lambda(M, R_{\lambda,n}) \simeq \mathrm{Hom}_{\Lambda'}(\Psi(M), R'_{\lambda,n})$$

of k-linear spaces.

Proof. An easy check on definitions. ■

5.3. Below we collect some (well known, see e.g. [8]) facts concerning the structure of the category $\mathrm{mod}\,\Lambda$, to be used later.

PROPOSITION.

(i) *The disjoint union $\mathcal{X} = \mathcal{P} \cup \mathcal{R} \cup \mathcal{I}$ is a complete family of pairwise nonisomorphic indecomposable Λ-modules.*

(ii) *For any $P \in \mathcal{P}$, $R \in \mathcal{R}$, $I \in \mathcal{I}$ we have*

$$(R, P) = (I, P) = (I, R) = 0.$$

(iii) *The classes $\mathcal{P}$ and $\mathcal{I}$ are connected components in the Auslander–Reiten quiver Γ_Λ of Λ of the form $(-\mathbb{N})Q^{\mathrm{op}}$ and $\mathbb{N}Q^{\mathrm{op}}$, respectively. The correspondence between vertices and modules is given by the mappings $(-n, i) \mapsto P_{-n,i}$ and $(n, i) \mapsto I_{n,i}$. In particular, $P_{-n,i} = \tau^{-n}P_{0,i}$ and $I_{n,i} = \tau^n I_{0,i}$ for every $n \geq 0$, $i = 1, \ldots, p + q$.*

(iv) *The regular modules form a 1-parameter family $\{\mathcal{T}_\lambda\}_{\lambda \in k \cup \{\infty\}}$ of pairwise orthogonal (in the* Hom*-sense) connected components in Γ_Λ. Each component $\mathcal{T}_\lambda$ for $\lambda \in k \setminus \{0\}$ is a rank 1 stable tube with vertices represented by $\{R_{\lambda,n}\}_{n \geq 1}$ (in particular, the almost split sequences are exactly of the same form as in Proposition* 4.1(v)*). The component $\mathcal{T}_0$ (resp. $\mathcal{T}_\infty$) is a rank p (resp. q) stable tube with vertices represented by $\mathcal{A}$ (resp. $\mathcal{B}$). More precisely, the almost split sequences in the tube $\mathcal{T}_0$ have the form*

$$0 \to A_{a,n} \to A_{a-1,n+1} \oplus A_{a,n-1} \to A_{a-1,n} \to 0$$

for all $a = 1, \ldots, p$, $n \geq 0$, where $A_{0,n} = A_{p,n}$ and $A_{a,0} = 0$. Similarly in the tube $\mathcal{T}_\infty$ we have the almost split sequences of the form

$$0 \to B_{a,n} \to B_{a-1,n-1} \oplus B_{a,n+1} \to B_{a-1,n} \to 0$$

for all $a = p+1, \ldots, p+q$, $n \geq 0$, where $B_{p,n} = B_{p+q,n}$ and $B_{a,0} = 0$. In particular, $\tau^{rp}A_{a,n} = A_{a,n}$ and $\tau^{rq}B_{a,n} = B_{a,n}$ for any $r \in \mathbb{Z}$.

COROLLARY. *Let M be an arbitrary Λ-module.*

(i) *For any $\lambda \in k \setminus \{0\}$, $n \geq 1$ we have*

$$m_{R_{\lambda,n}}(M) = m_{R'_{\lambda,n}}(\Psi(M)).$$

(ii) *For any* $n \geq 0$, $a = 1, \ldots, p$, $b = p+1, \ldots, p+q$ *we have*

$$m_{A_{a,n}}(M) = [M, A_{a-1,n}] - [M, A_{a-1,n+1}] - [M, A_{a,n-1}] + [M, A_{a,n}],$$
$$m_{B_{b,n}}(M) = [M, B_{b-1,n}] - [M, B_{b-1,n-1}] - [M, B_{b,n+1}] + [M, B_{b,n}].$$

Proof. (i) follows from Lemma 5.2 and the fact that Ψ preserves almost split sequences for indecomposables from the tubes $\mathcal{T}_\lambda$, $\lambda \in k \setminus \{0\}$ (see Proposition 5.3).

