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MULTIPLE CONJUGATE FUNCTIONS AND MULTIPLICATIVE LIPSCHITZ CLASSES

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Abstract. We extend the classical theorems of I. I. Privalov and A. Zygmund from single to multiple conjugate functions in terms of the multiplicative modulus of continuity. A remarkable corollary is that if a function f belongs to the multiplicative Lipschitz class $\operatorname{Lip}(\alpha_1, \dots, \alpha_N)$ for some $0 < \alpha_1, \dots, \alpha_N < 1$ and its marginal functions satisfy $f(\cdot, x_2, \dots, x_N) \in \operatorname{Lip}\beta_1, \dots, f(x_1, \dots, x_{N-1}, \cdot) \in \operatorname{Lip}\beta_N$ for some $0 < \beta_1, \dots, \beta_N < 1$ uniformly in the indicated variables x_l, $1 \le l \le N$, then $\widetilde{f}^{(\eta_1,\dots,\eta_N)} \in \operatorname{Lip}(\alpha_1, \dots, \alpha_N)$ for each choice of $(\eta_1, \dots, \eta_N)$ with $\eta_l = 0$ or 1 for $1 \le l \le N$.

1. Introduction: Conjugate functions of one variable. We briefly summarize the basic results known in the literature. Given a complex-valued function $f \in L^1(\mathbb{T})$, where $\mathbb{T} := (-\pi, \pi]$ is the one-dimensional torus, its Fourier series is defined by

$$f(x) \sim \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ikx}, \qquad x \in \mathbb{T}, \tag{1.1}$$

where

$$\widehat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx}\, dx, \qquad k \in \mathbb{Z},$$

are the Fourier coefficients of f.

We recall that the function $\widetilde{f}$ *conjugate* to f is defined by

$$\widetilde{f}(x) := \frac{1}{\pi}\ \text{P.V.} \int_{-\pi}^{\pi} f(x-t)\, \frac{1}{2} \cot \frac{t}{2}\, dt := \frac{1}{\pi} \lim_{h\to 0+} \int_{h\le|t|\le\pi} .$$

It is known (see, e.g., [6, Vol. I, p. 131]) that this principal value integral exists for almost every $x \in \mathbb{T}$, but $\widetilde{f} \notin L^1(\mathbb{T})$ in general. However, in case $\widetilde{f} \in L^1(\mathbb{T})$ the Fourier series of $\widetilde{f}$ is given by

$$\widetilde{f}(x) \sim \sum_{k\in\mathbb{Z}} (-i \operatorname{sign} k) \widehat{f}(k) e^{ikx}. \tag{1.2}$$

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The series in (1.2) is called the *conjugate series* to the Fourier series in (1.1). It is also known (see, e.g., [6, Vol. I, p. 253]) that $\widetilde{f}$ may be discontinuous in the case when f is continuous.

We recall that the modulus of continuity of a 2π-periodic continuous function f (in symbols: $f \in C(\mathbb{T})$) is defined by

$$\omega(f;\delta) := \sup_{0<|h|\le\delta} \max_{x\in\mathbb{T}} |f(x+h) - f(x)|, \qquad \delta > 0.$$

The following theorem is due to Zygmund [4] (see also [6, Vol. 1, p. 121]).

THEOREM 0. *If $f \in C(\mathbb{T})$ is such that*

$$t^{-1}\omega(f;t) \in L^1(0,\pi), \tag{1.3}$$

then the principal value integral defining $\widetilde{f}(x)$ exists in Lebesgue's sense for every $x \in \mathbb{T}$, $\widetilde{f} \in C(\mathbb{T})$, and

$$\omega(\widetilde{f};\delta) \le A\left[\int_0^\delta \frac{\omega(f;t)}{t}\,dt + \delta\int_\delta^\pi \frac{\omega(f;t)}{t^2}\,dt\right], \qquad \delta \in (0,\pi].$$

Here and below, by A we denote an absolute constant whose value may be different at each occurrence.

It follows immediately that if f is in $\operatorname{Lip}\alpha$ (or $\operatorname{lip}\alpha$) for some $0 < \alpha < 1$, then so is $\widetilde{f}$. Furthermore, if $f \in \operatorname{Lip} 1$, then

$$\omega(\widetilde{f};\delta) \le A\delta \log\frac{2\pi}{\delta}, \qquad \delta \in (0,\pi].$$

These particular cases were proved by Privalov [2].

Our aim is to extend the above results to functions of several variables. We will confine our attention basically to functions of two variables. In the last Section 5, we briefly summarize our results for functions $f \in C(\mathbb{T}^n)$, $n \ge 3$.

