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CIRCUMRADIUS VERSUS SIDE LENGTHS OF TRIANGLES IN LINEAR NORMED SPACES

BY

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Abstract. Given a planar convex body B centered at the origin, we denote by $\mathcal{M}^2(B)$ the Minkowski plane (i.e., two-dimensional linear normed space) with the unit ball B. For a triangle T in $\mathcal{M}^2(B)$ we denote by $R_B(T)$ the least possible radius of a Minkowskian ball enclosing T. We remark that in the terminology of location science $R_B(T)$ is the optimum of the minimax location problem with distance induced by B and vertices of T as existing facilities (see, for instance, [HM03] and the references therein). Using methods of linear algebra and convex geometry we find the lower and upper bound of $R_B(T)$ for the case when B is an arbitrary planar convex body centered at the origin and $T \subseteq \mathcal{M}^2(B)$ is an arbitrary triangle with given Minkowskian side lengths a_1, a_2, a_3. We also obtain some further results from the geometry of triangles in Minkowski planes, which are either corollaries of the main result or statements needed in the proof of the main result.

1. Introduction. Let $\mathbb{E}^2$ denote the Euclidean space with origin o, scalar product $\langle \cdot\,, \cdot\rangle$, and norm $|\cdot|$. Fixing an orthonormal basis e_1, e_2 in $\mathbb{E}^2$ we identify, in a standard manner, elements of $\mathbb{E}^2$ with elements of $\mathbb{R}^2$ (the space of column 2-vectors of real numbers). Thus, in analytic expressions elements of $\mathbb{E}^2$ are treated as column vectors. Given vectors u_1 and u_2 in $\mathbb{E}^2$, $\det(u_1, u_2)$ denotes the determinant of the 2×2 matrix with columns u_1 and u_2. The operation $(\cdot)^\top$ is the transposition of matrices. A compact convex set with a nonempty interior is called a *convex body*. The abbreviations bd, conv, and aff stand for boundary, convex hull, and affine hull, respectively.

Let B be a centrally symmetric convex body in $\mathbb{E}^2$ with center at the origin. We denote by $\mathcal{M}^2(B)$ the Minkowski plane (i.e., two-dimensional real normed space) with the norm $\|\cdot\|_B$ such that the unit ball induced by $\|\cdot\|_B$ is equal to B. Basic information on Minkowski planes is collected

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in [Tho96], [MS03], and [MSW01]. The set B is called the *gauge body* of $\mathcal{M}^2(B)$ or the *unit Minkowskian ball.* A homothetical copy $r \cdot B + c$, where $c \in \mathcal{M}^2(B)$ and $r > 0$, is called the Minkowskian ball of radius r with center at the point c.

Let K be a convex body in a Minkowski plane $\mathcal{M}^2(B)$. Then $\operatorname{perim}_B(K)$ denotes the *Minkowskian perimeter* of K, i.e., the Minkowskian length of the boundary of K. The Minkowskian diameter of K is denoted by $\operatorname{diam}_B(K)$. The least possible radius of a Minkowskian ball containing K is denoted by $R_B(K)$ and called the *minimal enclosing Minkowskian radius* of K. Minkowskian balls containing K and having Minkowskian radius $R_B(K)$ are called *minimal enclosing Minkowskian balls* of K. A Minkowskian ball whose boundary contains all three vertices of a triangle T in $\mathcal{M}^2(B)$ is called a *Minkowskian circumball* of T. The center of a Minkowskian circumball of a triangle T in $\mathcal{M}^2(B)$ is called a *Minkowskian circumcenter* of T, and its radius is a *Minkowskian circumradius* of T. We remark that in most sources the quantity $R_B(K)$ is called the Minkowskian circumradius of the convex body K. However, for a triangle T the quantity $R_B(T)$ is not always the Minkowskian circumradius of T according to the definition introduced above (see also Fig. 1). (The relationship of the minimal enclosing balls and Minkowskian circumballs of a triangle will be indicated below.) Thus, to avoid ambiguity, we have introduced another name for $R_B(K)$. In contrast to the Euclidean plane, there exist Minkowski planes in which some triangles have no Minkowskian circumball (see Fig. 2). Furthermore, there exist Minkowski planes where some triangles have several Minkowskian circumballs, possibly with different radii (see Fig. 3). A thorough discussion of existense and uniqueness of Minkowskian circumradii is postponed to the next section.

