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MAXIMAL OPERATORS OF FEJÉR MEANS OF DOUBLE VILENKIN–FOURIER SERIES

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Abstract. The main aim of this paper is to prove that the maximal operator $\sigma_0^* := \sup_n |\sigma_{n,n}|$ of the Fejér means of the double Vilenkin–Fourier series is not bounded from the Hardy space $H_{1/2}$ to the space weak-$L_{1/2}$.

Let $\mathbb{N}_+$ denote the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the additive group of integers modulo m_k. Define the group G_m as the complete direct product of the groups Z_{m_j}, with the product of the discrete topologies of Z_{m_j}'s. The direct product μ of the measures

$$\mu_k(\{j\}) := \frac{1}{m_k} \quad (j \in Z_{m_k})$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

If the sequence m is bounded, then G_m is called a *bounded Vilenkin group*, otherwise it is an *unbounded Vilenkin group*. The elements of G_m can be represented by sequences $x := (x_0, x_1, \ldots, x_j, \ldots)$ $(x_j \in Z_{m_j})$. It is easy to give a base of neighborhoods of $x \in G_m$:

$$I_0(x) := G_m, \qquad I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$$

for $n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{N}_+$.

The *generalized number system* based on m is defined in the following way: $M_0 := 1$, $M_{k+1} := m_k M_k$ $(k \in \mathbb{N})$. Then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ $(j \in \mathbb{N}_+)$ and only a finite number of n_j's are not zero. We use the following notations. For $n > 0$ let $|n| := \max\{k \in \mathbb{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$), $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$ and $n_{(k)} := n - n^{(k)}$.

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Denote by $L^p(G_m)$ the usual (one-dimensional) Lebesgue spaces, with norms $\|\cdot\|_p$ $(1 \le p \le \infty)$.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. First define the complex-valued functions $r_k : G_m \to \mathbb{C}$, called the *generalized Rademacher functions*, in this way:

$$r_k(x) := \exp \frac{2\pi \imath x_k}{m_k} \qquad (\imath^2 = -1,\, x \in G_m,\, k \in \mathbb{N}).$$

Now define the *Vilenkin system* $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as follows:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \qquad (n \in \mathbb{N}).$$

If $m = 2$, we call this system the *Walsh–Paley system*. The Vilenkin system is orthonormal and complete in $L^1(G_m)$ [8].

Now, we introduce analogues of the usual definitions of Fourier analysis. If $f \in L^1(G_m)$ we can make the following definitions:

- Fourier coefficients:

$$\widehat{f}(k) := \int_{G_m} f\overline{\psi}_k \, d\mu \qquad (k \in \mathbb{N}),$$

- partial sums:

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k \qquad (n \in \mathbb{N}_+,\, S_0 f := 0),$$

- Fejér means:

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_n f \qquad (n \in \mathbb{N}_+),$$

- Dirichlet kernels:

$$D_n := \sum_{k=0}^{n-1} \psi_k \qquad (n \in \mathbb{N}_+).$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n, \\ 0 & \text{if } x \in G_m \backslash I_n. \end{cases} \tag{1}$$

For $f \in L_1(G_m \times G_m)$, the rectangular partial sums of the double Vilenkin–Fourier series of f are defined as follows:

$$S_{M,N}(f; x^1, x^2) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i,j)\psi_i(x^1)\psi_j(x^2),$$

where the number

$$\widehat{f}(i,j) = \int\limits_{G_m \times G_m} f(x^1,x^2)\overline{\psi}_i(x^1)\overline{\psi}_j(x^2)\,\boldsymbol{\mu}(x^1,x^2).$$

is said to be the (i,j)th *Vilenkin–Fourier coefficient* of f ($\boldsymbol{\mu}$ is the product measure $\mu \times \mu$).

The norm (or quasinorm) of the space $L_p(G_m \times G_m)$ is defined by

$$\|f\|_p := \Bigg(\int\limits_{G_m \times G_m} |f(x^1,x^2)|^p\,\boldsymbol{\mu}(x^1,x^2)\Bigg)^{1/p} \qquad (0 < p < \infty).$$

The space weak-$L_p(G_m \times G_m)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak-}L_p(G_m \times G_m)} := \sup_{\lambda > 0} \lambda\,\boldsymbol{\mu}(|f| > \lambda)^{1/p} < \infty.$$

Let

$$I_{n,k}(x^1,x^2) := I_n(x^1) \times I_k(x^2).$$

The σ-algebra generated by the rectangles $\{I_{n,k}(x^1,x^2) : (x^1,x^2) \in G_m \times G_m\}$ will be denoted by $\mathfrak{F}_{n,k}$ $(n,k \in \mathbb{N})$.