2. Conjugate series and functions of two variables. Let $f(x,y)$ be a complex-valued function, 2π-periodic in each variable and integrable over the two-dimensional torus $\mathbb{T}^2$, in symbols: $f \in L^1(\mathbb{T}^2)$. We remind the reader that the double Fourier series of f is defined by

$$f(x,y) \sim \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \widehat{f}(k,l)e^{i(kx+ly)} =: S[f], \qquad (x,y) \in \mathbb{T}^2, \tag{2.1}$$

where the double Fourier coefficients $\widehat{f}(k,l)$ are defined by

$$\widehat{f}(k,l) := \frac{1}{(2\pi)^2}\iint_{\mathbb{T}^2} f(x,y)e^{-i(kx+ly)}\,dx\,dy, \qquad (k,l) \in \mathbb{Z}^2.$$

We recall that the three conjugate series to the Fourier series $S[f]$ in (2.1) are defined as follows:

$$\widetilde{S}^{(1,0)}[f] := \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} (-i\operatorname{sign} k)\widehat{f}(k,l)e^{i(kx+ly)} \tag{2.2}$$

(conjugate with respect to the first variable),

$$\widetilde{S}^{(0,1)}[f] := \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} (-i\operatorname{sign} l)\widehat{f}(k,l)e^{i(kx+ly)} \tag{2.3}$$

(conjugate with respect to the second variable), and

$$\widetilde{S}^{(1,1)}[f] := \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} (-i\operatorname{sign} k)(-i\operatorname{sign} l)\widehat{f}(k,l)e^{i(kx+ly)} \tag{2.4}$$

(conjugate with respect to both variables). We note that the series (2.1)–(2.4) are interrelated as follows:

$$\begin{aligned} &S[f] + i\widetilde{S}^{(1,0)}[f] + i\widetilde{S}^{(0,1)}[f] + i^2\widetilde{S}^{(1,1)}[f] \\ &\quad = \widehat{f}(0,0) + 2\sum_{k=1}^{\infty} \widehat{f}(k,0)z^k + 2\sum_{l=1}^{\infty} \widehat{f}(0,l)w^l + 4\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \widehat{f}(k,l)z^k w^l, \end{aligned} \tag{2.5}$$

where $z := e^{ix}$ and $w := e^{iy}$.

The corresponding conjugate functions are the following:

$$\widetilde{f}^{(1,0)}(x,y) := \frac{1}{\pi}\ \text{P.V.} \int_{\mathbb{T}} f(x-t,y)\,\frac{1}{2}\cot\frac{t}{2}\,dt,$$

$$\widetilde{f}^{(0,1)}(x,y) := \frac{1}{\pi}\ \text{P.V.} \int_{\mathbb{T}} f(x,y-t)\,\frac{1}{2}\cot\frac{t}{2}\,dt,$$

and

$$\begin{aligned} \widetilde{f}^{(1,1)}(x,y) &:= \frac{1}{\pi^2}\ \text{P.V.} \iint_{\mathbb{T}^2} f(x-t_1,y-t_2)\left(\frac{1}{2}\cot\frac{t_1}{2}\right)\left(\frac{1}{2}\cot\frac{t_2}{2}\right) dt_1\,dt_2 \\ &:= \frac{1}{\pi^2} \lim_{\substack{h_1\to 0+\\ h_2\to 0+}} \int_{h_1\le|t_1|\le\pi}\ \int_{h_2\le|t_2|\le\pi}. \end{aligned}$$

It follows from the corresponding one-dimensional theorem that if $f \in L^1(\mathbb{T}^2)$, then both $\widetilde{f}^{(1,0)}(x,y)$ and $\widetilde{f}^{(0,1)}(x,y)$ exist for almost every $(x,y) \in \mathbb{T}^2$. Furthermore, Sokół-Sokołowski [3] proved that if $f \in L\log^+ L(\mathbb{T}^2)$, then $\widetilde{f}^{(1,1)}(x,y)$ exists for almost every $(x,y) \in \mathbb{T}^2$.

3. Main results. Let $f(x,y)$ be a continuous function, 2π-periodic in each variable, in symbols: $f \in C(\mathbb{T}^2)$. We introduce the notion of the *mul-*

tiplicative modulus of continuity of f by setting

$$\omega(f;\delta_1,\delta_2) := \sup_{0<|h_j|\le\delta_j,\, j=1,2} \max_{(x,y)\in\mathbb{T}^2} |f(x+h_1,y+h_2) - f(x,y+h_2) - f(x+h_1,y) + f(x,y)|, \qquad \delta_1,\delta_2>0.$$

We note that in the particular case when

$$f(x,y) = f_1(x)f_2(y), \qquad (x,y)\in\mathbb{T}^2,$$

it is clear that

$$\omega(f;\delta_1,\delta_2) = \omega(f_1;\delta_1)\omega(f_2;\delta_2), \qquad \delta_1,\delta_2>0.$$

This explains the term "multiplicative modulus of continuity".