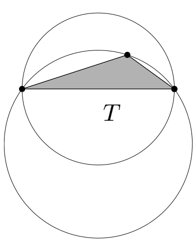

Fig. 1

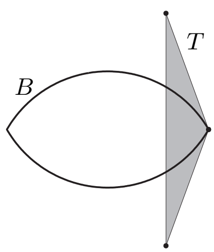

Fig. 2

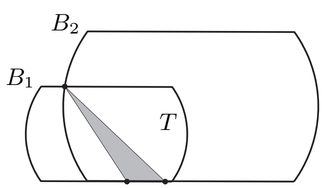

Fig. 3

Now let us formulate the geometric optimization problem that we will solve in this paper. Let a_1, a_2, a_3 be positive scalars. We wish to find the range of the quantity $R_B(T)$ where B is an arbitrary convex body in $\mathbb{E}^2$ centered at the origin and T is an arbitrary triangle with given

Minkowskian side lengths a_1, a_2, a_3 with respect to $\mathcal{M}^2(B)$. Furthermore, we want to characterize those pairs (T, B) that correspond to the optimal values of $R_B(T)$.

Of course, we first need to know which triples a_1, a_2, a_3 can be Minkowskian side lengths of a triangle in some Minkowski plane. Given a Minkowski plane $\mathcal{M}^2(B)$, positive scalars a_1, a_2, a_3 are side lengths of some triangle in $\mathcal{M}^2(B)$ if and only if they satisfy the triangle inequalities

$$2a_i \leq a_1 + a_2 + a_3, \qquad i = 1, 2, 3.$$

This well known statement is a straightforward consequence of the Jordan separation theorem. Moreover, one can see that there is a certain freedom in choosing the embedding.

We remark that $R_B(T) \geq \frac{1}{2}\operatorname{diam}_B(T)$ with equality if and only if two vertices of T lie in different parallel supporting lines of a minimal enclosing Minkowskian ball of T. Consequently, if B is a parallelogram, we have $R_B(T) = \frac{1}{2}\operatorname{diam}_B(T)$ for every triangle $T \subseteq \mathcal{M}^2(B)$. It is known that $\operatorname{diam}_B(T) = \max\{a_1, a_2, a_3\}$ (cf. Part II of Theorem 4 from [Ave03]). Thus, $R_B(T) \geq \frac{1}{2}\max\{a_1, a_2, a_3\}$ is the sharp lower bound.

Further on, we will present the least upper bound of $R_B(T)$, which is the main result of the paper.

The following lemma provides a representation of convex hexagons, which will be used later on in the proof and in the statement of our main theorem.

Lemma 1. *Let b_1, b_2, b_3 be nonzero vectors in $\mathbb{E}^2$. Then the following conditions are the equivalent*:

- (i) *The points b_1, b_2, b_3 are alternating vertices of a convex hexagon centered at the origin, i.e., $b_1, -b_3, b_2, -b_1, b_3, -b_2$ are vertices of a convex hexagon B, lying on the boundary of B in that order.*
- (ii) *There exist positive scalars a_1, a_2, a_3 satisfying the triangle inequalities $2a_i < a_1 + a_2 + a_3$ $(i = 1, 2, 3)$ and the vector equality $a_1 b_1 + a_2 b_2 + a_3 b_3 = o$.*

Now we are ready to formulate the main theorem of the paper.

Theorem 2. *Let T be a triangle in a Minkowski plane $\mathcal{M}^2(B)$ and let a_1, a_2, a_3 be Minkowskian side lengths of T. Then*

$$R_B(T) \leq \frac{2a_1a_2a_3}{a_1(a_2 + a_3 - a_1) + a_2(a_1 + a_3 - a_2) + a_3(a_1 + a_2 - a_3)} \tag{1}$$

with equality if and only if B is a hexagon with vertices $\pm b_1, \pm b_2, \pm b_3$, where $b_1, b_2, b_3 \in \mathbb{E}^2$ are such that $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$, and T is a *triangle whose sides are translates of the segments* $[o, a_1b_1], [o, a_2b_2], [o, a_3b_3]$ (*see Fig.* 4).

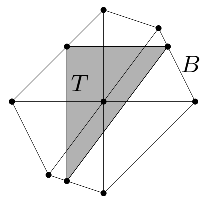

Fig. 4

We can transform (1) to the following equivalent but slightly more complicated inequality:

$$(2)\qquad 4R_B(T) \le (a_1 + a_2 + a_3) - \frac{(a_2 + a_3 - a_1)(a_1 + a_3 - a_2)(a_1 + a_2 - a_3)}{a_1(a_2 + a_3 - a_1) + a_2(a_1 + a_3 - a_2) + a_3(a_1 + a_2 - a_3)}.$$

From (2) we get the inequality $4R_B(T) \le \operatorname{perim}_B(T)$, which was also shown in [CK73] and [MZ05] (see also an open problem in [CK73]).

The following corollary of Theorem 2 gives a relation between the quantities $R_B(T), \operatorname{diam}_B(T)$, and $\operatorname{perim}_B(T)$ for a triangle $T \subseteq \mathcal{M}^2(B)$.