(ii) follows immediately from Proposition 5.3. ■

REMARK. Precise formulas giving the multiplicities for postprojective and preinjective indecomposables can be obtained as in Lemma 4.2 by applying the shape of the postprojective and preinjective component (see Proposition 5.3(iii)). We do not present them because of their rather complicated form. In particular, to compute the multiplicities of the form $m_{P_{0,a}}$ for all $a \in Q_0$, we use right minimal almost split homomorphisms in the "starting" full subquiver $\{0\} \times Q^{\mathrm{op}}$ of $-\mathbb{N}Q^{\mathrm{op}}$. We can inductively continue this procedure using the shape of the quiver $-\mathbb{N}Q^{\mathrm{op}}$.

5.4. Let M be an arbitrary Λ-module (notations as in 5.2). We have an analogous necessary and sufficient condition for a module from the tube $\mathcal{T}_\lambda$, $\lambda \in k \setminus \{0\}$, to be a direct summand of a given module M.

LEMMA. $R_{\lambda_0,n}$, $\lambda_0 \in k \setminus \{0\}$, *is a direct summand of* M, *for some* $n \geq 1$, *if and only if* λ_0 *is a common root of all* $(b_q - s)$*-minors of the matrix* $\overline{A} - \lambda\overline{B}$, *regarded as polynomials from* $k[\lambda]$, *where* $\overline{A} = A_p \cdots A_1$, $\overline{B} = B_q \cdots B_1$, *and* s *is the sum of the multiplicities of indecomposable postprojective direct summands of* M. *Moreover*, *the number of indecomposable direct summands of* M *from the tube* $\mathcal{T}_\lambda$ *is equal to* $b_q - \mathrm{r}(\overline{A} - \lambda\overline{B}) - s$ *for any* $\lambda \in k \setminus \{0\}$.

Proof. Since every postprojective module is of the form $\tau^{-m}P_{0,a}$ for some $m \geq 0$, $a \in Q_0$ and regular modules from $\mathcal{T}_\lambda$ for $\lambda \in k \setminus \{0\}$ are τ-invariants (see Proposition 5.3), we have $[P_{-i,a}, R_{\lambda,n}] = n$ for all $i \geq 0$, $a \in Q_0$, $\lambda \in k \setminus \{0\}$, $n \geq 1$. Then the argument from the proof of Proposition 4.4 yields $[M, R_{\lambda_0,1}] \geq s$ and the inequality is strict if and only if $R_{\lambda_0,n}$ is a direct summand of M for some $n \geq 1$. Now by the properties of the functor Ψ (see Lemma 5.2), we obtain the assertion of the lemma. ■

5.5. Now we modify the consecutive steps of Algorithm 4.5 and briefly outline the algorithm detecting the multiplicity vectors in the general $\widetilde{\mathbb{A}}_{p,q}$ case, under the assumption as in 4.5. We also assume that a Λ-module M is given by data as in 5.2.

ALGORITHM (the general case).

Input: A Λ-module M.

Output: The multiplicity vector $m(M)$.

(1+2) *Determining the multiplicity vector for postprojective and preinjective components*:

```
set n := 0
while there exists a = 1,...,p+q with dim_k P_{-n,a} ≤ codim_k M, do {
    for i = p+q downto 1 do
          compute the number m_{P_{-n,i}} (see Remark 5.3)
    n := n+1
}
```

Here $\underline{\mathrm{codim}}_k M$ is the difference between $\underline{\dim}_k M$ and the dimension vector of the direct sum of the indecomposable direct summands already detected by the algorithm.

For preinjective modules the procedure is an analogous generalization of that for the Kronecker case (i.e. computing the dimensions $[X, M]$ for $X \in \mathcal{I}$).

(3) *Determining the multiplicity vector for $\mathcal{T}_\lambda$ with $\lambda \in k \setminus \{0\}$*: Let $\{\lambda_1, \ldots, \lambda_t\}$ be all common roots of $(b_q - s)$-minors of the matrix $\overline{A} - \lambda \overline{B}$ treated as polynomials from $k[\lambda]$ (see Lemma 5.4).

By Corollary 5.3(i), to compute multiplicities, we apply the main part of Algorithm 4.5(3) for $\{\lambda_1, \ldots, \lambda_t\}$ and representation $\Psi(M)$ (now $j_i = b_q - \mathrm{r}(\overline{A} - \lambda_i \overline{B}) - s$ for $i = 1, \ldots, t$, see Lemma 5.4).

