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RINGEL–HALL ALGEBRAS OF HEREDITARY PURE SEMISIMPLE COALGEBRAS

BY

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Abstract. We define and investigate Ringel–Hall algebras of coalgebras (usually infinite-dimensional). We extend Ringel's results [Banach Center Publl. 26 (1990) and Adv. Math. 84 (1990)] from finite-dimensional algebras to infinite-dimensional coalgebras.

1. Introduction. Let K be a finite field, $\mathbb{C}$ be the field of complex numbers, and let C be a K-coalgebra. Denote by C-comod the category of all finite-dimensional left C-comodules. For X, Y, Z in C-comod, we denote by $F^X_{Z,Y} = F^X_{Z,Y}(C)$ the number of all C-subcomodules U of X such that $U \simeq Y$ and $X/U \simeq Z$. Analogously to [16], we define the *Ringel–Hall algebra* $\mathcal{H}(C)$ to be the $\mathbb{C}$-vector space with basis $\{u_M\}_{[M]}$ indexed by all isomorphism classes of finite-dimensional left C-modules and with multiplication given by the formula

$$u_{[M]}u_{[N]} = \sum_{[X]} F^X_{M,N} u_{[X]},$$

where the sum runs over all isomorphism classes of finite-dimensional left C-comodules.

In this paper we investigate the Ringel–Hall algebras $\mathcal{H}(C)$ and extend results given in [16]–[18] for finite-dimensional algebras to a class of coalgebras. In particular, we prove the existence of Hall polynomials for hereditary pure semisimple coalgebras and describe the corresponding Lie algebras.

The motivation for the study of Ringel–Hall algebras is their connection with generic extensions, Lie algebras and quantum groups (see [14]–[19]). Connections of Ringel–Hall algebras with Lie algebras are also studied in Section 5 of this paper.

The paper is organised as follows. In Section 2 we recall basic definitions and notation concerning algebras and coalgebras.

In Section 3 we recall the definition of the Ringel–Hall algebra $\mathcal{H}(A)$ of a finite-dimensional algebra A, give the definition of the Ringel–Hall algebra

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$\mathcal{H}(C)$ of a coalgebra C and prove the basic properties of the Ringel–Hall algebras of coalgebras.

In Section 4 we collect the basic facts on species and (co)tensor (co)algebras that we need in Section 5.

Section 5 contains the main results of this paper. In Proposition 5.8, we prove the existence of Hall polynomials for hereditary pure semisimple coalgebras. Moreover, we define the Ringel–Hall algebra $\mathcal{H}(Q,\mathbf{d})$, its specialisation $\mathcal{H}(Q,\mathbf{d})_1$, and the Lie subalgebra $\mathcal{K}(Q,\mathbf{d})_1$ of $\mathcal{H}(Q,\mathbf{d})_1$, for any valued quiver $(Q,\mathbf{d})$ from Table 1.2. Theorem 5.9 contains basic properties of the Lie algebra $\mathcal{K}(Q,\mathbf{d})_1$ and the $\mathbb{C}$-algebra $\mathcal{H}(Q,\mathbf{d})_1$ for each of the pure semisimple valued Dynkin quivers in Tables 1.1 and 1.2. In particular, we describe $\mathcal{K}(Q,\mathbf{d})_1$ by generators and relations and show that $\mathcal{H}(Q,\mathbf{d})_1$ is the universal enveloping algebra of $\mathcal{K}(Q,\mathbf{d})_1$. Moreover, we prove that $\mathcal{K}(Q,\mathbf{d})_1$ is isomorphic to the positive part $\mathfrak{n}_+$ of the infinite rank affine Lie algebra $\mathfrak{g}$ associated with $(Q,\mathbf{d})$ if $(Q,\mathbf{d})$ is any of the valued quivers in Table 1.2 (see [10, 7.11]).

In this paper we are mainly interested in coalgebras that are infinite-dimensional. We should mention that for finite-dimensional coalgebras the results of this paper follow from Ringel's papers [15]–[19]. However, we present all facts for arbitrary coalgebras, not only infinite-dimensional.