We recall that the (ordinary) *total modulus of continuity* of a function $f\in C(\mathbb{T}^2)$ is defined by

$$\widetilde{\omega}(f;\delta_1,\delta_2) := \sup_{0<|h_j|\le\delta_j,\, j=1,2} \max_{(x,y)\in\mathbb{T}^2} |f(x+h_1,y+h_2)-f(x,y)|, \qquad \delta_1,\delta_2>0.$$

In the particular case when

$$f(x,y) = f_1(x)+f_2(y), \qquad (x,y)\in\mathbb{T}^2,$$

it is clear that

$$\widetilde{\omega}(f;\delta_1,\delta_2) = \omega(f_1;\delta_1)+\omega(f_2;\delta_2), \qquad \delta_1,\delta_2>0.$$

On the other hand, in the multiplicative case only the following estimate is available:

$$\widetilde{\omega}(f_1(x)f_2(y);\delta_1,\delta_2) \le \|f_2\|\omega(f_1;\delta_1)+\|f_1\|\omega(f_2;\delta_2), \qquad \delta_1,\delta_2>0,$$

where $\|\cdot\|$ is the usual maximum norm in $C(\mathbb{T})$. This is why we use the multiplicative modulus of continuity in place of the total modulus in the case of multiple conjugate functions.

Our first new result reads as follows.

THEOREM 1. *If $f\in C(\mathbb{T}^2)$ is such that*

$$t^{-1}\omega(f(\cdot,y);t)\in L^1(0,\pi) \quad \textit{for every } y\in\mathbb{T} \tag{3.1}$$

and

$$t^{-1}\omega(f;t,\delta_2)\in L^1(0,\pi) \quad \textit{for every } \delta_2\in(0,\pi], \tag{3.2}$$

then the principal value integral defining $\widetilde{f}^{(1,0)}(x,y)$ exists in Lebesgue's sense for every $(x,y)\in\mathbb{T}^2$ and

$$\omega(\widetilde{f}^{(1,0)};\delta_1,\delta_2) \le A\left[\int_0^{\delta_1}\frac{\omega(f;t,\delta_2)}{t}\,dt + \delta_1\int_{\delta_1}^{\pi}\frac{\omega(f;t,\delta_2)}{t^2}\,dt\right], \qquad \delta_1,\delta_2\in(0,\pi]. \tag{3.3}$$

If, in addition,

$$\lim_{\delta_2\to 0+}\int_0^\pi \frac{\omega(f;t,\delta_2)}{t}\,dt = 0, \tag{3.4}$$

then $\widetilde{f}^{(1,0)} \in C(\mathbb{T}^2)$.

REMARK 1. It is not difficult to check that if condition (3.2) is satisfied, then

$$\lim_{t\to 0+} t\int_t^\pi \frac{\omega(f;u,\delta_2)}{u^2}\,du = 0 \qquad \text{for every } \delta_2 \in (0,\pi].$$

Thus, under the conditions (3.1) and (3.2) (without (3.4)), we have

$$\omega(\widetilde{f}^{(1,0)};\delta_1,\delta_2) \to 0 \qquad \text{as } \delta_1,\delta_2 \to 0$$

independently of one another.

REMARK 2. We say that a function $f \in C(\mathbb{T}^2)$ belongs to the *multiplicative Lipschitz class* $\mathrm{Lip}(\alpha,\beta)$ for some $\alpha,\beta > 0$ if

$$\omega(f;\delta_1,\delta_2) \le A\delta_1^\alpha\delta_2^\beta, \qquad \delta_1,\delta_2 > 0.$$

Furthermore, we say that a function $f \in \mathrm{Lip}(\alpha,\beta)$ belongs to the *multiplicative little Lipschitz class* $\mathrm{lip}(\alpha,\beta)$ if

$$\delta_1^{-\alpha}\delta_2^{-\beta}\omega(f;\delta_1,\delta_2) \to 0 \qquad \text{as } \delta_1,\delta_2 \to 0$$

independently of one another.

Both notions were first used in [1], on the present author's suggestions, to investigate the continuity behavior of the sum of double trigonometric series with nonnegative coefficients.