(4) *Determining the multiplicity vector for $\mathcal{T}_0$ and $\mathcal{T}_\infty$*:

```
set n := 1
while there exists a = 1,...,p such that dim_k A_{a,n} ≤ codim_k M, do {
    for i = 1 to p do
          compute the number m_{A_{i,n}} (apply Corollary 5.3(ii))
    n := n+1
}
```

For modules from the tube $\mathcal{T}_\infty$ we proceed analogously. ■

REMARK. (i) The correctness and stop property for the algorithm formulated above follow by arguments analogous to those from Remark 4.5.

(ii) To determine the number $m_{P_{-n,a}}$, in each loop execution we have to compute only one new dimension $[M, P_{-n,a}]$ (the remaining needed dimensions are already computed in the previous loop execution). Analogously, to determine $m_{A_{a-1,n}}$ for any $n \geq 2$ and $a = 1, \ldots, p$, we compute just one new dimension $[M, A_{a-1,n+1}]$ and use the already computed dimensions $[M, A_{a-1,n}]$, $[M, A_{a,n-1}]$, $[M, A_{a,n}]$.

Notice that also in the general case, the shape and nature of the above procedures motivate searching for an "inductive" method of computing the dimensions $h(M)_X = [M, X]$. In the next paragraphs we present our proposal of handling this problem.

5.6. We start by observing that (from the construction) for any $n \geq 0$ and $a = 1, \ldots, p+q$ there exist uniquely determined $m \geq 0$, $i \in \{2, \ldots, p+1\}$, $j \in \{2, \ldots, q+1\}$ (and vice versa) such that $P_{-n,a} = X(\alpha_{i,p}(\beta^{-1}\alpha)^m \beta_{j,q}^{-1})$, where $\alpha_{p+1,p} = \beta_{q+1,q}^{-1} = (p+q)$. Analogously, for any $n' \geq 0$ and $a' = 1, \ldots, p+q$ there exist uniquely determined $m' \geq 0$, $i' \in \{0, \ldots, p-1\}$, $j' \in \{0, \ldots, q-1\}$ (and vice versa) such that $I_{n',a'} = X(\beta_{1,j'}^{-1}(\alpha\beta^{-1})^{m'}\alpha_{1,i'})$, where $\alpha_{1,0} = \beta_{1,0}^{-1} = (1)$. Similarly, for any $m \geq 1$ and $a = 1, \ldots, p$, we have $\overline{\alpha}_{a,m} = \alpha_{s,p}(\beta^{-1}\alpha)^n \beta^{-1}\alpha_{1,t}$ for some $n \geq 0$, $s \in \{2, \ldots, p+1\}$, $t \in \{0, \ldots, p-1\}$ or $\overline{\alpha}_{a,m} = \alpha_{i,j}$ for some $2 \leq i \leq j \leq p-1$ or $\overline{\alpha}_{a,m} = (a)$ (if $m = 1$, $a \neq 1$). Analogously for any $m' \geq 1$ and $a' = p+1, \ldots, p+q$ we have $\overline{\beta}_{a',m'} = \beta_{1,t'}^{-1}\alpha(\beta^{-1}\alpha)^{n'}\beta_{s',q}^{-1}$ for some $n' \geq 0$, $s' \in \{2, \ldots, q+1\}$, $t' \in \{0, \ldots, q-1\}$ or $\overline{\beta}_{a',m'} = \beta_{i,j}^{-1}$ for some $2 \leq i \leq j \leq q-1$ or $\overline{\beta}_{a',m'} = (a')$ (if $m' = 1$, $a' \neq p+q$).

To describe an inductive method of computing, for a given M, the dimensions $[M, X]$ (resp. $[M, X]$) for all indecomposables X from an individual component, we proceed as in the Kronecker algebra case. (We use the notation for M established in 5.2.)