2. Preliminaries on coalgebras and finite-dimensional algebras. In this section we collect basic information on algebras and coalgebras. For coalgebra representations we use the notation and terminology of [11], [24] and [26]. The reader is referred to [1], [2], [9], [23], [29], and [30] for the terminology and notation of representation theory, and to [12] and [31] for background on coalgebras and comodules.

We fix an arbitrary field K. Let A be a finite-dimensional K-algebra. Let C be a K-coalgebra (usually infinite-dimensional), with comultiplication Δ and counity ε. We recall that a *left C-comodule* is a K-vector space X together with a K-linear map $\delta_X : X \to C \otimes X$ such that $(\Delta \otimes \mathrm{id}_X)\delta_X = (\mathrm{id}_C \otimes \delta_X)\delta_X$ and $(\varepsilon \otimes \mathrm{id}_X)\delta_X$ is the canonical isomorphism $X \cong K \otimes X$, where $\otimes = \otimes_K$. A K-linear map $f : X \to Y$ between left C-comodules is a *C-comodule homomorphism* if $\delta_Y f = (\mathrm{id}_C \otimes f)\delta_X$.

We denote by C-Comod (resp. $\mathrm{Mod}(A)$) the category of all left C-comodules (resp. right A-modules), and by C-comod (resp. $\mathrm{mod}(A)$) the full subcategory of C-Comod (resp. $\mathrm{Mod}(A)$) formed by C-comodules (resp. A-modules) of finite K-dimension. Unless stated otherwise, all modules and comodules considered are assumed to be finite-dimensional.

Given a coalgebra C, let C^* be the associated algebra, that is, $C^* = \mathrm{Hom}_K(C,K)$, where the multiplication in C^* is given by the convolution product

$$C^* \otimes C^* \to (C \otimes C)^* \xrightarrow{\Delta^*} C^*$$

and $\varepsilon^* : K \to C^*$ is the identity element of C^* (see [12] and [31]).

Let X be a left C-comodule. The composite K-linear map

$$X \otimes C^* \xrightarrow{\delta_X \otimes \mathrm{id}} C \otimes X \otimes C^* \xrightarrow{\mathrm{id} \otimes \tau} C \otimes C^* \otimes X \xrightarrow{\mathrm{ev} \otimes \varepsilon} K \otimes X \cong X$$

defines a right C^*-module structure on X, where $\tau : X \otimes C^* \to C^* \otimes X$ is the twist isomorphism and $\mathrm{ev} : C \otimes C^* \to K$ is the evaluation map $c \otimes \varphi \mapsto \varphi(c)$. If C is a finite-dimensional coalgebra, this correspondence gives an equivalence of categories (see [31])

$$C\text{-comod} \cong \mathrm{mod}(C^*). \tag{2.1}$$

A finite-dimensional algebra A is called *basic* if

$$A = \bigoplus_{j \in I_A} P(j), \tag{2.2}$$

where $\{P(j);\ j \in I_A\}$ is a complete set of pairwise non-isomorphic projective right A-modules (see [1] and [2]). We call C *basic* if there is a decomposition

$$\mathrm{soc}\,_C C = \bigoplus_{j \in I_C} S(j) \tag{2.3}$$

of the left socle $\mathrm{soc}\,_C C$ of C, such that $\{S(j);\ j \in I_C\}$ is a complete set of pairwise non-isomorphic simple left C-comodules (see [3], [4], [22], [24]).

In the present paper, all algebras and coalgebras are assumed to be basic.

Recall (see [1] and [2]) that a finite-dimensional algebra A is of *finite representation type* if there are only finitely many isomorphism classes of finite-dimensional indecomposable A-modules. We recall (see [20], [21] and [24]) that a K-coalgebra is said to be *left pure semisimple* if every left C-comodule is a direct sum of finite-dimensional C-comodules, or equivalently, if every infinite sequence $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} \cdots$ of monomorphisms between finite-dimensional indecomposable left C-comodules terminates, that is, there exists $m_0 \geq 1$ such that f_j is bijective for all $j \geq m_0$. A coalgebra C (resp. algebra A) is called *hereditary* if the category C-Comod (resp. $\mathrm{Mod}(A)$) is hereditary, i.e. $\mathrm{Ext}^2_C(X,Y) = 0$ for all C-comodules X, Y (resp. $\mathrm{Ext}^2_A(X,Y) = 0$ for all A-modules X, Y).