Now, it is easy to check that if $f \in \mathrm{Lip}(\alpha,\beta)$ for some $0 < \alpha < 1$ and $\beta > 0$, and condition (3.1) is satisfied, then it follows from (3.3) that $\widetilde{f}^{(1,0)} \in \mathrm{Lip}(\alpha,\beta)$. Furthermore, if $f \in \mathrm{lip}(\alpha,\beta)$ for some $0 < \alpha < 1$ and $\beta > 0$, and condition (3.1) is satisfied, then $\widetilde{f}^{(1,0)} \in \mathrm{lip}(\alpha,\beta)$. We note that if $f \in \mathrm{Lip}(1,\beta)$ for some $\beta > 0$ and condition (3.1) is satisfied, then

$$\omega(\widetilde{f}^{(1,0)},\delta_1,\delta_2) \le A\delta_1\delta_2^\beta\log\frac{2\pi}{\delta_1}, \qquad 0 < \delta_1,\delta_2 \le \pi.$$

REMARK 3. The proof of the symmetric counterpart of Theorem 1 for the conjugate function $\widetilde{f}^{(0,1)}$ runs along the lines analogous to those in Theorem 1.

THEOREM 1′. *If* $f \in C(\mathbb{T}^2)$ *is such that*

$$t^{-1}\omega(f(x,\cdot);t) \in L^1(0,\pi) \qquad \textit{for every } x \in \mathbb{T}$$

and

$$t^{-1}\omega(f;\delta_1,t) \in L^1(0,\pi) \qquad \textit{for every } \delta_1 \in (0,\pi],$$

then the principal value integral defining $\widetilde{f}^{(0,1)}$ *exists in Lebesgue's sense for every* $(x,y) \in \mathbb{T}^2$ *and*

$$\omega(\widetilde{f}^{(0,1)}; \delta_1, \delta_2) \le A \left[\int_0^{\delta_2} \frac{\omega(f; \delta_1, t)}{t} \, dt + \delta_2 \int_{\delta_2}^{\pi} \frac{\omega(f; \delta_1, t)}{t^2} \, dt \right], \qquad \delta_1, \delta_2 \in (0, \pi].$$

If, in addition,

$$\lim_{\delta_1 \to 0+} \int_0^{\pi} \frac{\omega(f; \delta_1, t)}{t} \, dt = 0,$$

then $\widetilde{f}^{(0,1)} \in C(\mathbb{T}^2)$.

Each of the statements in Remarks 1 and 2 can be reformulated for $\widetilde{f}^{(0,1)}$ in place of $\widetilde{f}^{(1,0)}$.

REMARK 4. We say that the marginal function $f(\cdot, y)$ is in $\operatorname{Lip} \alpha_1$ for some $\alpha_1 > 0$ uniformly in $y \in \mathbb{T}$ if the inequality

$$\omega(f(\cdot, y); \delta_1) \le A \delta_1^{\alpha_1}, \qquad \delta_1 > 0,$$

holds for every $y \in \mathbb{T}$ with the same Lipschitz constant A.

The following Corollary 1 is a simple consequence of Theorems 1 and 1′, on taking into account Remarks 1 and 2 and their symmetric counterparts.

COROLLARY 1. (a) *If* $f \in \operatorname{Lip}(\alpha, \beta)$ *for some* $0 < \alpha, \beta < 1$, *and* $f(\cdot, y) \in \operatorname{Lip} \alpha_1$ *for some* $0 < \alpha_1 < 1$ *uniformly in* $y \in \mathbb{T}$, *and* $f(x, \cdot) \in \operatorname{Lip} \beta_1$ *for some* $0 < \beta_1 < 1$ *uniformly in* $x \in \mathbb{T}$, *then*

$$\widetilde{f}^{(1,0)}, \widetilde{f}^{(0,1)} \in \operatorname{Lip}(\alpha, \beta). \tag{3.5}$$

Furthermore,

$$\begin{aligned} &\widetilde{f}^{(1,0)}(\cdot, y), \widetilde{f}^{(0,1)}(\cdot, y) \in \operatorname{Lip} \alpha_1, \\ &\widetilde{f}^{(1,0)}(x, \cdot), \widetilde{f}^{(0,1)}(x, \cdot) \in \operatorname{Lip} \beta_1, \end{aligned} \tag{3.6}$$

and the memberships in (3.6) *hold uniformly in* $y \in \mathbb{T}$ *or* $x \in \mathbb{T}$, *respectively.*

(b) *Part* (a) *remains valid if each* Lip *is replaced by* lip.

REMARK 5. Finally, we recall the following well-known fact. If $f \in L^2(\mathbb{T}^2)$, then each of the conjugate functions $\widetilde{f}^{(1,0)}$, $\widetilde{f}^{(0,1)}$, $\widetilde{f}^{(1,1)}$ also belongs to $L^2(\mathbb{T}^2)$, and

$$(\widetilde{f}^{(1,0)})^{\sim(0,1)}(x, y) = (\widetilde{f}^{(0,1)})^{\sim(1,0)} x, y) = \widetilde{f}^{(1,1)}(x, y)$$

for almost every $(x, y) \in \mathbb{T}^2$. This is a straightforward consequence of the unicity of Fourier series of functions in $L^2(\mathbb{T}^2)$.