LEMMA. *Let M be a finite-dimensional Λ-module given by data as in* 4.3. *Then*

$$\text{(i)} \qquad [M, X(\alpha_{i,p}(\beta^{-1}\alpha)^m \beta_{j,q}^{-1})] = (m+1)b_q - \mathrm{r}(M_P^{i,j,m}),$$

$$\text{(ii)} \qquad [X(\beta_{1,j'}^{-1}(\alpha\beta^{-1})^{m'}\alpha_{1,i'}), M] = (m'+1)a_1 - \mathrm{r}(M_I^{j',i',m'}),$$

$$\text{(iii)} \qquad [M, X(\alpha_{s,p}(\beta^{-1}\alpha)^n \beta^{-1}\alpha_{1,t})] = (n+1)b_q + a_{t+1} - \mathrm{r}(M_A^{s,t,n}),$$

$$\text{(iv)} \qquad [M, X(\beta_{1,t'}^{-1}\alpha(\beta^{-1}\alpha)^{n'}\beta_{s',q}^{-1})] = (n'+1)b_q + b_{t'} - \mathrm{r}(M_B^{t',s',n}),$$

for all $m, m', n, n' \geq 0$, $i, s \in \{2, \ldots, p+1\}$, $j, s' \in \{2, \ldots, q+1\}$, $t, i' \in \{0, \ldots, p-1\}$, $j', t' \in \{0, \ldots, q-1\}$, *where*

$$M_P^{i,j,m} \in \mathbb{M}_{((m+1)b_q)\times(ma_1+a_{i-1}+b_{j-2})}(k),$$
$$M_I^{j',i',m'} \in \mathbb{M}_{((m'+1)a_1)\times(m'b_q+a_{i'+2}+b_{j'+1})}(k),$$
$$M_A^{s,t,n} \in \mathbb{M}_{((n+1)b_q+a_{t+1})\times((n+1)a_1+a_{s-1})}(k),$$
$$M_B^{t',s',n'} \in \mathbb{M}_{((n'+1)b_q+b_{t'})\times((n'+1)a_1+b_{s'-2})}(k)$$

are the block matrices

$$M_P^{i,j,m} = \begin{bmatrix} \overline{A}_{p,i-1} & \overline{B} & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\overline{A} & \overline{B} & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\overline{A} & \overline{B} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\overline{A} & \overline{B} & 0 \\ 0 & 0 & \cdots & 0 & 0 & -\overline{A} & \overline{B}_{q,j-1} \end{bmatrix},$$

$$M_I^{j',i',m'} = \begin{bmatrix} \overline{A}^t_{i'+1,1} & \overline{B}^t & 0 & 0 & 0 & \cdots & 0 \\ 0 & -\overline{A}^t & \overline{B}^t & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\overline{A}^t & \overline{B}^t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\overline{A}^t & \overline{B}^t & 0 \\ 0 & 0 & \cdots & 0 & 0 & -\overline{A}^t & \overline{B}^t_{j'+1,1} \end{bmatrix},$$

$$M_A^{s,t,n} = \begin{bmatrix} \overline{A}_{p,s-1} & \overline{B} & 0 & 0 & \cdots & 0 \\ 0 & -\overline{A} & \overline{B} & 0 & \cdots & 0 \\ 0 & 0 & -\overline{A} & \overline{B} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\overline{A} & \overline{B} \\ 0 & 0 & \cdots & 0 & 0 & -\overline{A}_{t,1} \end{bmatrix},$$

$$M_B^{t',s',n'} = \begin{bmatrix} \overline{B}_{q,s'-1} & \overline{A} & 0 & 0 & \cdots & 0 \\ 0 & -\overline{B} & \overline{A} & 0 & \cdots & 0 \\ 0 & 0 & -\overline{B} & \overline{A} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -\overline{B} & \overline{A} \\ 0 & 0 & \cdots & 0 & 0 & -\overline{B}_{t',1} \end{bmatrix},$$

where $\overline{A}_{s,t} = A_s A_{s-1} \cdots A_t$ *and* $\overline{B}_{s,t} = B_s B_{s-1} \cdots B_t$ *for* $s \geq t$, *and* $\overline{A} = \overline{A}_{p,1}$, $\overline{B} = \overline{B}_{q,1}$, $\overline{A}_{0,1} = \mathrm{I}$, $\overline{B}_{0,1} = I$.

SUBLEMMA.

