COLLOQUIUM MATHEMATICUM

VOL. 115 2009 NO. 1

INFINITE MEASURE PRESERVING FLOWS WITH INFINITE ERGODIC INDEX

BY

ALEXANDRE I. DANILENKO and ANTON V. SOLOMKO (Kharkov)

Abstract. We construct a rank-one infinite measure preserving flow $(T_r)_{r\in\mathbb{R}}$ such that for each $p > 0$, the "diagonal" flow $(T_r \times \cdots \times T_r)_{r\in\mathbb{R}}$ (p times) on the product space is ergodic.

0. Introduction. In [KP], Kakutani and Parry constructed an infinite measure preserving transformation T such that $T\times T$ is ergodic but $T\times T\times T$ is not. This is in stark contrast with the "classical" weak mixing property for finite measure preserving systems. Since then there has been interest in understanding dynamical properties of infinite measure preserving systems that are analogous to the classical weak mixing: trivial L^∞-spectrum [ALW], infinite ergodic index [KP], power weak mixing [AFS], [Da2], double ergodicity [B–S], etc. A number of examples of ergodic infinite measure preserving transformations with unusual weak mixing were constructed in those papers. In [Da1] and [DS1] these examples have been extended to the case of infinite measure preserving (and non-singular) actions of countable discrete Abelian groups. In a recent work [I–W], Iams, Kats, Silva, Street and Wickelgren start to investigate the weak mixing properties for infinite measure preserving actions of *continuous* Abelian groups as $\mathbb{R}$ and $\mathbb{R}^p$. They

(i) show by example that the trivial L^∞-spectrum does not imply double ergodicity,
(ii) construct an infinite measure preserving flow with ergodic Cartesian square.

The following has remained open so far: is there an infinite measure preserving flow whose Cartesian powers are all ergodic? In the present work we answer this question in the affirmative:

THEOREM 0.1. *There exists a rank-one infinite σ-finite measure preserving flow $T = (T_r)_{r\in\mathbb{R}}$ on a standard measure space such that for each $q \in \mathbb{Q}\setminus\{0\}$ and $p > 0$, the transformation $T_q \times \cdots \times T_q$ (p times) is ergodic.*

2000 *Mathematics Subject Classification*: 37A40.
Key words and phrases: infinite measure preserving transformation, ergodic.

DOI: 10.4064/cm115-1-2

It follows that every Cartesian power of T is ergodic. Given $m > 0$, we denote by $T \otimes \cdots \otimes T$ (m times) the action $(r_1, \ldots, r_m) \mapsto T_{r_1} \times \cdots \times T_{r_m}$ of $\mathbb{R}^m$. Then $T \otimes \cdots \otimes T$ is a rank-one infinite measure preserving action of $\mathbb{R}^m$ whose Cartesian powers are all ergodic. This refines the example of doubly ergodic $\mathbb{R}^m$-action from [I–W, Section 6].

We now recall the definition of rank one in the σ-finite measure preserving case. Let $S = (S_g)_{g \in \mathbb{R}^m}$ be a measure preserving action of $\mathbb{R}^m$ on a standard σ-finite measure space $(Y, \mathfrak{C}, \nu)$.

(i) A *Rokhlin tower* or *column* for S is a triple (A, f, F), where $A \in \mathfrak{C}$, F is a cube in $\mathbb{R}^m$ and $f : A \to F$ is a measurable mapping such that for any Borel subset $H \subset F$ and an element $g \in \mathbb{R}^m$ with $g + H \subset F$, one has $f^{-1}(g + H) = S_g f^{-1}(H)$.

(ii) S is said to be of *rank one* (*by cubes*) if there exists a sequence of Rokhlin towers (A_n, f_n, F_n) such that the volume of F_n goes to infinity and for any subset $C \in \mathfrak{C}$ of finite measure, there is a sequence of Borel subsets $H_n \subset F_n$ such that

$$\lim_{n\to\infty} \nu(C \triangle f_n^{-1}(H_n)) = 0.$$

We note that our example is quite different from the one that appeared in [I–W]. Iams, Kats, Silva, Street and Wickelgren use the geometrical cutting-and-stacking techniques with four cuts at every step while we use a more abstract (C, F)-construction with an “unbounded” sequence of cuts [Da1], [Da3]. This permits us to reduce the calculations.