We recall the notion of a valued quiver. By a *quiver* we mean an oriented graph $Q = (Q_0, Q_1)$, where Q_0 is the set of vertices and Q_1 the set of arrows. A *valued quiver* is a pair $(Q, \mathbf{d})$, where Q is a quiver such that each $\alpha \in Q_1$ is a valued arrow $\alpha : i \xrightarrow{(d'_\alpha, d''_\alpha)} j$, where d'_α, d''_α are positive integers. If $d'_\alpha = d''_\alpha = 1$, then we simply write $i \to j$ instead of $i \xrightarrow{(d'_\alpha, d''_\alpha)} j$. A valued subquiver $(\overline{Q}, \overline{\mathbf{d}})$ of $(Q, \mathbf{d})$ is said to be *convex* if for any vertices i, j of $\overline{Q}$, there exists a valued arrow $i \xrightarrow{(d'_\alpha,d''_\alpha)} j$ in $\overline{Q}_1$ if and only if there exists a valued arrow $i \xrightarrow{(d'_\alpha,d''_\alpha)} j$ in Q_1.

The *left* (*Gabriel*) *valued quiver* of C is the valued quiver $({}_CQ, {}_C\mathbf{d})$, where ${}_CQ_0 = I_C$ and, given $i, j \in {}_CQ_0$, there exists a unique valued arrow $i \xrightarrow{({}_Cd'_{ij}, {}_Cd''_{ij})} j$ from i to j in ${}_CQ_1$ if and only if $\mathrm{Ext}^1_C(S(i), S(j)) \neq 0$ and

$$ {}_Cd'_{ij} = \dim_{F_j} \mathrm{Ext}^1_C(S(i), S(j)), \qquad {}_Cd''_{ij} = \dim \mathrm{Ext}^1_C(S(i), S(j))_{F_i}, $$

where $F_i = \mathrm{End}_C S(i)$ (see [11, Definition 4.3]).

For every X in C-comod, let

$$ \mathbf{lgth}\, X = (x(j))_{j \in I_C} \in \mathbb{Z}^{(I_C)} $$

be the composition length vector, where $x(j)$ is the number of simple composition factors of X isomorphic to $S(j)$ (see [24, (6.2)]).

With any hereditary Ext-finite K-coalgebra C (i.e. $\mathrm{Ext}^1_C(S', S'')$ is finite-dimensional for all simple C-comodules S', S'', see [25]), with a fixed decomposition (2.3), we associate the *Euler quadratic form*

$$ q_C : \mathbb{Z}^{(I_C)} \to \mathbb{Z} \tag{2.4} $$

by the formula

$$ q_C(v) = \sum_{i \in I_C} \mathbf{s}^0_i v_i^2 - \sum_{i,j \in I_C} \mathbf{s}^1_{ij} v_i v_j, \tag{2.5} $$

where $v \in \mathbb{Z}^n$, $\mathbf{s}^0_i = \dim_K \mathrm{End}_C S(i)$ and $\mathbf{s}^1_{ij} = \dim_K \mathrm{Ext}^1_C(S(i), S(j))$ and $\mathbb{Z}^{(I_C)}$ is the direct sum of I_C copies of the free abelian group $\mathbb{Z}$ (see [24], [25] and [28]).

If an indecomposable coalgebra C is hereditary and left pure semisimple, then $({}_CQ, {}_C\mathbf{d})$ is one of the valued quivers in Tables 1.1 and 1.2 below (see [7], [11, Theorem 4.14], [13]). Moreover, in this case the map

$$ \mathbf{lgth} : C\text{-comod} \to \mathbb{Z}^{(I_C)} $$

defines a bijection between the set of isomorphism classes of finite-dimensional indecomposable left C-comodules and the set

$$ \mathcal{R}^+_C = \{v \in \mathbb{N}^{(Q_0)};\ q_C(v) = \mathbf{s}^0_i \text{ for some } i\} $$

of positive roots of the Euler quadratic form q_C.