COROLLARY 3. *Let* $\mathcal{M}^2(B)$ *be an arbitrary Minkowski plane*, *and* T *be a triangle in* $\mathcal{M}^2(B)$. *Then*

$$(3)\qquad R_B(T) \le \frac{2\operatorname{diam}_B(T)^2}{6\operatorname{diam}_B(T) - \operatorname{perim}_B(T)}$$

with equality if and only if one of the following two conditions is satisfied:

(i) $2\operatorname{diam}_B(T) = \operatorname{perim}_B(T)$,
(ii) $2\operatorname{diam}_B(T) < \operatorname{perim}_B(T)$, B *is a hexagon with vertices* $\pm b_1, \pm b_2, \pm b_3$ *such that the vectors* $b_1, b_2, b_3 \in \mathbb{E}^2$ *satisfy the equality* $a_1b_1 + a_2b_2 + a_3b_3 = 0$, *where* $a_1 = \operatorname{perim}_B(T) - 2\operatorname{diam}_B(T)$, $a_2 = a_3 = \operatorname{diam}_B(T)$, *and* T *is a triangle whose sides are translates of the segments* $[o, a_ib_i]$ $(i = 1, 2, 3)$.

The proof of Theorem 2 uses Lemma 1 and Theorem 5 from the next section. Corollary 3 is derived directly from Theorem 2 using elementary algebraic considerations.

2. Further results on Minkowskian circumradii of triangles. This section contains results on the existence and uniqueness of triangle circumradii (Theorem 4), presents an auxiliary result about the possible location of circumcenters (Theorem 5) and answers the question about the range of values of circumradii (see Theorem 6, a counterpart of Theorem 2).

Part I of the following theorem follows directly from the results in [KN75] (see also the survey [MSW01, Proposition 41]); the proof of sufficiency in Part II is implicitly given in the proof of Proposition 14 from [MSW01].

THEOREM 4. *Let $\mathcal{M}^2(B)$ be an arbitrary Minkowski plane. Then*:

I. *Every triangle T in $\mathcal{M}^2(B)$ has at least one Minkowskian circumradius if and only if B is smooth.*
II. *Every triangle T in $\mathcal{M}^2(B)$ has at most one Minkowskian circumradius if and only if B is strictly convex.*

Clearly, finding the upper bound of $R_B(T)$ given its side lengths a_1, a_2, a_3, we can restrict our considerations to T and B for which $R_B(T) > \frac{1}{2}\operatorname{diam}_B(T)$. In the nondegenerate case the minimal enclosing Minkowskian ball of T turns out to have some nice geometric properties:

THEOREM 5. *Let T be a triangle in a Minkowski plane $\mathcal{M}^2(B)$ such that $R_B(T) > \frac{1}{2}\operatorname{diam}_B(T)$. Then the Minkowskian circumball and the minimal enclosing Minkowskian ball of T are uniquely determined and coincide. Moreover*, *their common center lies in the interior of T.*

Applying Theorem 2 for the case $a_1 = a_2 = a_3$ we see that if T is an equilateral triangle in $\mathcal{M}^2(B)$, then $R_B(T) \le \frac{2}{3}\operatorname{diam}_B(T)$ with equality if and only if B is an affine regular hexagon which is homothetic to the difference body of T. Results extending this statement can be found in [FL03], [FL04], [Boh38], and [Egg58].

In Theorem 2 we gave an estimate for the radius of a minimal enclosing Minkowskian ball of a triangle T. In fact, in view of Theorem 5, inequality (1) is also an estimate for the Minkowskian circumradius of T provided the Minkowskian circumcenter lies in the interior of T. It turns out that in the general case, i.e., when we do not impose any restrictions on the position of the Minkowskian circumcenter, every value R which is not less than $\frac{1}{2}\operatorname{diam}_B(T)$ can be a Minkowskian circumradius of T with respect to some Minkowski plane $\mathcal{M}^2(B)$. This is shown by the following

PROPOSITION 6. *Let $B \subseteq \mathbb{E}^2$ be a parallelogram centered at the origin*, *and let R, a_1, a_2, a_3 be scalars such that $2a_i \le a_1 + a_2 + a_3$ $(i = 1, 2, 3)$ and $R \ge \frac{1}{2}\max\{a_1, a_2, a_3\}$. Then there exists a triangle T in $\mathcal{M}^2(B)$ with side lengths a_1, a_2, a_3 and Minkowskian circumradius R.*

3. The proofs. We now prove all new statements of the previous sections.

Proof of Theorem 4. In view of the remarks before the statement of the theorem, the only new part is the necessity in Part II. Assume that the unit Minkowskian ball B is not strictly convex, i.e., B has a one-dimensional face I. Let us construct two homothetic triangles T and T' with vertices p_1, p_2, p_3 and p'_1, p'_2, p'_3, respectively, such that T and T' are not translates of each other and B is the Minkowskian circumball of both T and T'. Clearly, the triangle T above has two different Minkowskian circumradii. Let p_1 be an endpoint of I and p_2 be in the relative interior of $-I$. Then for a point p'_1 in $\operatorname{bd} B \setminus (-I \cup I)$ which is sufficiently close to p_1, the line $p'_1 + \mathbb{R} \cdot (p_2 - p_1)$ intersects $-I$; let p'_2 denote the intersection point. Clearly for a point p_3 in $-I$ which is sufficiently close to $-p_2$, the line $p'_1 + \mathbb{R} \cdot (p_3 - p_1)$ also intersects $-I$, and we set p'_3 to be the intersection point.