(i) *The dimensions of the solution spaces for the systems of linear equations*

$$\begin{cases} x_1 C_0 &= 0, \\ x_2 C_1 &= x_1, \\ \vdots & \vdots \\ x_{n+1} C_n &= x_n, \end{cases}$$

and

$$x_{n+1} C_n \cdots C_0 = 0$$

are equal for any $n \geq 1$ *and any matrices* $C_0, \ldots, C_n$ *of suitable sizes, where* $x_1, \ldots, x_{n+1}$ *are the unknown row vectors.*

(ii) *The dimensions of solution spaces of the systems of linear equations*

$$\begin{cases} y_m D_m &= y_{m-1}, \\ \vdots & \vdots \\ y_2 D_2 &= y_1, \\ y_1 D_1 &= x_1, \\ x_2 C_1 &= x_1, \\ \vdots & \vdots \\ x_{n+1} C_n &= x_n, \end{cases}$$

and

$$y_m D_m \cdots D_1 = x_{n+1} C_n \cdots C_1$$

are the same for any $n, m \geq 1$ *and any matrices* $C_1, \ldots, C_m$, $D_1, \ldots, D_n$ *of suitable sizes, where* $x_1, \ldots, x_{n+1}, y_1, \ldots, y_m$ *are the unknown row vectors.*

Proof. The assertions follow easily by applying appropriate elementary transformations and dropping the equations containing those vectors x_i that are determined by the other ones (see the proof of Lemma 4.6). ■

Proof of Lemma. Fix a postprojective indecomposable module of the form $X(w)$ with $w = \alpha_{i,p}(\beta^{-1}\alpha)^m \beta_{j,q}^{-1}$. By the properties of the functors F_λ and $\mathrm{F}_\bullet$ (cf. 3.1) we have

$$\mathrm{Hom}_\Lambda(M, X(w)) \simeq \mathrm{Hom}_{\widetilde{\Lambda}}(\mathrm{F}_\bullet(M), V(\widetilde{w})),$$

therefore to compute $[M, X(w)]$, we consider the space $\mathrm{Hom}_{\widetilde{\Lambda}}(\mathrm{F}_\bullet(M), V(\widetilde{w}))$, where $\widetilde{w}$ is a fixed lifting of the walk w ($\mathrm{F}_\lambda(V(\widetilde{w})) = X(w)$). Any homomorphism $f : \mathrm{F}_\bullet(M) \to V(\widetilde{w})$ is given by a collection

$$f = \{x_{a_i}, x_{a_{i+1}}, \ldots, x_{a_p}\} \cup \{y^s_{b_1}, \ldots, y^s_{b_q}\}_{s=1}^m \cup \{x^s_{a_1}, \ldots, x^s_{a_p}\}_{s=1}^m \cup \{y_{b_{j-1}}, \ldots, y_{b_q}\}$$

of row vectors satisfying the commutative diagram

$$\begin{array}{ccccccccccccccccccc}
k^{a_{i-1}} & \xrightarrow{A_{i-1}} & k^{a_i} \cdots & \xrightarrow{A_p} & k^{b_q} & \xleftarrow{B_q} & \cdots & \xleftarrow{B_1} & k^{a_1} & \xrightarrow{A_1} \cdots & \xrightarrow{A_p} & k^{b_q} \cdots & \xrightarrow{A_p} & k^{b_q} & \xleftarrow{B_q} & \cdots & \xleftarrow{B_j} & k^{b_{i-1}} & \xleftarrow{B_{j-1}} k^{b_{j-2}} \\
\downarrow & & \downarrow{\scriptstyle x_{a_i}} & & \downarrow{\scriptstyle y^1_{b_q}} & & & & \downarrow{\scriptstyle x^1_{a_1}} & & & \downarrow{\scriptstyle y^2_{b_q}} & & \downarrow{\scriptstyle y_{b_q}} & & & & \downarrow{\scriptstyle y_{b_{j-1}}} & \downarrow \\
0 & \xrightarrow{0} & k \cdots & \xrightarrow{1} & k & \xleftarrow{1} & \cdots & \xleftarrow{1} & k & \xrightarrow{1} \cdots & \xrightarrow{1} & k \cdots & \xrightarrow{1} & k & \xleftarrow{1} & \cdots & \xleftarrow{1} & k & \xleftarrow{0} 0
\end{array}$$