Keeping this fact in mind, we can conclude that, under the conditions of Corollary 1, the conclusions (3.5) and (3.6) also hold true for $\widetilde{f}^{(1,1)}$ in place of $\widetilde{f}^{(1,0)}$ or $\widetilde{f}^{(0,1)}$.

4. Proof of Theorem 1. (i) Due to (3.1), Theorem 0 applies to the marginal function $f(\cdot, y)$, where $y \in \mathbb{T}$ is fixed. As a result, $\widetilde{f}^{(1,0)}(x, y)$ is continuous in $x \in \mathbb{T}$; furthermore, by (2.5) and (3.1), we have

$$\begin{aligned}\widetilde{f}^{(1,0)}(x, y) &= -\frac{1}{\pi} \lim_{h \to 0+} \int_h^\pi [f(x + t, y) - f(x - t, y)] \frac{1}{2} \cot \frac{t}{2}\, dt \\ &= -\frac{1}{\pi} \int_0^\pi [f(x + t, y) - f(x - t, y)] \frac{1}{2} \cot \frac{t}{2}\, dt,\end{aligned}$$

since the integrand is majorized by $t^{-1}\omega(f(\cdot, y); 2t)$ which is Lebesgue integrable on the interval $(0, \pi)$. Thus, the principal value integral defining $\widetilde{f}^{(1,0)}(x, y)$ exists in Lebesgue's sense for every $(x, y) \in \mathbb{T}^2$.

It follows from the above representation that for any real number h_2 we have

$$\begin{aligned}&|\widetilde{f}^{(1,0)}(x, y + h_2) - \widetilde{f}^{(1,0)}(x, y)| \\ &= \left| -\frac{1}{\pi} \int_0^\pi [f(x+t, y+h_2) - f(x-t, y+h_2) - f(x+t, y) + f(x-t, y)] \frac{1}{2} \cot \frac{t}{2}\, dt \right| \\ &\qquad \le \frac{1}{\pi} \int_0^\pi \frac{\omega(f; 2t, |h_2|)}{t}\, dt \to 0 \quad \text{as } h_2 \to 0,\end{aligned}$$

independently of (x, y), where the limit relation is due to (3.4). This proves that the family of functions $\{\widetilde{f}^{(1,0)}(x, y) : x \in \mathbb{T}\}$ is equicontinuous in the second variable y. Since we have seen above that $\widetilde{f}^{(1,0)}$ is continuous in the first variable x, we conclude that $\widetilde{f}^{(1,0)} \in C(\mathbb{T}^2)$.

(ii) Now, we turn to the proof of (3.3). To this end, we introduce the auxiliary functions

$$\phi_{y,h_2}(x) := f(x, y + h_2) - f(x, y), \qquad x \in \mathbb{T}, \tag{4.1}$$

where $y \in \mathbb{T}$ and $0 < |h_2| \le \pi$ are fixed. We claim that condition (1.3) in Theorem 0 is satisfied with ϕ_{y,h_2} in place of f. Indeed, it follows from (3.1) that

$$t^{-1}\omega(\phi_{y,h_2}; t) \le t^{-1}[\omega(f(\cdot, y + h_2); t) + \omega(f(\cdot, y); t)] \in L^1(0, \pi).$$

Applying Theorem 0 yields $\phi_{y,h_2} \in C(\mathbb{T})$ for all fixed $y \in \mathbb{T}$ and $0 < |h_2| \le \pi$, and

$$\omega(\widetilde{\phi}_{y,h_2}; \delta_1) \le A\Big[\int_0^{\delta_1} t^{-1}\omega(\phi_{y,h_2}; t)\, dt + \delta_1 \int_{\delta_1}^\pi t^{-2}\omega(\phi_{y,h_2}; t)\, dt\Big], \qquad \delta_1 \in (0, \pi]. \tag{4.2}$$

By (4.1) and (i) above, we have

$$(4.3)\qquad \widetilde{\phi}_{y,h_2}(x) = \frac{1}{\pi}\int_{\mathbb{T}} \phi_{y,h_2}(x-t)\,\frac{1}{2}\cot\frac{t}{2}\,dt$$
$$= \widetilde{f}^{(1,0)}(x, y+h_2) - \widetilde{f}^{(1,0)}(x,y), \qquad (x,y)\in\mathbb{T}^2,\ 0<|h_2|\le\pi.$$