However, we are unable to verify whether for the flow T constructed in this paper, the transformation $T_r \times \cdots \times T_r$ (p times) is ergodic for every $0 \neq r \in \mathbb{R}$ and $p > 0$. Thus the problem of existence of such flows is open.

1. (C, F)-actions of $\mathbb{R}^p$. The (C, F)-construction of measure preserving actions for discrete countable groups was introduced in [dJ] and [Da1]. It was extended to the case of locally compact second countable Abelian groups in [DS2]. (See also a recent survey [Da3].) Here we outline it briefly for $\mathbb{R}^p$, $p \in \mathbb{N}$.

Given two subsets $E, F \subset \mathbb{R}^p$, we denote by $E + F$ their algebraic sum, i.e. $E + F = \{e + f \mid e \in E,\ f \in F\}$. The algebraic difference $E - F$ is defined in a similar way. If F is a singleton, say $F = \{f\}$, then we will write $E + f$ for $E + F$. Two subsets E and F of $\mathbb{R}^p$ are called *independent* if $(E - E) \cap (F - F) = \{0\}$, i.e. if $e + f = e' + f'$ for some $e, e' \in E$ and $f, f' \in F$ then $e = e'$ and $f = f'$.

Fix $p \in \mathbb{N}$ and consider two sequences $(F_n)_{n=0}^\infty$ and $(C_n)_{n=1}^\infty$ of subsets in $\mathbb{R}^p$ such that F_n is a cube $[-a_n, a_n) \times \cdots \times [-a_n, a_n)$ (p times) for some

$a_n \in \mathbb{Q}$ and $C_n \subset \mathbb{Q}^p$ is a finite set with $\#C_n > 1$; moreover,

$$F_n \text{ and } C_{n+1} \text{ are independent;} \tag{1-1}$$

$$F_n + C_{n+1} \subset F_{n+1}. \tag{1-2}$$

This means that $F_n + C_{n+1}$ consists of $\#C_{n+1}$ mutually disjoint "copies" $F_n + c$, $c \in C_{n+1}$, of F_n and all these copies are contained in F_{n+1}. We equip F_n with the measure $(\#C_1 \cdots \#C_n)^{-1}(\lambda \restriction F_n)$, where λ denotes Lebesgue measure on $\mathbb{R}^p$. Endow C_n with the equidistributed probability measure. Let $X_n := F_n \times \prod_{k>n} C_k$ stand for the product of measure spaces. Define an embedding $X_n \to X_{n+1}$ by setting

$$(f_n, c_{n+1}, c_{n+2}, \ldots) \mapsto (f_n + c_{n+1}, c_{n+2}, \ldots).$$

It is easy to see that it is measure preserving. Then $X_0 \subset X_1 \subset \cdots$. Let $X := \bigcup_{n=0}^\infty X_n$ denote the inductive limit of the measure spaces X_n and let μ denote the corresponding measure on X. Then μ is σ-finite. It is infinite if and only if

$$\lim_{n\to\infty} \frac{\lambda(F_n)}{\#C_1 \cdots \#C_n} = \infty. \tag{1-3}$$

Assume in addition that

$$\text{given } g \in \mathbb{R}^p, \text{ there is } m \in \mathbb{N} \text{ with } g + F_n + C_{n+1} \subset F_{n+1} \text{ for all } n > m. \tag{1-4}$$

Given $g \in \mathbb{R}^p$ and $n \in \mathbb{N}$, we set

$$L_g^{(n)} := (F_n \cap (F_n - g)) \times \prod_{k>n} C_k, \qquad R_g^{(n)} := (F_n \cap (F_n + g)) \times \prod_{k>n} C_k.$$

Clearly, $L_g^{(n)} \subset L_g^{(n+1)}$ and $R_g^{(n)} \subset R_g^{(n+1)}$. Define $T_g^{(n)} \colon L_g^{(n)} \to R_g^{(n)}$ by setting

$$T_g^{(n)}(f_n, c_{n+1}, \ldots) := (f_n + g, c_{n+1}, \ldots).$$

Put

$$L_g := \bigcup_{n=1}^\infty L_g^{(n)} \subset X \quad \text{and} \quad R_g := \bigcup_{n=1}^\infty R_g^{(n)} \subset X.$$