or equivalently the system

$$(*)\qquad \begin{cases}
x_{a_i} A_{i-1} & = 0, \\
x_{a_{i+1}} A_i & = x_{a_i}, \\
\vdots & \quad \vdots \\
y^1_{b_q} A_p & = x_{a_p}, \\
y^1_{b_q} B_q & = y^1_{b_{q-1}}, \\
\vdots & \quad \vdots \\
y^1_{b_1} B_1 & = x^1_{a_1}, \\
x^1_{a_2} A_1 & = x^1_{a_1}, \\
\vdots & \quad \vdots \\
y^2_{b_q} A_p & = x^1_{a_p}, \\
\vdots & \quad \vdots \\
y_{b_q} A_p & = x^m_{a_p}, \\
y_{b_q} B_q & = y_{b_{q-1}}, \\
\vdots & \quad \vdots \\
y_{b_j} B_j & = y_{b_{j-1}}, \\
y_{b_{j-1}} B_{j-1} & = 0,
\end{cases}$$

of linear equations. By the sublemma (with (i) applied to the first $p-i+2$ equations and the last $q-j+2$ equations, and (ii) to the remaining part), the dimension of the solution space for $(*)$ is the same as that for the system

$$(*)'\qquad \begin{cases}
y^1_{b_q} \overline{A}_{p,i-1} & = 0, \\
y^1_{b_q} \overline{B} & = y^2_{b_q} \overline{A}, \\
\vdots & \quad \vdots \\
y^{m-1}_{b_q} \overline{B} & = y^m_{b_q} \overline{A}, \\
y^m_{b_q} \overline{B} & = y_{b_q} \overline{A}, \\
y_{b_q} \overline{B}_{q,j-1} & = 0.
\end{cases}$$

As a block matrix equation, $(*)'$ has the form

$$[y_{b_q}^1, \dots, y_{b_q}^m, y_{b_q}] \cdot M_P^{i,j,m} = 0,$$

and consequently, (i) is proved.

It is easy to check that applying the standard duality $\mathrm{D} : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{\mathrm{op}}$ we obtain (ii) (notice that $\Lambda^{\mathrm{op}} = kQ^{\mathrm{op}}$ and Q^{op} is a quiver of type $\widetilde{\mathbb{A}}_{q,p}$).

Formulas (iii) and (iv) follow by applying similar arguments. ■

Remark. It remains to show how to compute the dimension $[M, A_{a,n}]$ for $A_{a,n} = X(\overline{\alpha}_{a,n})$, when $\overline{\alpha}_{a,n} = \alpha_{s,t}$ or $\overline{\alpha}_{a,n} = (a)$ for $2 \le s \le t \le p-1$, $2 \le a \le p-1$. In the first case $[M, A_{a,n}]$ is equal to the dimension of the solution space for the system $A_t \cdots A_s A_{s-1} x = 0$, and in the second case to that for $A_{a-1} x = 0$. Analogously, if $\overline{\beta}_{a,n} = \beta_{s,t}^{-1}$ or $\overline{\beta}_{a,n} = (a)$ for $2 \le s \le t \le q-1$, $p+1 \le a \le p+q-1$, then $[M, B_{a,n}]$ is the dimension of the solution space of $B_t \cdots B_s B_{s-1} x = 0$ or $B_{a-p} x = 0$, respectively.

5.7. The fact below, just as before, is crucial for improving efficiency of computing coordinates of the vector $h(M)$, and indicates an inductive method of rank computation for the family of matrices from Lemma 5.6.

Lemma. *Let $\mathcal{N} = \mathcal{N}(\overline{A}_{p,i-1}, \overline{A}, \overline{B})$ be a family of generalized upper triangular matrices N_l, $l \in \mathbb{N}$, defined inductively, for a fixed i, by setting $N_1 = [\widehat{\overline{A}_{p,i-1}|B}]$ and*

$$N_{l+1} = \begin{bmatrix} N_{11}^{(l)} & [N_{12}^{(l)}|0] \\ 0 & \widehat{U}_l \end{bmatrix}$$

where $N_l = \begin{bmatrix} N_{11}^{(l)} & N_{12}^{(l)} \\ N_{21}^{(l)} & N_{22}^{(l)} \end{bmatrix}$ with maximal zero block $N_{21}^{(l)}$ containing la_1 columns and $U_l = \begin{bmatrix} N_{22}^{(l)} & 0 \\ -\bar{A} & \bar{B} \end{bmatrix}$. Then $\mathrm{r}(N_{l+1}) = \mathrm{r}(N_l) + \mathrm{r}(\widehat{U}_l) - \mathrm{r}(N_{22}^{(l)})$; moreover, $N_{22}^{(l)} = U_{22}^{(l-1)}$, where $\widehat{U}_l = \begin{bmatrix} U_{11}^{(l)} & U_{12}^{(l)} \\ U_{21}^{(l)} & U_{22}^{(l)} \end{bmatrix}$ with maximal zero block $U_{21}^{(l)}$ containing a_1 columns. Moreover, for any m, n and fixed j, t we have