Replacing x by $x+h_1$ in (4.3), we find that

$$(4.4)\qquad \widetilde{\phi}_{y,h_2}(x+h_1) - \widetilde{\phi}_{y,h_2}(x)$$
$$= \widetilde{f}^{(1,0)}(x+h_1, y+h_2) - \widetilde{f}^{(1,0)}(x+h_1, y)$$
$$- \widetilde{f}^{(1,0)}(x, y+h_2) + \widetilde{f}^{(1,0)}(x,y)$$
$$:= \Delta(\widetilde{f}^{(1,0)}; x, y; h_1, h_2), \qquad (x,y)\in\mathbb{T}^2,\ 0<|h_j|\le\pi,\ j=1,2.$$

Analogously, we also have

$$(4.5)\qquad \phi_{y,h_2}(x+h_1) - \phi_{y,h_2}(x) = \Delta(f; x, y; h_1, h_2).$$

By (4.4) and (4.5), inequality (4.2) can be rewritten as

$$(4.6)\qquad \sup_{0<|h_1|\le\delta_1}\ \max_{x\in\mathbb{T}} |\Delta(\widetilde{f}^{(1,0)}; x, y; h_1, h_2)|$$
$$= \sup_{0<|h_1|\le\delta_1}\ \max_{x\in\mathbb{T}} |\widetilde{\phi}_{y,h_2}(x+h_1) - \widetilde{\phi}_{y,h_2}(x)|$$
$$\le A\Big[\int_0^{\delta_1} t^{-1} \sup_{0<|h_1|\le\delta_1}\ \max_{x\in\mathbb{T}} |\Delta(f; x, y; h_1, h_2)|\,dt$$
$$+ \delta_1 \int_{\delta_1}^{\pi} t^{-2} \sup_{0<|h_1|\le\delta_1}\ \max_{x\in\mathbb{T}} |\Delta(f; x, y; h_1, h_2)|\,dt\Big].$$

Now, taking the supremum over $y\in\mathbb{T}$ and $0<|h_2|\le\delta_2$ on both sides of (4.6) gives (3.1) to be proved.

5. Extension to functions of several variables. We recall (see, e.g., [5] or [6, Vol. II, Ch. 17]) that the multiple Fourier series of a function $f\in L^1(\mathbb{T}^N)$ is given by

$$(5.1)\qquad f(x_1,\dots,x_N)$$
$$\sim \sum_{k_1\in\mathbb{Z}} \cdots \sum_{k_N\in\mathbb{Z}} \widehat{f}(k_1,\dots,k_N) e^{i(k_1x_1+\dots+k_Nx_N)}, \qquad (x_1,\dots,x_N)\in\mathbb{T}^N,$$

where the multiple Fourier coefficients $\widehat{f}(k_1,\dots,k_N)$ are defined by

$$\widehat{f}(k_1,\dots,k_N) := \frac{1}{(2\pi)^N} \int\!\cdots\!\int_{\mathbb{T}^N} f(x_1,\dots,x_N) e^{-i(k_1x_1+\dots+k_Nx_N)}\,dx_1\dots dx_N,$$
$$(k_1,\dots,k_N)\in\mathbb{Z}^N.$$

Let $j_1, \dots, j_n$ and n be natural numbers such that
$$1 \le j_1 < \dots < j_n \le N, \qquad 1 \le n \le N.$$
The N-fold series
$$\sum_{k_1 \in \mathbb{Z}} \dots \sum_{k_N \in \mathbb{Z}} \prod_{m=1}^{n} (-i \operatorname{sign} k_{j_m}) \widehat{f}(k_1, \dots, k_N) e^{i(k_1 x_1 + \dots + k_N x_N)} \tag{5.2}$$
is called the *conjugate series* to the Fourier series in (5.1) with respect to the specified variables $x_{j_1}, \dots, x_{j_n}$. Clearly, altogether there are $2^N - 1$ conjugate series to the Fourier series in (5.1).