Then a Borel one-to-one map $T_g \colon L_g \to R_g$ is well defined by $T_g \restriction L_g^{(n)} = T_g^{(n)}$. It follows from (1-4) that $L_g = R_g = X$ for each $g \in \mathbb{R}^p$. It is easy to verify that $T := (T_g)_{g\in\mathbb{R}^p}$ is a Borel μ-preserving action of $\mathbb{R}^p$. The reader can verify that μ is the only (up to scaling) T-invariant σ-finite Borel measure on X such that $\mu(X_n) < \infty$ for all n. T is called the (C, F)-*action of* $\mathbb{R}^p$ *associated with* $(C_{n+1}, F_n)_{n\ge 0}$. Each (C, F)-action is of rank one.

Given a Borel subset $A \subset F_n$, we set $[A]_n := \{x = (x_i)_{i=n}^{\infty} \in X_n \mid x_n \in A\}$ and call it an *n-cylinder* in X. Clearly,

$$[A]_n = \bigsqcup_{c \in C_{n+1}} [A + c]_{n+1}.$$

Notice also that

$$T_g[A]_n = [A + g]_n \text{ for all } g \in \mathbb{R}^p \text{ and } A \subset F_n \cap (F_n - g),\ n \in \mathbb{N}. \tag{1-5}$$

The sequence of all n-cylinders approximates the entire Borel σ-algebra on X when $n \to \infty$.

Now for each $n \in \mathbb{N}$, we fix a finite partition $\mathfrak{F}_n$ of F_n into smaller cubes of equal volume: $F_n = \bigsqcup_{\Delta \in \mathfrak{F}_n} \Delta$. If $\Delta = [d_1^1, d_1^2) \times \cdots \times [d_p^1, d_p^2)$ then we denote by $l(\Delta)$ the "left" vertex $(d_1^1, \dots, d_p^1)$ of Δ. We say that $\mathfrak{F}_{n+1}$ is a *chopping* of $\mathfrak{F}_n$ (and write $\mathfrak{F}_n \prec \mathfrak{F}_{n+1}$) if for each $\Delta \in \mathfrak{F}_n$ and $c \in C_{n+1}$, the set $\Delta + c$ is a union of several (more than one!) $\mathfrak{F}_{n+1}$-elements. Note that if $\Delta + c = \bigsqcup_{i=1}^m \Delta_i$ with $\Delta_i \in \mathfrak{F}_{n+1}$ then for each $\Delta' \in \mathfrak{F}_n$,

$$\Delta' + c = \bigsqcup_{i=1}^{m} (\Delta_i + l(\Delta') - l(\Delta)) \tag{1-6}$$

and $\Delta_i + l(\Delta') - l(\Delta) \in \mathfrak{F}_{n+1}$ for all i. From now on we will assume that

$$\mathfrak{F}_0 \prec \mathfrak{F}_1 \prec \cdots . \tag{1-7}$$

The following measure-theoretical fact is standard. We state it without proof.

LEMMA 1.1. *Let $(Y, \mathfrak{C}, \nu)$ be a standard measure space with $\nu(Y) < \infty$. Let $\mathfrak{G}_n$ be a sequence of finite partitions of Y into subsets of equal measure. Assume that $\mathfrak{G}_n$ refines and approximates $\mathfrak{C}$ as $n \to \infty$. Then for each measurable subset $A \subset Y$ and $\delta < 1$,*

$$\lim_{n \to \infty} \frac{\#\{\Delta \in \mathfrak{G}_n \mid \nu(A \cap \Delta) > \delta \nu(\Delta)\}}{\#\mathfrak{G}_n} = \frac{\nu(A)}{\nu(Y)}.$$

The following lemma will be crucial in the proof of Theorem 0.1.