By construction, T and T' are homothetic and B is the Minkowskian circumball of both T and T'. It is easy to see that T and T' are not translates of each other, since the side $[p'_1, p'_2]$ of T' has strictly smaller length than the side $[p_1, p_2]$ of T. ∎

Proof of Theorem 5. Without loss of generality let B be a minimal enclosing Minkowskian ball of T. Then $R_B(T) = 1$. First let us show that all three vertices of T lie in $\operatorname{bd} B$. It is not hard to see that at least two vertices of T lie in $\operatorname{bd} B$. Indeed, otherwise one could find a larger positive homothetical copy of T lying in B, which would contradict $R_B(T) = 1$. Let p_1, p_2 be distinct vertices of T lying in $\operatorname{bd} B$. We now show that the remaining vertex p_3 also lies in $\operatorname{bd} B$. Assume the contrary. Let $[-u, u]$, $u \in \operatorname{bd} B$, be the diameter of B parallel to $[p_1, p_2]$, and v be a nonzero vector in $\mathcal{M}^2(B)$ such that the line $u + \mathbb{R} \cdot v$ supports B. We remark that the set $\{p_1, p_2\}$ is not contained in $(u + \mathbb{R} \cdot v) \cup (-u + \mathbb{R} \cdot v)$, since otherwise we would have $2 = \|p_1 - p_2\|_B = \operatorname{diam}_B(T)$, which contradicts $R_B(T) < \frac{1}{2} \operatorname{diam}_B(T)$. Consequently, we can find a slight translate $T' := T + \alpha \cdot v$, $\alpha \in \mathbb{R}$, of T such that T lies in B, while the vertex $p'_3 := p_3 + \alpha \cdot v$ and at least one of the two vertices $p'_i := p_i + \alpha \cdot v$ $(i = 1, 2)$ do not lie in $\operatorname{bd} B$. This means that at least two vertices of T are not in $\operatorname{bd} B$. But then $R_B(T') = R_B(T) < 1$, a contradiction (see also Figs. 5 and 6). In a similar way we can also show that every vertex p_i and the origin lie on different sides of the line $\operatorname{aff} I_i$, where I_i is the side of T opposite to I_i (see also Fig. 7). This means that the barycentric coordinates of the origin with respect to T' are all positive. Consequently, the origin lies in the interior of T'.

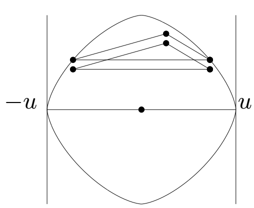

Fig. 5

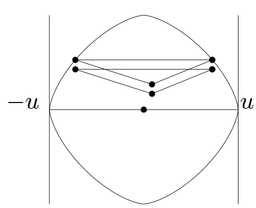

Fig. 6

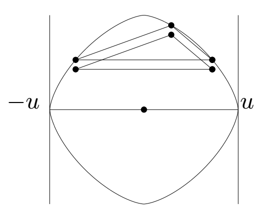

Fig. 7

Now let us show the uniqueness of the Minkowskian circumball of T. Let T' be a positive homothetical copy of T with $\operatorname{ext} T' \subseteq \operatorname{bd} B$. In the proof of Proposition 14 from [MSW01] it was shown that in this case one side I_i of T lies on the boundary of B. Since the origin lies in T', we see that $-p_i$ lies in I_i. Consequently, $R_B(T) = \frac{1}{2}\operatorname{diam}_B(T)$, a contradiction. Hence the Minkowskian circumball of T is unique. ■

Proof of Lemma 1. Let us start with the sufficiency. Without loss of generality we assume that $a_3 = \max\{a_1, a_2, a_3\}$. Then

$$-b_3 = \frac{a_1}{a_3}\, b_1 + \frac{a_2}{a_3}\, b_2 = s_1 b_1 + s_2 b_2 + s_3(b_1 + b_2),$$

where $s_1 := 1 - a_2/a_3$, $s_2 := 1 - a_1/a_3$, and $s_3 := a_1/a_3 + a_2/a_3 - 1$. Clearly, $s_1 + s_2 + s_3 = 1$, i.e., s_1, s_2, s_3 are barycentric coordinates of the point $-b_3$ with respect to the triangle $\operatorname{conv}\{b_1, b_2, b_1 + b_2\}$. By assumption s_1, s_2 are nonnegative and s_3 is strictly positive, which means that $-b_3$ lies in the triangle $\operatorname{conv}\{b_1, b_2, b_1 + b_2\}$, but not on its side $[b_1, b_2]$. Consequently, $\operatorname{conv}\{\pm b_1, \pm b_2, \pm b_3\}$ is a convex hexagon.