Denote by $f = (f^{(n,k)} : n,k \in \mathbb{N})$ a martingale with respect to $(\mathfrak{F}_{n,k} : n,k \in \mathbb{N})$ (for details see, e.g., [9, 13]). The maximal function and the diagonal maximal function of a martingale f are defined by

$$f^* = \sup_{n,k \in \mathbb{N}} |f^{(n,k)}|, \qquad f^{\square} = \sup_{n \in \mathbb{N}} |f^{(n,n)}|,$$

respectively. In case $f \in L_1(G_m \times G_m)$, the maximal functions are also given by

$$f^*(x^1,x^2) = \sup_{n,k \in \mathbb{N}} \frac{1}{\boldsymbol{\mu}(I_{n,k}(x^1,x^2))} \Bigg| \int\limits_{I_{n,k}(x^1,x^2)} f(u^1,u^2)\,\boldsymbol{\mu}(u^1,u^2)\Bigg|,$$

$$f^{\square}(x^1,x^2) = \sup_{n \in \mathbb{N}} \frac{1}{\boldsymbol{\mu}(I_{n,n}(x^1,x^2))} \Bigg| \int\limits_{I_{n,n}(x^1,x^2)} f(u^1,u^2)\,\boldsymbol{\mu}(u^1,u^2)\Bigg|.$$

for $(x^1,x^2) \in G_m \times G_m$.

The *Hardy martingale spaces* $H_p(G_m \times G_m)$ and $H_p^{\square}(G_m \times G_m)$ $(0 < p < \infty)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty \quad \text{and} \quad \|f\|_{H_p^{\square}} := \|f^{\square}\|_p < \infty,$$

respectively.

If $f \in L_1(G_m \times G_m)$ then it is easy to show that the sequence $(S_{M_n,M_k}(f) : n,k \in \mathbb{N})$ is a martingale. If f is a martingale, that is, $f = (f^{(n,k)} : n,k \in \mathbb{N})$, then the Vilenkin–Fourier coefficients must be defined in a slightly different way:

$$\widehat{f}(i,j)=\lim_{k,l\to\infty}\int\limits_{G_m\times G_m} f^{(k,l)}(x^1,x^2)\overline{\psi}_i(x^1)\overline{\psi}_j(x^2)\,\boldsymbol{\mu}(x^1,x^2).$$

The Vilenkin–Fourier coefficients of $f\in L_1(G_m\times G_m)$ are the same as those of the martingale $(S_{M_n,M_k}(f):n,k\in\mathbb{N})$ obtained from f.

For $n,k\in\mathbb{N}_+$ and a martingale f the *Fejér mean* of order (n,k) of the double Vilenkin–Fourier series of f is given by

$$\sigma_{n,k}(f;x^1,x^2)=\frac{1}{nk}\sum_{i=0}^{n-1}\sum_{j=0}^{k-1}S_{i,j}(f;x^1,x^2).$$

For a martingale f the restricted and unrestricted maximal operators of the Fejér means are defined by

$$\sigma^*_\lambda f(x^1,x^2)=\sup_{1/M_\lambda\le n/k\le M_\lambda}|\sigma_{n,k}(f;x^1,x^2)|,$$
$$\sigma^* f(x^1,x^2)=\sup_{n,k\in\mathbb{N}}|\sigma_{n,k}(f;x^1,x^2)|.$$

In the one-dimensional case the weak type inequality

$$\mu(\sigma^*f>\lambda)\le\frac{c}{\lambda}\|f\|_1\qquad(\lambda>0)$$

can be found in Zygmund [15] for the trigonometric series, in Schipp [5] for Walsh series and in Pál and Simon [4] for bounded Vilenkin series. Again in one dimension, Fujii [2] and Simon [7] verified that σ^* is bounded from H_1 to L_1. Weisz [10, 12] generalized this by proving the boundedness of σ^* from the martingale Hardy space H_p to L_p for $p>1/2$. Simon [6] gave a counterexample to show that this does not hold for $0<p<1/2$. In the endpoint case $p=1/2$ Weisz [14] proved that σ^* is bounded from $H_{1/2}$ to weak-$L_{1/2}$. By interpolation it follows that σ^* is not bounded from H_p to weak-L_p for any $0<p<1/2$. It is an open question whether σ^* is bounded from $H_{1/2}$ to $L_{1/2}$ or not. (We think the answer is no.)