The underlying valued graphs obtained from the valued quivers $\mathbb{A}^{(s)}_\infty$, ${}_\infty\mathbb{A}^{(s)}_\infty$, $\mathbb{B}^{(s)}_\infty$, $\mathbb{C}^{(s)}_\infty$ and $\mathbb{D}^{(s)}_\infty$ of Table 1.2 by forgetting their orientation are denoted by $\mathbb{A}_\infty$, ${}_\infty\mathbb{A}_\infty$, $\mathbb{B}_\infty$, $\mathbb{C}_\infty$ and $\mathbb{D}_\infty$, respectively.

In the last part of this section we investigate classes of (co)algebra homomorphisms, which play an important role in this paper.

Table 1.1. Valued Dynkin quivers

$$\mathbb{A}_n: \quad \mathbf{1} \;\text{——}\; \mathbf{2} \;\text{——}\; \cdots \;\text{——}\; \mathbf{n-1} \;\text{——}\; \mathbf{n} \qquad (n \text{ vertices, } n \geq 1);$$

$$\mathbb{B}_n: \quad \mathbf{1} \overset{(1,2)}{\text{——}} \mathbf{2} \;\text{——}\; \mathbf{3} \;\text{——}\; \cdots \;\text{——}\; \mathbf{n-1} \;\text{——}\; \mathbf{n} \qquad (n \text{ vertices, } n \geq 2);$$

$$\mathbb{C}_n: \quad \mathbf{1} \overset{(2,1)}{\text{——}} \mathbf{2} \;\text{——}\; \mathbf{3} \;\text{——}\; \cdots \;\text{——}\; \mathbf{n-1} \;\text{——}\; \mathbf{n} \qquad (n \text{ vertices, } n \geq 2);$$

$$\mathbb{D}_n: \quad \begin{array}{ccccccccccc} & & \mathbf{n} & & & & & & & & \\ & & | & & & & & & & & \\ \mathbf{1} & \text{——} & \mathbf{2} & \text{——} & \mathbf{3} & \text{——} & \cdots & \text{——} & \mathbf{n-2} & \text{——} & \mathbf{n-1} \end{array} \qquad (n \text{ vertices, } n \geq 4);$$

$$\mathbb{E}_6: \quad \begin{array}{ccccccccc} & & & & \mathbf{6} & & & & \\ & & & & | & & & & \\ \mathbf{1} & \text{——} & \mathbf{2} & \text{——} & \mathbf{3} & \text{——} & \mathbf{4} & \text{——} & \mathbf{5} \end{array};$$

$$\mathbb{E}_7: \quad \begin{array}{ccccccccccc} & & & & \mathbf{7} & & & & & & \\ & & & & | & & & & & & \\ \mathbf{1} & \text{——} & \mathbf{2} & \text{——} & \mathbf{3} & \text{——} & \mathbf{4} & \text{——} & \mathbf{5} & \text{——} & \mathbf{6} \end{array};$$

$$\mathbb{E}_8: \quad \begin{array}{ccccccccccccc} & & & & \mathbf{8} & & & & & & & & \\ & & & & | & & & & & & & & \\ \mathbf{1} & \text{——} & \mathbf{2} & \text{——} & \mathbf{3} & \text{——} & \mathbf{4} & \text{——} & \mathbf{5} & \text{——} & \mathbf{6} & \text{——} & \mathbf{7} \end{array};$$

$$\mathbb{F}_4: \quad \mathbf{1} \;\text{——}\; \mathbf{2} \overset{(2,1)}{\text{——}} \mathbf{3} \;\text{——}\; \mathbf{4};$$

$$\mathbb{G}_2: \quad \mathbf{1} \overset{(3,1)}{\text{——}} \mathbf{2},$$

where $\mathbf{t} \text{——} \mathbf{r}$ means $\mathbf{t} \leftarrow \mathbf{r}$ or $\mathbf{t} \rightarrow \mathbf{r}$.