Now let us show the necessity. Let B be a convex hexagon in $\mathbb{E}^2$ centered at the origin. Let b_1, b_2, b_3 be alternating vertices of B. Without loss of generality we assume that $\operatorname{conv}\{\pm b_1, \pm b_2\}$ is the parallelogram of maximal area contained in B. Then, as is well known, B is contained in the parallelogram $[-b_1, b_1] + [-b_2, b_2]$, which implies that $-b_3 = a_1 b_1 + a_2 b_2$ for some scalars a_1, a_2 with $0 \le a_1 \le 1$, $0 \le a_2 \le 1$, and $a_1 + a_2 \ge 1$. But then we just put $a_3 := 1$ and get the assertion. ■

Proof of Theorem 2. Let $T \subseteq \mathcal{M}^2(B)$ be a triangle with vertices p_1, p_2, p_3 such that a_i $(i = 1, 2, 3)$ is the Minkowskian length of the side of T opposite to p_i. Without loss of generality we may assume that $R \cdot B$, where $R := R_B(T)$, is a minimal enclosing Minkowskian ball of T. Above it was noticed that $\frac{1}{2}\max\{a_1, a_2, a_3\}$ is the least possible value of R. Therefore, since we wish to estimate R from above, we may assume that $R > \frac{1}{2}\{a_1, a_2, a_3\}$. Then, by Theorem 5, all three vertices of T lie on the boundary of $R \cdot B$, and the origin (which is the center of $R \cdot B$) lies in the interior of T. Also, we remark that for $R > \frac{1}{2}\max\{a_1, a_2, a_3\}$ all triangle inequalities $2a_i \le a_1 + a_2 + a_3$ become strict. Indeed, if say $a_3 = a_1 + a_2$, then $\frac{1}{2}(p_1 + p_2) + (a_3/2) \cdot B$ turns out to be the minimal enclosing Minkowskian ball of T: the vertices p_1 and p_2 of T are at Minkowskian distance $a_3/2$ to $\frac{1}{2}(p_1 + p_2)$, and

$$\begin{aligned}\left\|\frac{1}{2}\,(p_1 + p_2) - p_3\right\|_B &= \frac{1}{2}\,\|(p_1 - p_3) + (p_2 - p_3)\|_B \\ &\le \frac{1}{2}\,(\|p_1 - p_3\|_B + \|p_2 - p_3\|_B) = \frac{1}{2}\,(a_1 + a_2) = a_3.\end{aligned}$$

Thus, $R_B(T) = a_3/2$, a contradiction. Consequently, $2a_i < a_1 + a_2 + a_3$ for every $i = 1, 2, 3$.

Let s_1, s_2, s_3 be the barycentric coordinates of the origin, i.e., s_i is the ratio of the area of the triangle $\operatorname{conv}(\{o, p_1, p_2, p_3\} \setminus \{p_i\})$ to the area of T. Clearly, we have

$$s_1, s_2, s_3 > 0, \qquad s_1 + s_2 + s_3 = 1.$$

Let us consider all possible differences of points in $\{o, p_1, p_2, p_3\}$ and normalize these differences with respect to the norm of $\mathcal{M}^2(B)$. We obtain points

$$\begin{array}{llll} p_1/R, & (p_1 - p_2)/a_3, & -p_2/R, & (p_3 - p_2)/a_1, \\ p_3/R, & (p_3 - p_1)/a_2, & -p_1/R, & (-p_1 + p_2)/a_3, \\ p_2/R, & (-p_3 + p_2)/a_1, & -p_3/R, & (-p_3 + p_1)/a_2, \end{array} \tag{4}$$

lying on the boundary of B in that order with respect to some orientation of B. Without loss of generality we may assume that this orientation is positive.