For the two-dimensional Vilenkin–Fourier series Weisz [11] proved the following results:

Theorem A (Weisz [11]). *Let $p>1/2$. Then the maximal operator σ^*_λ is bounded from $H_p^\square$ to L_p.*

Theorem B (Weisz [11]). *Let $p>1/2$. Then the maximal operator σ^* is bounded from H_p to L_p.*

The main aim of this paper is to prove that for any bounded Vilenkin system the maximal operator σ^* (resp. σ^*_λ) is not bounded from $H_{1/2}$ (resp. $H^\square_{1/2}$) to weak-$L_{1/2}$. Moreover, we prove that the following is true.

THEOREM 1. *For any bounded Vilenkin system the maximal operator* σ_0^* *is not bounded from* $H_{1/2}$ *to weak-*$L_{1/2}$.

Thus, as regards boundedness of σ^* and σ_λ^*, the case of double Vilenkin–Fourier series differs from that of one-dimensional Vilenkin–Fourier series.

By Theorem 1 and interpolation it follows that σ_0^* is not bounded from H_p to weak-L_p for any $0 < p < 1/2$. In particular, in Theorems A and B the assumption $p > 1/2$ is essential. On the other hand, it would be interesting to find a decent space to replace weak-$L_{1/2}$ in order to have the relevant boundedness. However, this question does not seem to be easy.

The *Fejér kernel* of order n of the Vilenkin–Fourier series is defined by

$$K_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} D_k(x).$$

Set

$$K_{s,l}(x) := \sum_{j=s}^{s+l-1} D_k(x).$$

In order to prove the theorem we need the following lemmas.

LEMMA 1 ([3]). *Suppose that* $s,t,n \in \mathbb{N}$ *and* $x \in I_t \backslash I_{t+1}$. *If* $t \le s \le |n|$, *then*

$$K_{n^{s+1},M_s}(x) = \begin{cases} M_t M_s \psi_{n^{s+1}}(x)\,\dfrac{1}{1-r_t(x)} & \text{if } x - x_t e_t \in I_s,\\ 0, & \text{otherwise.}\end{cases}$$

LEMMA 2. *Let* $2 < A \in \mathbb{N}_+$, $k \le s < A$ *and* $n_A^* := M_{2A} + M_{2A-2} + \dots + M_2 + M_0$. *Then*

$$n_{A-1}^*|K_{n_{A-1}^*}(x)| \ge M_{2k}M_{2s}/4$$

for $x \in I_{2A}(0,\dots,0,x_{2k} \ne 0,0,\dots,0,x_{2s} \ne 0,x_{2s+1},\dots,x_{2A-1})$, $k = 0,1,\dots,A-3$, $s = k+2,k+3,\dots,A-1$.

Proof. Let $n \in \mathbb{N}_+$. It is known [1] that

$$D_n(x) = \psi_n(x)\Big(\sum_{j=0}^{\infty} D_{M_j}(x)\sum_{u=m_j-n_j}^{m_j-1} r_j^u(x)\Big),$$

thus

$$|D_n(x)| \le \sum_{j=0}^{\infty} n_j D_{M_j}(x).$$

Since for $x \in I_l \setminus I_{l+1}$,

$$\sum_{j=0}^{\infty} n_j D_{M_j}(x) = \sum_{j=0}^{l} n_j M_j \le m_l M_l = M_{l+1},$$

if $s \le l$ we obtain

$$|K_{n^{s+1},M_s}(x)| = \Big| \sum_{u=n^{s+1}}^{n^{s+1}+M_s-1} D_u(x) \Big| \le M_{l+1} M_s.$$

From Lemma 1 we see that

$$K_{n^{2l+1},M_{2l}}(x) = 0 \quad \text{ for } l = s+1, s+2, \ldots, A-1.$$

If $l < s \le |n|$, $x \in I_l \setminus I_{l+1}$ and $x - x_l e_l \in I_s$, then also from Lemma 1 we get

$$1 \le \frac{|K_{n^{s+1},M_s}(x)|}{M_l M_s} = \frac{1}{2|\sin(\pi x_l / m_l)|} \le \frac{m_l}{\pi}.$$