LEMMA 1.2. *Fix a map $\delta : \mathbb{Q}^p \to (0,1)$ and $g \in \mathbb{R}^p$. If for infinitely many—call them "good"—numbers $n \in \mathbb{N}$ and any $\Delta_1, \Delta_2 \in \mathfrak{F}_n$ there exist $A \subset [\Delta_1]_n$ and $l \in \mathbb{Z}$ such that $\mu(A) > \delta(l(\Delta_1) - l(\Delta_2))\mu([\Delta_1]_n)$ and $T_{lg} A \subset [\Delta_2]_n$, then the transformation T_g is ergodic.*

Proof. Let $A_1, A_2 \subset X$ be two subsets of positive finite measure. First, choose $n > 0$ and $\Delta_1, \Delta_2 \in \mathfrak{F}_n$ with $\mu(A_i \cap [\Delta_i]_n) > 0.99\mu([\Delta_i]_n)$, $i = 1, 2$. We let $d := l(\Delta_1) - l(\Delta_2)$. For $m > n$, let $\mathfrak{G}_m \subset \mathfrak{F}_m$ be such that $[\Delta_1]_n = \bigsqcup_{\Delta \in \mathfrak{G}_m} [\Delta]_m$. Then $[\Delta_2]_n = \bigsqcup_{\Delta \in \mathfrak{G}_m} [\Delta - d]_m$ by (1-6). The sequence $\mathfrak{G}_m$ of finite partitions of the finite measure space $[\Delta_1]_n$ refines and approximates the entire Borel σ-algebra on $[\Delta_1]_n$. We now set

$$\mathfrak{G}_{m,1} := \{\Delta \in \mathfrak{G}_m \mid \mu(A_1 \cap [\Delta]_m) > (1 - \delta(d)/2)\mu([\Delta]_m)\},$$
$$\mathfrak{G}_{m,2} := \{\Delta \in \mathfrak{G}_m \mid \mu(A_2 \cap [\Delta - d]_m) > (1 - \delta(d)/2)\mu([\Delta - d]_m)\}.$$

By Lemma 1.1, $\#\mathfrak{G}_{m,i}/\#\mathfrak{G}_m \to \mu(A_i \cap [\Delta_i]_n)/\mu([\Delta_i]_n)$ as $m \to \infty$, and therefore $\#\mathfrak{G}_{m,i} > \frac{1}{2}\#\mathfrak{G}_m$ for large m and $i = 1, 2$. Fix a large and "good" m. Take any $\Delta_1' \in \mathfrak{G}_{m,1} \cap \mathfrak{G}_{m,2}$ (the intersection is not empty) and apply the assumption of Lemma 1.2 to Δ_1' and $\Delta_2' := \Delta_1' - d$. Then there are $A \subset [\Delta_1']_m$ and $l \in \mathbb{Z}$ such that $\mu(A \cap [\Delta_1']_m) > \delta(d)\mu([\Delta_1']_m)$ and $T_{lg}A \subset [\Delta_2']_m$. Therefore $\mu(A \cap A_1) > \frac{1}{2}\delta(d)\mu([\Delta_1']_m)$. This implies $\mu(T_{lg}A_1 \cap A_2) > 0$. ∎

2. Proof of Theorem 0.1. We construct the flow T as a (C, F)-action of $\mathbb{R}$ associated with specially selected sequences $(C_{n+1}, F_n, \mathfrak{F}_n)_{n=0}^{\infty}$ satisfying (1-1)–(1-4) and (1-7). These sequences will be constructed inductively. Suppose that we already have C_{n-1}, F_{n-1} and $\mathfrak{F}_{n-1}$. Our purpose is to construct C_n, F_n and $\mathfrak{F}_n$. Fix a map $\delta\colon \mathbb{Q} \to (0,1)$ with $\sum_{q\in\mathbb{Q}} \delta(q) < 1/2$. Let $D_{n-1} := \{l(\Delta) \mid \Delta \in \mathfrak{F}_{n-1}\} \subset \mathbb{Q}$. We enumerate the elements of $D_{n-1} - D_{n-1}$ as $d_i^{(n)}$, $i = 1, \dots, k_n$. Select integers $h_1^{(n)}, \dots, h_{k_n}^{(n)}$ in such a way that $h_i^{(n)} - 1 > \delta(d_i^{(n)})h^{(n)}$, where $h^{(n)} := h_1^{(n)} + \dots + h_{k_n}^{(n)}$. Let us enumerate the nonzero rationals as $\mathbb{Q} \setminus \{0\} = \{g^{(n)} \mid n \in \mathbb{N}\}$. Set

$$A_i := \{m(d_i^{(n)} + q_n g^{(n)}) \mid m = 1, \dots, h_i^{(n)}\}, \qquad i = 1, \dots, k_n,$$
$$C_n := \bigcup_{i=1}^{k_n} (A_i + r_i),$$

where $q_n \in \mathbb{Z}$ and $r_i \in \mathbb{Q}$, $i = 1, \dots, k_n$, are some "large" parameters chosen so that