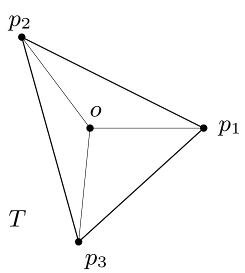

Fig. 8

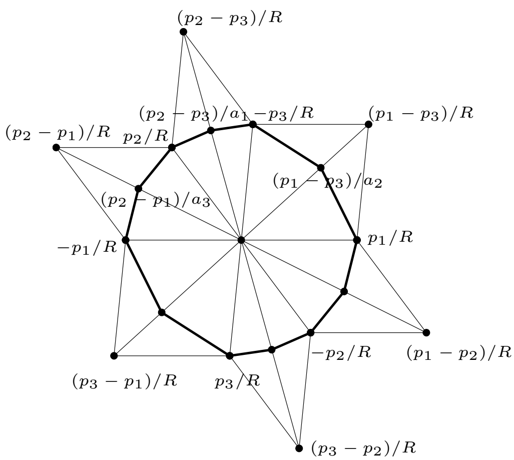

Fig. 9

Due to the convexity of B, the segment $[(p_1 - p_3)/a_2, (p_2 - p_3)/a_1]$, joining two alternating points of (4), lies in the polygon $Q := \operatorname{conv}\{o, (p_1 - p_3)/a_2, (p_2 - p_3)/a_1, -p_3/R\}$. We remark that Q is either a quadrilateral or a triangle. It follows that the area of the triangle T' with vertices $(p_1 - p_3)/a_2$, $(p_2 - p_3)/a_1$, and o is not larger than the area of Q. We express the area of T' as a determinant:

$$\begin{aligned} V(T') &= \frac{1}{2} \det((p_1 - p_3)/a_2, (p_2 - p_3)/a_1) = \frac{1}{2a_1 a_2} \det(p_1 - p_3, p_2 - p_3) \\ &= \frac{1}{2a_1 a_2} \left(\det(p_1, p_2) + \det(p_2, p_3) + \det(p_3, p_1)\right) = \frac{1}{a_1 a_2} V(T). \end{aligned}$$

Analogously,

$$\begin{aligned} V(Q) &= \frac{1}{2}(\det((p_1-p_3)/a_2, -p_3/R) + \det(-p_3/R, (p_2-p_3)/a_1)) \\ &= \frac{1}{2a_2R}\det(p_3,p_1) + \frac{1}{2a_1R}\det(p_2,p_3) = \left(\frac{1}{a_2R}\, s_2 + \frac{1}{a_1R}\, s_1\right)V(T). \end{aligned}$$

Consequently, the inequality $V(Q) \geq V(T')$ amounts to

$$\frac{1}{a_1R}\, s_2 + \frac{1}{a_2R}\, s_1 \geq \frac{1}{a_1a_2},$$

which is equivalent to

$$a_2s_1 + a_1s_2 \geq R. \tag{5}$$

Two further inequalities $a_2s_3 + a_3s_2 \geq R$ and $a_1s_3 + a_3s_1 \geq R$ are obtained analogously. The above three inequalities can be written in matrix form as

$$As \geq Re, \tag{6}$$

where

$$A := \begin{bmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & a_1 \\ a_2 & a_1 & 0 \end{bmatrix}, \quad s := \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}, \quad e := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

In order to obtain (1) we examine the linear program

$$\langle x, e\rangle \to \min \tag{7}$$

for $x \in \mathbb{R}^3$ subject to

$$Ax \geq Re. \tag{8}$$

The set $\{x \in \mathbb{R}^3 : Ax \geq Re\}$ forms a cone C with three facets having outward normals $-(0,a_3,a_2)^\top$, $-(a_3,0,a_1)^\top$, and $-(a_2,a_1,0)^\top$. The apex of this cone is the solution of the equation $Ax = Re$, i.e., the point $RA^{-1}e$. The vector $A^{-1}e$ can be evaluated using Cramer's rule, that is, $A^{-1}e = \varDelta^{-1}(\varDelta_1,\varDelta_2,\varDelta_3)^\top$, where

$$\begin{aligned} \varDelta &:= \det A = 2a_1a_2a_3, \\ \varDelta_1 &:= a_1(a_2+a_3-a_1), \\ \varDelta_2 &:= a_2(a_1+a_3-a_2), \\ \varDelta_3 &:= a_3(a_1+a_2-a_3). \end{aligned}$$

Using the triangle inequality we see that the values $\varDelta_i$ and hence also the coordinates of $A^{-1}e$ are positive. Now let us show that $RA^{-1}e$ is the optimal solution of the linear program (7), (8). We apply the Farkas lemma (see, for instance, [Sch86]). The gradient of the optimized linear function is e. Thus, we have to show that $-e$ belongs to the normal cone of C at the point $RA^{-1}e$. The apex $RA^{-1}e$ of C is incident to all facets of C, which have outward normal vectors $-(0, a_3, a_2)^\top$, $-(a_3, 0, a_1)^\top$, and $(a_2, a_1, 0)^\top$. Hence we just have to show that the solution $x := (x_1, x_2, x_3)^\top$ of the equation $-x_1(0, a_3, a_2)^\top - x_2(a_3, 0, a_1)^\top - x_3(a_2, a_1, 0)^\top = -e$ has positive coordinates. This equation can be written in matrix form as