Using these facts, the equality from [3, p. 16]

$$nK_n = \sum_{h=0}^{|n|} \sum_{j=0}^{n_h-1} K_{n^{h+1}+jM_h,M_h},$$

and

$$(n^*_{A-1})_h = \begin{cases} 1 & \text{if } 2 \,|\, h, h < 2A, \\ 0 & \text{otherwise} \end{cases}$$

we estimate

$$\begin{aligned} n^*_{A-1}|K_{n^*_{A-1}}(x)| &= \Big| \sum_{h=0}^{|n^*_{A-1}|} \sum_{j=0}^{(n^*_{A-1})_h-1} K_{(n^*_{A-1})^{h+1}+jM_h,M_h}(x) \Big| \\ &= \Big| \sum_{h=0,\,2|h}^{2A-2} K_{(n^*_{A-1})^{h+1},M_h}(x) \Big| = \Big| \sum_{l=0}^{s} K_{(n^*_{A-1})^{2l+1},M_{2l}}(x) \Big| \\ &= \Big| \sum_{l=0}^{s} K_{(n^*_{A-1})^{2l+2},M_{2l}}(x) \Big| \\ &\ge |K_{(n^*_{A-1})^{2s+2},M_{2s}}(x)| - \Big| \sum_{l=0}^{s-1} K_{(n^*_{A-1})^{2l+2},M_{2l}}(x) \Big| \\ &\ge M_{2s}M_{2k} - \sum_{l=0}^{s-1} |K_{(n^*_{A-1})^{2l+2},M_{2l}}(x)| \ge M_{2s}M_{2k} - \sum_{l=0}^{s-1} M_{2l+1}M_{2k}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \sum_{l=0}^{s-1} M_{2l+1} &= \sum_{l=0}^{s-2} M_{2l+1} + M_{2s-1} \le M_{2s-2} + M_{2s-1} \\ &= \frac{M_{2s}}{m_{2s-1}m_{2s-2}} + \frac{M_{2s}}{m_{2s-1}} \le \frac{3M_{2s}}{4}. \end{aligned}$$

Summarizing,

$$n_{A-1}^*|K_{n_{A-1}^*}(x)| \ge \frac{M_{2s}M_{2k}}{4}. \ \blacksquare$$

Proof of Theorem 1. Let $A \in \mathbb{N}_+$ and

$$f_A(x^1,x^2) := (D_{M_{2A+1}}(x^1) - D_{M_{2A}}(x^1))(D_{M_{2A+1}}(x^2) - D_{M_{2A}}(x^2)).$$

It is evident that

$$\widehat{f}_A(i,k) = \begin{cases} 1 & \text{if } i,k = M_{2A},\dots,M_{2A+1}-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can write

$$(2) \qquad S_{i,j}(f_A;x^1,x^2) = \begin{cases} (D_i(x^1) - D_{M_{2A}}(x^1))(D_j(x^2) - D_{M_{2A}}(x^2)), \\ \qquad\qquad \text{if } i,j = M_{2A}+1,\dots,M_{2A+1}-1, \\ f_A(x^1,x^2) \quad \text{if } i,j \ge M_{2A+1}, \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Since

$$f_A^*(x^1,x^2) = \sup_{n,k\in\mathbb{N}} |S_{M_n,M_k}(f_A;x^1,x^2)| = |f_A(x^1,x^2)|,$$

from (1) we get

$$\begin{aligned} (3) \qquad \|f_A\|_{H_p} &= \|f_A^*\|_p = \|D_{M_{2A+1}} - D_{M_{2A}}\|_p^2 \\ &= \Big(\Big(\int_{I_{2A}\setminus I_{2A+1}} M_{2A}^p + \int_{I_{2A+1}} |M_{2A+1} - M_{2A}|^p\Big)^{1/p}\Big)^2 \\ &= \bigg(\bigg(\frac{m_{2A}-1}{M_{2A+1}}\, M_{2A}^p + \frac{(m_{2A}-1)^p}{M_{2A+1}}\, M_{2A}^p\bigg)^{1/p}\bigg)^2 \\ &\le 2^{2/p} m_{2A}^2 M_{2A}^{2-2/p} \le c M_{2A}^{2-2/p}. \end{aligned}$$

Since

$$D_{k+M_{2A}} - D_{M_{2A}} = \psi_{M_{2A}} D_k, \qquad k = 1,\dots,M_{2A},$$

from (2) we obtain

$$\begin{aligned} (4) \qquad \sigma_0^* f_A(x^1,x^2) &= \sup_{n\in\mathbb{N}} |\sigma_{n,n}(f_A;x^1,x^2)| \ge |\sigma_{n_A^*,n_A^*}(f_A;x^1,x^2)| \\ &= \frac{1}{(n_A^*)^2} \Big| \sum_{i=0}^{n_A^*-1} \sum_{j=0}^{n_A^*-1} S_{i,j}(f_A;x^1,x^2) \Big| \end{aligned}$$

$$= \frac{1}{(n_A^*)^2}\Big|\sum_{i=M_{2A}+1}^{n_A^*-1}\sum_{j=M_{2A}+1}^{n_A^*-1}(D_i(x^1)-D_{M_{2A}}(x^1))(D_j(x^2)-D_{M_{2A}}(x^2))\Big|$$