Set
$$\eta_l := \begin{cases} 1 & \text{if } l = j_m \text{ for some } 1 \le m \le n, \\ 0 & \text{otherwise, where } l = 1, \dots, N. \end{cases}$$
The conjugate function $\widetilde{f}^{(\eta_1, \dots, \eta_N)}$ to f with respect to the variables $x_{j_1}, \dots, x_{j_n}$ is defined by means of a principal value integral as follows:
$$\widetilde{f}^{(\eta_1, \dots, \eta_N)}(x_1, \dots, x_N) := \frac{1}{\pi^n} \text{ P.V. } \int \dots \int_{\mathbb{T}^{(j_1, \dots, j_n)}} f(x_1 - \eta_1 t_1, \dots, x_N - \eta_N t_N)$$
$$\times \prod_{m=1}^{n} \left(\frac{1}{2} \cot \frac{t_{j_m}}{2} \right) dt_{j_1} \dots dt_{j_n} := \frac{1}{\pi^n} \lim_{h_{j_1}, \dots, h_{j_n} \to 0+} \int \dots \int_{\mathbb{T}(h_{j_1}, \dots, h_{j_n})},$$
where
$$\mathbb{T}^{(j_1, \dots, j_n)} := \{(t_{j_1}, \dots, t_{j_n}) : t_{j_m} \in \mathbb{T},\ m = 1, \dots, n\},$$
$$\mathbb{T}(h_{j_1}, \dots, h_{j_n}) := \{(t_{j_1}, \dots, t_{j_n}) : h_{j_m} \le |t_{j_m}| \le \pi,\ m = 1, \dots, n\}.$$
The following theorem is due to Zygmund [5]: If
$$\int \cdots \int_{\mathbb{T}^N} |f(x_1, \dots, x_N)| (\log^+ |f(x_1, \dots, x_N)|)^{n-1} \, dx_1 \dots dx_N < \infty,$$
then the principal value integral defining $\widetilde{f}^{(\eta_1, \dots, \eta_N)}(x_1, \dots, x_N)$ exists for almost every $(x_1, \dots, x_N)$; furthermore, if
$$\int \dots \int_{\mathbb{T}^N} |f(x_1, \dots, x_N)| (\log^+ |f(x_1, \dots, x_N)|)^{n} \, dx_1 \dots dx_N < \infty,$$
then $\widetilde{f}^{(\eta_1, \dots, \eta_N)} \in L^1(\mathbb{T}^N)$ and the conjugate series (5.2) is the Fourier series of the conjugate function $\widetilde{f}^{(\eta_1, \dots, \eta_N)}$ for each N-tuple $(\eta_1, \dots, \eta_N)$ with $\eta_l = 0$ or 1 for $1 \le l \le N$.

Finally, given a function f defined on $\mathbb{T}^N$, $(x_1, \dots, x_N) \in \mathbb{T}^N$, and $h_j \ne 0$, $j = 1, \dots, N$, we use the notation
$$\Delta(f; x_1, \dots, x_N; h_1, \dots, h_N)$$
$$:= \sum_{\eta_1 = 0}^{1} \dots \sum_{\eta_N = 0}^{1} (-1)^{\eta_1 + \dots + \eta_N} f(x_1 + \eta_1 h_1, \dots, x_N + \eta_N h_N).$$

We introduce the notion of the multiplicative modulus of continuity of $f \in C(\mathbb{T}^N)$ as follows: for $\delta_1, \dots, \delta_N > 0$, set

$$\omega(f; \delta_1, \dots, \delta_N) := \sup_{0<|h_j|\le\delta_j,\, j=1,\dots,N} \ \max_{(x_1,\dots,x_N)\in\mathbb{T}^N} |\Delta(f; x_1, \dots, x_N; h_1, \dots, h_N)|.$$

Now, the extension of Theorem 1 reads as follows.

THEOREM 2. *If* $f \in C(\mathbb{T}^N)$ *is such that*

$$t^{-1}\omega(f(t, x_2, \dots, x_N); t) \in L^1(0, \pi) \quad \text{for every } (x_2, \dots, x_N) \in \mathbb{T}^{N-1}$$

and

$$t^{-1}\omega(f; \cdot, \delta_2, \dots, \delta_N) \in L^1(0, \pi) \quad \text{for every } \delta_2, \dots, \delta_N \in (0, \pi],$$

then the principal value integral defining $\widetilde{f}^{(1,0,\dots,0)}$ *exists in Lebesgue's sense at every* $(x_1, \dots, x_N) \in \mathbb{T}^N$ *and*

$$\omega(\widetilde{f}^{(1,0,\dots,0)}; \delta_1, \delta_2, \dots, \delta_N) \le A\Big[\int_0^{\delta_1} \frac{\omega(f; t, \delta_2, \dots, \delta_N)}{t}\, dt + \delta_1 \int_{\delta_1}^{\pi} \frac{\omega(f; t, \delta_2, \dots, \delta_N)}{t^2}\, dt\Big], \quad \delta_1, \delta_2, \dots, \delta_N \in (0, \pi]. \tag{5.3}$$

If, in addition,

$$\lim_{\delta_2,\dots,\delta_N\to 0+} \int_0^{\pi} \frac{\omega(f; t, \delta_2, \dots, \delta_N)}{t}\, dt = 0,$$

then $\widetilde{f}^{(1,0,\dots,0)} \in C(\mathbb{T}^N)$.