(i) the sets $A_i + r_i$, $i = 1, \dots, k_n$, are mutually disjoint (and hence $\#C_n = h^{(n)}$),
(ii) F_{n-1} and C_n are independent.

Now set $F_n := [-a_n, a_n)$ for a rational a_n large so that $F_n \supset F_{n-1} + C_n$ and $a_n/(\#C_1 \cdots \#C_n) > n$. Finally, let $\mathfrak{F}_n$ be a partition of F_n into subintervals of equal length such that $\mathfrak{F}_n \succ \mathfrak{F}_{n-1}$. Thus all the conditions (1-1)–(1-4) and (1-7) are satisfied for $(C_{n+1}, F_n, \mathfrak{F}_n)_{n=0}^{\infty}$. Denote the associated infinite measure preserving (C, F)-action of $\mathbb{R}$ by $T = (T_r)_{r\in\mathbb{R}}$.

Fix $p \in \mathbb{N}$ and $g \in \mathbb{Q}$, $g \neq 0$. Clearly, $(T_{r_1} \times \dots \times T_{r_p})_{(r_1,\dots,r_p)\in\mathbb{R}^p}$ is the (C, F)-action of $\mathbb{R}^p$ associated with the sequence $(C_{n+1}^p, F_n^p)_{n=0}^{\infty}$. The upper index p here means the p-fold Cartesian power. Moreover, if we set

$$\mathfrak{F}_n^p := \{\Delta_1 \times \dots \times \Delta_p \mid \Delta_i \in \mathfrak{F}_n,\ i = 1, \dots, p\}$$

then $\mathfrak{F}_n^p$ is a finite partition of F_n^p and $\mathfrak{F}_0^p \prec \mathfrak{F}_1^p \prec \cdots$. We are going to apply Lemma 1.2 to show that the transformation $T_g \times \cdots \times T_g$ (p times) is ergodic. Pick any n so that $g' := q_n g^{(n)}$ is a multiple of g. (There exist infinitely many such n.) Given any $\Delta^{(1)} = \Delta_1^{(1)} \times \cdots \times \Delta_p^{(1)}$ and $\Delta^{(2)} = \Delta_1^{(2)} \times \cdots \times \Delta_p^{(2)} \in \mathfrak{F}_{n-1}^p$, there exist numbers $i_1, \dots, i_p \in \{1, \dots, k_n\}$ such that $l(\Delta^{(1)}) - l(\Delta^{(2)}) = (d_{i_1}^{(n)}, \dots, d_{i_p}^{(n)})$. This yields $\Delta_s^{(1)} - d_{i_s}^{(n)} = \Delta_s^{(2)}$, $s = 1, \dots, p$. We now define subsets $A_s \subset [\Delta_s^{(1)}]_{n-1} \subset X$, $s = 1, \dots, p$, as unions of n-cylinders:

$$A_s := \bigsqcup_{m=1}^{h_{i_s}^{(n)}-1} [\Delta_s^{(1)} + m(d_{i_s}^{(n)} + g') + r_{i_s}]_n.$$

Counting the number of these n-cylinders, we obtain

$$\mu(A_s) = \frac{h_{i_s}^{(n)} - 1}{h^{(n)}}\, \mu([\Delta_s^{(1)}]_{n-1}) > \delta(d_{i_s}^{(n)}) \mu([\Delta_s^{(1)}]_{n-1}).$$