$$= \frac{1}{(n_A^*)^2}\Big|\sum_{i=1}^{n_{A-1}^*-1}\sum_{j=1}^{n_{A-1}^*-1}(D_{i+M_{2A}}(x^1)-D_{M_{2A}}(x^1))(D_{j+M_{2A}}(x^2)-D_{M_{2A}}(x^2))\Big|$$

$$= \frac{(n_{A-1}^*)^2}{(n_A^*)^2}\,|K_{n_{A-1}^*}(x^1)|\,|K_{n_{A-1}^*}(x^2)|.$$

Let $q := \sup\{m_i : i \in \mathbb{N}\}$. For every $l = 1, \dots, [\frac{1}{4}\log_q \sqrt{A}] - 1$ (A is supposed to be large enough) let k_l^1 and k_l^2 be the smallest natural numbers for which

$$M_{2A}\sqrt{A}\,\frac{1}{q^{4l}} \le M^2_{2k_l^1} < M_{2A}\sqrt{A}\,\frac{1}{q^{4l-4}},$$
$$M_{2A}\sqrt{A}\,q^{4l} \le M^2_{2k_l^2} < M_{2A}\sqrt{A}\,q^{4l+4}.$$

Define

$$I_{2A}^{k,s}(x) := I_{2A}(0,\dots,0,x_{2k}\ne 0,0,\dots,0,x_{2s}\ne 0,x_{2s+1},\dots,x_{2A-1})$$

and let

$$(x^1,x^2) \in I_{2A}^{k_l^1,k_l^1+1}(x^1)\times I_{2A}^{k_l^2,k_l^2+1}(x^2).$$

Then from Lemma 2 and (4) we obtain

$$\sigma_0^* f_A(x^1,x^2) \ge c\,\frac{M^2_{2k_l^1}M^2_{2k_l^2}}{M^2_{2A}} \ge cM_{2A}\sqrt{A}\,\frac{1}{q^{4l}}\,\frac{M_{2A}\sqrt{A}\,q^{4l}}{M^2_{2A}} \ge cA.$$

On the other hand,

$$\boldsymbol{\mu}\{(x^1,x^2)\in G_m\times G_m : |\sigma_0^* f_A(x^1,x^2)| \ge cA\}$$
$$\ge c\sum_{l=1}^{[\frac{1}{4}\log_q\sqrt{A}]}\sum_x \boldsymbol{\mu}(I_{2A}^{k_l^1,k_l^1+1}(x^1)\times I_{2A}^{k_l^2,k_l^2+1}(x^2))$$

$$\Big(\sum_x := \sum_{x^1_{2k_l^1+3}=0}^{m_{2k_l^1+3}-1}\cdots\sum_{x^1_{2A-1}=0}^{m_{2A-1}-1}\sum_{x^2_{2k_l^2+3}=0}^{m_{2k_l^2+3}-1}\cdots\sum_{x^2_{2A-1}=0}^{m_{2A-1}-1}\Big)$$

$$\ge c\sum_{l=1}^{[\frac{1}{4}\log_q\sqrt{A}]}\frac{m_{2k_l^1+3}\cdots m_{2A-1}m_{2k_l^2+3}\cdots m_{2A-1}}{M^2_{2A}}$$

$$= c \sum_{l=1}^{[\frac{1}{4}\log_q \sqrt{A}]} \frac{1}{M_{2k_l^1+2} M_{2k_l^2+2}} r \geq \sum_{l=1}^{[\frac{1}{4}\log_q \sqrt{A}]} \frac{1}{M_{2k_l^1} M_{2k_l^2}}$$

$$\geq c \sum_{l=1}^{[\frac{1}{4}\log_q \sqrt{A}]} \frac{1}{(M_{2A}\sqrt{A}\, q^{-4l+1})^{1/2} (M_{2A}\sqrt{A}\, q^{4l+4})^{1/2}} \geq c \frac{\log_q A}{M_{2A}\sqrt{A}}.$$

Combining this with (3) we obtain

$$\frac{cA(\boldsymbol{\mu}\{(x^1,x^2) \in G_m \times G_m : |\sigma_0^* f_A(x^1,x^2)| \geq cA\})^2}{\|f_A\|_{H_{1/2}}} \geq \frac{cA \log_q^2 A}{M_{2A}^2 A} M_{2A}^2 = c \log_q^2 A \to \infty \quad \text{as } A \to \infty.$$

The Theorem is proved. ■

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