Theorem 2 can be proved by induction on N. Indeed, according to Theorems 0 and 1, the conclusion is true for $N = 1$ and $N = 2$. Let $N \ge 3$ and assume that the conclusion has been proved for $N - 1$. From this induction hyphothesis, the validity of (5.3) can be justified in the same way as in the proof of Theorem 1 by relying on Theorem 0. In particular, this time we consider the auxiliary functions

$$\phi_{x_N,h_N}(x_1, \dots, x_{N-1}) := f(x_1, \dots, x_{N-1}, x_N + h_N) - f(x_1, \dots, x_{N-1}, x_N),\ (x_1, \dots, x_{N-1}) \in \mathbb{T}^{N-1},$$

where $x_N \in \mathbb{T}$ and $0 < |h_N| \le \pi$ are fixed. The additional statement in Theorem 2 can also be justified along the same lines as in the case of the analogous statement in Theorem 1.

REMARK 6. It is evident that a theorem analogous to Theorem 2 holds true for each conjugate function $\widetilde{f}^{(\eta_1,\dots,\eta_N)}$ in place of $\widetilde{f}^{(1,0,\dots,0)}$, where $\eta_m = 1$ for a certain $2 \le m \le N$ and $\eta_l = 0$ for every $l \ne m$, $1 \le l \le N$.

REMARK 7. We say that $f \in C(\mathbb{T}^N)$ belongs to the *multiplicative Lipschitz class* $\mathrm{Lip}(\alpha_1, \dots, \alpha_N)$ for some $\alpha_1, \dots, \alpha_N > 0$ if

$$\omega(f; \delta_1, \dots, \delta_N) \le A\delta_1^{\alpha_1} \dots \delta_N^{\alpha_N}, \qquad \delta_1, \dots, \delta_N > 0;$$

and that $f \in \mathrm{Lip}(\alpha_1, \dots, \alpha_N)$ belongs to the *multiplicative little Lipschitz class* $\mathrm{lip}(\alpha_1, \dots, \alpha_N)$ if

$$\delta_1^{-\alpha_1} \dots \delta_N^{-\alpha_N} \omega(f; \delta_1, \dots, \delta_N) \to 0 \quad \text{ as } \delta_1, \dots, \delta_N \to 0$$

independently of one another.

Now, the following Corollary 2 of Theorem 2 is the extension of Corollary 1 from double to multiple conjugate functions.

COROLLARY 2. (a) *If* $f \in \mathrm{Lip}(\alpha_1, \dots, \alpha_N)$ *for some* $0 < \alpha_1, \dots, \alpha_N < 1$, *and the marginal functions satisfy* $f(\cdot, x_2, \dots, x_N) \in \mathrm{Lip}\,\beta_1$ *for some* $0 < \beta_1 < 1$ *uniformly in* $(x_2, \dots, x_N) \in \mathbb{T}^{N-1}, \dots,$ $f(x_1, \dots, x_{N-1}, \cdot) \in \mathrm{Lip}\,\beta_N$ *for some* $0 < \beta_N < 1$ *uniformly in* $(x_1, \dots, x_{N-1}) \in \mathbb{T}^{N-1}$, *then* $\widetilde{f}^{(\eta_1, \dots, \eta_N)} \in \mathrm{Lip}(\alpha_1, \dots, \alpha_N)$ *for each* N*-tuple* $(\eta_1, \dots, \eta_N)$ *with* $\eta_l = 0$ *or* 1, $1 \le l \le N$.

(b) *Part* (a) *remains valid if each* Lip *is replaced by* lip.

Corollary 2 can also be proved by induction on N, making use of the uniqueness of Fourier series of functions in $L^2(\mathbb{T}^N)$. By the uniqueness, if $f \in L^2(\mathbb{T}^N)$, then $f^{(\eta_1, \dots, \eta_N)} \in L^2(\mathbb{T}^N)$ for each N-tuple $(\eta_1, \dots, \eta_N)$, where $\eta_{j_1} = \dots = \eta_{j_m} = 1$ for $1 \le j_1 < \dots < j_m \le N$, $1 \le m \le N$, and $\eta_l = 0$ otherwise, $1 \le l \le N$; furthermore, for this N-tuple $(\eta_1, \dots, \eta_N)$ we have

$$\widetilde{f}^{(\eta_1, \dots, \eta_N)}(x_1, \dots, x_N) = (\widetilde{f}^{(0, \dots, \eta_{j_1}, \dots, 0)})^{\sim \dots \sim (0, \dots, \eta_{j_m}, \dots, 0)}(x_1, \dots, x_N)$$

for almost every $(x_1, \dots, x_N) \in \mathbb{T}^N$.

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