Now we apply (1-5) to show that

$$\begin{aligned} T_{g'} A_s &= \bigsqcup_{m=1}^{h_{i_s}^{(n)}-1} [\Delta_s^{(1)} + m(d_{i_s}^{(n)} + g') + r_{i_s} + g']_n \\ &= \bigsqcup_{m=1}^{h_{i_s}^{(n)}-1} [\Delta_s^{(1)} - d_{i_s}^{(n)} + (m+1)(d_{i_s}^{(n)} + g') + r_{i_s}]_n \\ &= \bigsqcup_{m=2}^{h_{i_s}^{(n)}} [\Delta_s^{(2)} + m(d_{i_s}^{(n)} + g') + r_{i_s}]_n \subset [\Delta_s^{(2)}]_{n-1}. \end{aligned}$$

If we now set $A := A_1 \times \cdots \times A_p \subset [\Delta^{(1)}]_{n-1}$ then $(T_{g'} \times \cdots \times T_{g'}) A \subset [\Delta^{(2)}]_{n-1}$ and

$$\mu^p(A) > \delta(d_{i_1}^{(n)}) \cdots \delta(d_{i_p}^{(n)}) \mu^p([\Delta^{(1)}]_{n-1}).$$

Define a map $\widehat{\delta}\colon \mathbb{Q}^p \to (0,1)$ by setting $\widehat{\delta}(d_1, \dots, d_p) := \delta(d_1) \cdots \delta(d_p)$ for all $d_1, \dots, d_p \in \mathbb{Q}$. Then by Lemma 1.2 (with $\widehat{\delta}$ instead of δ), the transformation $T_g \times \cdots \times T_g$ (p times) is ergodic.

REFERENCES

[ALW] J. Aaronson, M. Lin and B. Weiss, *Mixing properties of Markov operators and ergodic transformations, and ergodicity of Cartesian products*, Israel J. Math. 33 (1979), 198–224.

[AFS] T. Adams, N. Friedman and C. E. Silva, *Rank one power weak mixing nonsingular transformations*, Ergodic Theory Dynam. Systems 21 (2001), 1321–1332.

[B–S] A. Bowles, L. Fidkowski, A. Marinello and C. E. Silva, *Double ergodicity of nonsingular transformations and infinite measure-preserving staircase transformations*, Illinois J. Math. 45 (2001), 999–1019.

[Da1] A. I. Danilenko, *Funny rank-one weak mixing for nonsingular Abelian actions*, Israel J. Math. 121 (2001), 29–54.

[Da2] —, *Infinite rank one actions and nonsingular Chacon transformations*, Illinois J. Math. 48 (2004), 769–786.

[Da3] —, (C, F)*-actions in ergodic theory*, in: Geometry and Dynamics of Groups and Spaces, Progr. Math. 265, Birkhäuser, 2008, 325–351.

[DS1] A. I. Danilenko and C. E. Silva, *Multiple and polynomial recurrence for Abelian actions in infinite measure*, J. London Math. Soc. (2) 69 (2004), 183–200.

[DS2] —, —, *Mixing rank-one actions of locally compact Abelian groups*, Ann. Inst. H. Poincaré Probab. Statist. 43 (2007), 375–398.

[HK] A. Hajian and S. Kakutani, *An example of an ergodic measure preserving transformation defined on an infinite measure space*, in: Lecture Notes in Math. 160, Springer, 1970, 45–52.

[I–W] S. Iams, B. Katz, C. E. Silva, B. Street and K. Wickelgren, *On weakly mixing and doubly ergodic nonsingular actions*, Colloq. Math. 103 (2005), 247–264.

[dJ] A. del Junco, *A simple map with no prime factors*, Israel J. Math. 104 (1998), 301–320.

[KP] S. Kakutani and W. Parry, *Infinite measure preserving transformations with "mixing"*, Bull. Amer. Math. Soc. 69 (1963), 752–756.

Alexandre I. Danilenko
Institute for Low Temperature
Physics & Engineering
National Academy of Sciences of Ukraine
47 Lenin Ave.
Kharkov, 61164, Ukraine
E-mail: danilenko@ilt.kharkov.ua

Anton V. Solomko
Department of Mathematics
and Mechanical Engineering
Kharkov National University
4 Freedom sq.
Kharkov, 61077, Ukraine
E-mail: solomko.anton@gmail.com

Received 26 November 2007 (4988)