$$\mathrm{r}(M_P^{i,j,m}) = \mathrm{r}(N_m) - \mathrm{r}(N_{22}^{(m)}) + \mathrm{r}(\widehat{U'_m}),$$

$$\mathrm{r}(M_A^{i,t,n}) = \mathrm{r}(N_{n+1}) - \mathrm{r}(N_{22}^{(n+1)}) + \mathrm{r}(\widehat{Z}'_{n+1}),$$

where $U'_m = \begin{bmatrix} N_{22}^{(m)} & 0 \\ -\bar{A} & \bar{B}_{q,j-1} \end{bmatrix}$ and $Z'_{n+1} = \begin{bmatrix} N_{22}^{(n+1)} \\ -\bar{A}_{t,1} \end{bmatrix}$.

Proof. This follows easily by analysis of Gauss elimination.

5.8. Finally, we summarize previous remarks concerning efficiency of our procedures, by estimating briefly the pessimistic complexity of the algorithms for $\widetilde{\mathbb{A}}_{p,q}$-algebras and comparing it to the "naive" approach. We also discuss perspectives and possibilies for generalizations of our methods to other classes of algebras.

Notice first that, except for solving polynomial equations, Algorithms 4.5 and 5.5 can be "automatized". One can also consider the situation of k being a finite field; in this case the algorithms can be "fully automatized".

LEMMA. *Excluding the problem of solving polynomial equations, the pessimistic complexity of Algorithm* 5.5 (*for fixed* p, q) *is* $\mathcal{O}(m^4)$, *where* m *is the dimension of the given module* M.

Proof. Set $m = \dim_k M$. First we consider the Kronecker algebra case (i.e. Algorithm 4.5). It is obvious that the loop in step 1 is executed at most m times (see the loop condition), and similarly in steps 2 and 4. Analyzing step 3, we see that $m_1+\cdots+m_s$ cannot be greater than m ($\bigoplus_{i=1}^{s}\bigoplus_{j=1}^{m_i} R_{\lambda_i,j}^{a_j^i}$ is a direct summand of M), so the inner loop in step 3 is also executed at most m times. All these executions rely on computing the dimensions $[M, P_i]$ (resp. $[I_i, M]$, $[M, R_{\lambda,n}]$), i.e. performing Gauss elimination for matrices of size at most $2n_2\times 2n_1$ (see Lemma 4.7 and Remark 4.7(ii)). Since $m = n_1+n_2$ and the complexity of Gauss elimination on an $n \times n$ matrix is $\mathcal{O}(n^3)$, the assertion holds for the Kronecker algebra case.

The analysis of the complexity for Algorithm 5.5 is similar, since we have Lemma 5.7 at our disposal. In the new estimation one should only multiply the parameters from the previous one by $p + q$ (a constant which does not affect the complexity). ∎

REMARK. (i) The estimations in the proof above are very rough, so in practice the relevant algorithms can be much more efficient. In particular, this is the case if the support of M does not contain all vertices of Q, or more generally, the disposition of coordinates in $\underline{\dim}_k M$ is not enough "homogeneous".

(ii) Note that Lemmata 4.7 and 5.7 essentially improve the efficiency of the algorithms. In general, without this improvement, the rank computations are realized by Gauss elimination for matrices of increasing sizes estimated by $im\times im$, for $i = 1, \ldots, m$ (see Remark 4.7(ii)). In that case the complexity is $\mathcal{O}(\sum_{i=1}^{m}(im)^3) = \mathcal{O}(m^7)$.

A final comment. The method of determining multiplicity vectors for modules, proposed in this paper, can be adopted for other classes of algebras; in particular, for domestic canonical algebras and hereditary tame algebras. The expected pessimistic complexities of the relevant algorithms in these situations are similar to that in the case of $\widetilde{\mathbb{A}}_{p,q}$-algebras. We strongly believe that the existence of such an algorithm with reasonably low polynomial complexity depends on the structure and shape of the module category considered, rather than on a precise description of canonical forms for indecomposables. We have already obtained some results in this direction. They will be presented in forthcoming publications.

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