Table 1.2. Infinite pure semisimple locally Dynkin valued quivers

$$\mathbb{A}_\infty^{(s)}: \quad \mathbf{0} \;\text{——}\; \mathbf{1} \;\text{——}\; \mathbf{2} \;\text{——}\; \cdots \;\text{——}\; \mathbf{s-1} \leftarrow \mathbf{s} \rightarrow \mathbf{s+1} \rightarrow \cdots$$

$$_\infty\mathbb{A}_\infty^{(s)}: \quad \cdots \leftarrow \mathbf{-2} \leftarrow \mathbf{-1} \leftarrow \mathbf{0} \;\text{——}\; \mathbf{1} \;\text{——}\; \mathbf{2} \;\text{——}\; \cdots \;\text{——}\; \mathbf{s-1} \leftarrow \mathbf{s} \rightarrow \mathbf{s+1} \rightarrow \cdots$$

$$\mathbb{B}_\infty^{(s)}: \quad \mathbf{0} \overset{(1,2)}{\text{——}} \mathbf{1} \;\text{——}\; \mathbf{2} \;\text{——}\; \cdots \;\text{——}\; \mathbf{s-1} \leftarrow \mathbf{s} \rightarrow \mathbf{s+1} \rightarrow \cdots$$

$$\mathbb{C}_\infty^{(s)}: \quad \mathbf{0} \overset{(2,1)}{\text{——}} \mathbf{1} \;\text{——}\; \mathbf{2} \;\text{——}\; \cdots \;\text{——}\; \mathbf{s-1} \leftarrow \mathbf{s} \rightarrow \mathbf{s+1} \rightarrow \cdots$$

$$\mathbb{D}_\infty^{(s)}: \quad \begin{array}{ccccccccccccc} & & \mathbf{-1} & & & & & & & & & & \\ & & | & & & & & & & & & & \\ \mathbf{0} & \text{——} & \mathbf{1} & \text{——} & \mathbf{2} & \text{——} & \cdots & \text{——} & \mathbf{s-1} & \leftarrow & \mathbf{s} & \rightarrow & \mathbf{s+1} \rightarrow \cdots \end{array}$$

where $0 \leq s < \infty$ and $\mathbf{t} \text{——} \mathbf{r}$ means $\mathbf{t} \leftarrow \mathbf{r}$ or $\mathbf{t} \rightarrow \mathbf{r}$.

Let A, B be finite-dimensional K-algebras and let $f : A \to B$ be a surjective homomorphism of algebras. The homomorphism f induces a functor

$$\Phi_f : \mathrm{mod}(B) \to \mathrm{mod}(A),$$

given by $M \mapsto M$ and $g \mapsto g$ for all B-modules M and B-module homomorphisms g, where any B-module M_B has the A-module structure M_A given by $ma = mf(a)$ for $a \in A$ and $m \in M$. It is easy to see that Φ_f is full, faithful and exact. We identify $\mathrm{mod}(B)$ with the subcategory $\Phi_f(\mathrm{mod}(B))$ of mod(A). We say that a surjective homomorphism $f : A \to B$ has *idempotent kernel* if $(\operatorname{Ker} f)^2 = \operatorname{Ker} f$.

Let C, D be K-coalgebras and let $f : D \to C$ be an injective homomorphism of coalgebras. The homomorphism f induces a functor

$$\Psi_f : D\text{-comod} \to C\text{-comod},$$

given by $M \mapsto M$ and $g \mapsto g$ for all D-comodules M and D-comodule homomorphisms g, where a D-comodule ${}_DM = (M, \delta_M)$ is a C-comodule ${}_CM$ via

$$M \xrightarrow{\delta_M} D \otimes M \xrightarrow{f \otimes \mathrm{id}} C \otimes M.$$

It is easy to see that Ψ_f is full, faithful and exact. We identify the category D-comod with the subcategory $\Psi_f(D\text{-comod})$ of C-comod. We call an inclusion $f : D \hookrightarrow C$ *of coidempotent type* if $D = \Delta^{-1}(C \otimes D + D \otimes C)$, where Δ defines the coalgebra structure on C.

EXAMPLE 2.6. We give a family of examples of surjective algebra homomorphisms with idempotent kernel and injective coalgebra homomorphisms of coidempotent type. These types of (co)algebra homomorphisms are essential in this paper.

Let Q be a finite quiver which is a tree (i.e., an acyclic quiver without multiple edges). Let $A = KQ$ and $C = KQ$ be the path algebra and path coalgebra of Q, respectively (see [1], [24], [25]). Let $\overline{Q}$ be a convex subquiver of Q, and let $B = K\overline{Q}$ and $D = K\overline{Q}$ be the path algebra and path coalgebra of $\overline{Q}$, respectively.