$$-Ax = -e. \tag{9}$$

The solution of (9) is $A^{-1}e$. But above we have already derived that $A^{-1}e$ has positive coordinates. Consequently, by the Farkas lemma, $RA^{-1}e$ is the optimal solution of (7), (8). Since $\langle s, e\rangle = 1$ and $As \le Re$, we see that the optimum $\langle RA^{-1}e, e\rangle$ of (7), (8) is bounded from above by one:

$$\langle RA^{-1}e, e\rangle \le 1. \tag{10}$$

It is easy to see that

$$\begin{aligned} &\langle RA^{-1}e, e\rangle \\ &\quad = \frac{R}{2a_1a_2a_3}\left(a_1(a_2+a_3-a_1)+a_2(a_1+a_3-a_2)+a_3(a_1+a_2-a_3)\right), \end{aligned} \tag{11}$$

which shows that (10) amounts to (1).

It remains to characterize those B which yield equality in (1).

If equality in (1) is attained, then $\langle RA^{-1}e, e\rangle = 1$, i.e., the optimum of the problem (7), (8) is equal to one. But then, since $\langle s, e\rangle = 1$ and $A \cdot s \le R \cdot e$, we see that s is the optimal solution of (7), (8), and due to the uniqueness of the optimal solution of our linear program we infer that $s = RA^{-1}e$. The latter can be rewritten as the equality $As = Re$, or as three scalar equalities $a_1s_2 + a_2s_1 = R$, $a_1s_3 + a_3s_1 = R$, and $a_2s_3 + a_3s_2 = R$. The equality $a_1s_2 + a_2s_1 = R$ implies that the areas of the quadrilateral Q and the triangle T' defined above are equal. Consequently, $Q = T'$, i.e., Q degenerates to a triangle. This means that the point $-p_3R$ lies in the relative interior of the segment $[(p_2 - p_3)/a_1, (p_1 - p_3)/a_2]$, which implies that the segment $[(p_2 - p_3)/a_1, (p_1 - p_3)/a_2]$ is contained in $\operatorname{bd} B$. Using two further equalities $a_2s_3 + a_3a_2 = R$ and $a_1s_3 + a_3s_1 = R$ we also find that the segments $[(p_3 - p_1)/a_2, (p_2 - p_1)/a_3]$ and $[(p_3 - p_2)/a_1, (p_1 - p_2)/a_3]$ are contained in $\operatorname{bd} B$. Thus, we conclude that B is a hexagon with vertices $\pm b_1, \pm b_2, \pm b_3$ given by

$$b_1 := (p_2 - p_3)/a_1, \tag{12}$$

$$b_2 := (p_3 - p_1)/a_2, \tag{13}$$

$$b_3 := (p_1 - p_2)/a_3. \tag{14}$$

It is then easily verified that $a_1b_1 + a_2b_2 + a_3b_3 = o$.

Now let us show the converse implication. Let B be a convex hexagon with vertices $\pm b_1, \pm b_2, \pm b_3$ such that $a_1b_1 + a_2b_2 + a_3b_3 = o$. Then there exists a triangle T with vertices p_1, p_2, p_3 such that (12)–(14) are satisfied. Indeed, we may put $p_1 := a_1b_1 + a_3b_3$, $p_2 = a_1b_1$, $p_3 = o$. Let us show that for this choice of B and T equality in (1) is attained. Let the scalar R be defined from the equality $\langle RA^{-1}e, e\rangle = 1$. Let $s := RA^{-1}e$. Without loss of generality we replace T by an appropriate translate such that the origin has barycentric coordinates s_1, s_2, s_3 with respect to T (here $s := (s_1, s_2, s_3)^\top$). Let us show that R is the circumradius of T with respect to the Minkowski plane $\mathcal{M}^2(B)$. We have $As = Re$, which implies, in view of the corresponding geometric interpretations of this equality, that R is indeed the Minkowskian circumradius of T with respect to $\mathcal{M}^2(B)$. ■

Proof of Corollary 3. The degenerate case $2\,\mathrm{diam}_B(T) = \mathrm{perim}_B(T)$ is trivial.