Note that there exist:

- a surjective K-algebra homomorphism $f : A \to B$ induced by $f(\alpha) = \alpha$ for any $\alpha \in \overline{Q}_0 \cup \overline{Q}_1$, and $f(\alpha) = 0$ for any $\alpha \in (Q_0 \cup Q_1) \setminus (\overline{Q}_0 \cup \overline{Q}_1)$,
- an inclusion $g : D \hookrightarrow C$ of K-coalgebras induced by $g(\alpha) = \alpha$ for any $\alpha \in \overline{Q}_0 \cup \overline{Q}_1$.

It is straightforward to prove (using convexity of $\overline{Q}$ and definitions of path (co)algebras) that f has idempotent kernel and g is of coidempotent type.

LEMMA 2.7. (a) *Let A, B be finite-dimensional K-algebras and let $f : A \to B$ be a surjective homomorphism with idempotent kernel. Then the subcategory $\Phi_f(\mathrm{mod}(B))$ of* mod(A) *is closed under extensions.*

(b) *Let C, D be K-coalgebras and let $f : D \hookrightarrow C$ be an inclusion of coidempotent type. Then the subcategory $\Psi_f(D\text{-comod})$ of C-comod is closed under extensions. If, in addition, C has finite dimension over K, then the K-algebra surjection $f^* : C^* \to D^*$ has idempotent kernel.*

Proof. (a) Let $\operatorname{Ker} f = I = I^2$. Let X, Y be B-modules. The remarks above show that X, Y may be viewed as A-modules. Consider an exact sequence of A-modules

$$0 \to X \xrightarrow{i} Z \xrightarrow{p} Y \to 0.$$

We claim that Z is a B-module. Indeed, it is enough to prove that Z is annihilated by the ideal I of A. Since $p(I \cdot Z) = I \cdot p(Z) = I \cdot Y$ and Y is a B-module, we have $p(I \cdot Z) = I \cdot Y = 0$. Then $I \cdot Z \subseteq \operatorname{Ker} p = \operatorname{Im} i$. Let $X' \subseteq X$ be such that $i(X') = I \cdot Z$. We have $I \cdot i(X') = 0$, because X is a B-module. Therefore $I \cdot Z = I^2 \cdot Z = I \cdot (I \cdot Z) = I \cdot i(X') = 0$, because f has idempotent kernel.

(b) Let X, Y be D-comodules, which we consider as C-comodules. Let

$$0 \to X \hookrightarrow Z \xrightarrow{p} Y \to 0$$

be an exact sequence of C-comodules. We claim that Z is a D-comodule. Indeed, it is enough to prove that $\delta(Z) \subseteq D \otimes Z$, where δ gives the C-comodule structure on Z. We choose a K-basis $\{x_i, y_j\}_{i\in\mathcal{I}, j\in\mathcal{J}}$ of Z such that $\{x_i\}_{i\in\mathcal{I}}$ is a K-basis of X. Let $z \in Z$ and consider

$$\delta(z) = \sum c_i \otimes x_i + \sum \overline{c}_i \otimes y_i.$$

Since (Z, δ) is a C-comodule, we have $(\Delta \otimes \mathrm{id})\delta = (\mathrm{id} \otimes \delta)\delta$. Therefore

$$\sum \Delta(c_i) \otimes x_i + \sum \Delta(\overline{c}_i) \otimes y_i = \sum c_i \otimes \delta(x_i) + \sum \overline{c}_i \otimes \delta(y_i).$$

Note that $\delta(x_i) \in D \otimes X$ for all $i \in \mathcal{I}$, because X is a D-comodule. On the other hand, $(\mathrm{id} \otimes p)\delta = \delta_Y p : Z \to D \otimes Y$, because (Y, δ_Y) is a C-comodule, $\operatorname{Im} \delta_Y \subseteq D \otimes Y$ and p is a homomorphosim of C-comodules. Therefore

$$(\mathrm{id} \otimes p)\delta(z) = (\mathrm{id} \otimes p)\Big(\sum c_i \otimes x_i + \sum \overline{c}_i \otimes y_i\Big) = \sum \overline{c}_i \otimes p(y_i) \in D \otimes Y,$$

because $x_i \in \operatorname{Ker} p$ for all $i \in \mathcal{I}$. Then

$$\begin{aligned}(\Delta \otimes \mathrm{id})\delta(z) &= \sum c_i \otimes \delta(x_i) + \sum \overline{c}_i \otimes \delta(y_i) \in C \otimes D \otimes X + D \otimes C \otimes Z \\ &\subseteq (C \otimes D + D \otimes C) \otimes Z.\end{aligned}$$