Let us consider the case $2\,\mathrm{diam}_B(T) < \mathrm{perim}_B(T)$. Let a_1, a_2, a_3 be the side lengths of T such that $0 < a_1 \le a_2 \le a_3$. We set

$$\begin{aligned} d &:= a_3 = \mathrm{diam}_B(T),\\ p &:= a_1 + a_2 + a_3 = \mathrm{perim}_B(T),\\ x &:= a_2 - a_1. \end{aligned}$$

Then the side lengths a_1, a_2, a_3 can be expressed by

$$a_1 = \frac{1}{2}\,(p - x - d), \tag{15}$$

$$a_2 = \frac{1}{2}\,(p + x - d), \tag{16}$$

$$a_3 = d. \tag{17}$$

The inequalities $a_1 \le a_2 \le a_3$ are equivalent to

$$0 \le x \le 3d - p. \tag{18}$$

We can see that the inequalities (18) are sharp in the following sense. For each x satisfying (18) there exists a triangle with side lengths a_1, a_2, a_3 given by (15)–(17). The triangle inequalities for a_1, a_2, a_3 given by (15)–(17) are satisfied in view of $2d < p$, which is equivalent to $a_3 < a_1 + a_2$. Thus, it only remains to show that $a_1 > 0$. In view of (18) and $2d < p$, we have $2a_1 = p - x - d \ge p - d > d > 3d - p \ge 0$. Hence $a_1 > 0$.

Let us give a sharp upper bound of the right hand side of (1) in terms of the quantities p and d. First we express the term $2a_1a_2$, appearing in the numerator, as

$$2a_1a_2 = \frac{1}{2}\,(a_1 + a_2)^2 - \frac{1}{2}\,(a_1 - a_2)^2 = \frac{1}{2}\,(p - d)^2 - \frac{1}{2}\,x^2.$$

It is easy to see that the denominator of the right hand side of (1) is equal to

$$(a_1 + a_2 + a_3)^2 - 2a_1^2 - 2a_2^2 - 2a_3^2 = p^2 - 2d^2 - 2a_1^2 - 2a_2^2.$$

We express the term $2(a_1^2 + a_2^2)$ by the equality

$$2(a_1^2 + a_2^2) = (a_1 + a_2)^2 + (a_1 - a_2)^2 = (p - d)^2 + x^2.$$

Thus,

$$R_B(T) \le \frac{d}{2} \cdot \frac{(p-d)^2 - x^2}{p^2 - 2d^2 - (p-d)^2 - x^2} = \frac{d}{2} \cdot \frac{(p-d)^2 - x^2}{d(2p-3d) - x^2}.$$

Introducing the notations $C_0 := d/2$, $C_1 := (p-d)^2$, and $C_2 := d \cdot (2p-3d)$, we rewrite the latter inequality in the form

$$R_B(T) \le C_0 \cdot \frac{C_1 - x^2}{C_2 - x^2} = C_0 \cdot \left(\frac{C_1 - C_2}{C_2 - x^2} + 1 \right).$$

It is easily verified that $C_1 - C_2 = (p-2d)^2 \ge 0$.

In view of (18) we get

$$C_2 - (3d-p)^2 = (p-2d)(6d-p) \le C_2 - x^2 \le C_2 = (p-2d)^2.$$

The left and right hand sides above are both positive. Consequently,

$$C_0 \cdot \left(\frac{C_1 - C_2}{C_2 - x^2} + 1 \right) \le C_0 \cdot \left(\frac{C_1 - C_2}{C_2 - (3d-p)^2} + 1 \right) = \frac{2d^2}{6d-p}$$

with equality if and only if $x = 3d - p$, i.e., if and only if $a_1 = p - 2d$ and $a_2 = a_3 = d$. ■

Proof of Proposition 6. We assume that $\|x\|_B := \max\{|x_1|, |x_2|\}$, where $x = (x_1, x_2) \in \mathbb{E}^2$. Without loss of generality we also assume that the side lengths are arranged in increasing order, i.e., $0 < a_1 \le a_2 \le a_3$. Define $T := \operatorname{conv}\{p_1, p_2, p_3\}$, where

$$p_1 := \begin{bmatrix} a_3 \\ a_2 \end{bmatrix}, \quad p_2 := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad p_3 := \begin{bmatrix} a_1 \\ 0 \end{bmatrix}.$$

Then it is easily verified that $\|p_1 - p_2\|_B = a_3$, $\|p_2 - p_3\|_B = a_1$, and $\|p_3 - p_1\|_B = a_2$ (see also Fig. 10). It can be verified that for every $R \ge \frac{1}{2} a_3$ the point $c = \left[\begin{smallmatrix} a_3 - R \\ R \end{smallmatrix} \right]$ is at Minkowskian distance R to the points p_1, p_2, and p_3 (see Fig. 11), which yields the assertion. ■

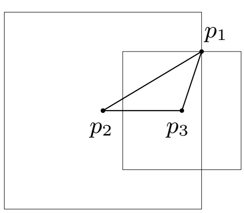

Fig. 10

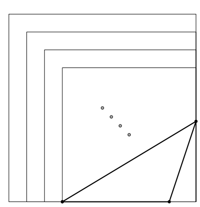

Fig. 11

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Chapter 4 of the PhD thesis [Düv06] of Nico Düvelmeyer, which is concerned with description of the metric spaces that can be isometrically embedded into an appropriate Minkowski space of a given dimension.

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