Finally, $\delta(z) \in D \otimes Z$, because $D = \Delta^{-1}(C \otimes D + D \otimes C)$. Therefore $\Psi_f(D\text{-comod})$ is closed under extensions.

Assume that C is finite-dimensional. Note that

$$\operatorname{Ker} f^* = \{h \in C^*;\ h \circ f = 0\} = \{h \in C^*;\ h(c) = 0 \text{ for all } c \in D\} = D^\perp$$

and $(D^\perp)^\perp = D$, where for any subset X of C^*, $X^\perp = \{c \in C;\ h(c) = 0$ for all $h \in X\}$ (see [31]). On the other hand,

$$D = \Delta^{-1}(C \otimes D + D \otimes C) = D \wedge D = (D^\perp D^\perp)^\perp,$$

where $\wedge$ is the wedge product (see [31, Proposition 9.0.0]). By [31, p. 181], we have $D^\perp = (D \wedge D)^\perp = D^\perp D^\perp$, because C has finite dimension. Therefore

$$\operatorname{Ker} f^* = D^\perp = D^\perp D^\perp = (\operatorname{Ker} f^*)^2. \ \blacksquare$$

REMARK 2.8. Note that for an arbitrary surjective homomorphism $f : A \to B$ of algebras (resp. injective homomorphism $f : D \to C$ of coalgebras) the subcategory $\Phi_f(\mathrm{mod}(B))$ (resp. $\Psi_f(D\text{-comod})$) is not necessairly closed under extensions. Indeed, consider the quiver

$$Q : \ 1 \xrightarrow{\alpha} 2$$

and its path algebra $A = KQ$ and path coalgebra $C = KQ$ (see [1], [24], [25]). Put $B = A/\mathcal{J}_A$, where $\mathcal{J}_A$ is the Jacobson radical of A, and $D = C_0$, where C_0 is the socle of C. In this case B is the path algebra, and D is the path coalgebra of the quiver

$$\overline{Q} : \ 1 \quad 2,$$

which is not connected. It is easy to see that $(\mathcal{J}_A)^2 = 0 \neq \mathcal{J}_A$ and $\Delta^{-1}(C \otimes D + D \otimes C) = C \neq D$. Moreover, $\Phi_f(\mathrm{mod}(B))$ (resp. $\Psi_f(D\text{-comod})$) is not closed under extensions.

In Sections 4 and 5 we will see that convex valued subquivers of Gabriel valued quivers of algebras and coalgebras induce surjective homomorphisms of algebras with idempotent kernel and inclusions of coalgebras of coidempotent type, respectively.

3. Ringel–Hall algebras of coalgebras and finite-dimensional algebras. Let K be a finite field, C be a basic K-coalgebra, A a basic finite dimensional K-algebra, and let X, Y, Z be finite-dimensional left C-comodules (resp. finite-dimensional right A-modules). We define $F^Y_{Z,X} = F^Y_{Z,X}(C)$ (resp. $F^Y_{Z,X} = F^Y_{Z,X}(A)$) to be the number of subcomodules (resp. submodules) $U \subseteq Y$ such that $U \simeq X$ and $Y/U \simeq Z$.

Analogously to [16], we define $\mathcal{H}(C)$ (resp. $\mathcal{H}(A)$) to be the $\mathbb{C}$-vector space with basis $\{u_{[M]}\}_{[M]}$ indexed by the set of all isomorphism classes of finite-dimensional left C-comodules (resp. finite-dimensional right A-modules) and the multiplication

$$u_{[N]}u_{[M]} = \sum_{[X]} F^X_{N,M} u_{[X]}, \tag{3.1}$$

where the sum runs over all isomorphism classes of left C-comodules (resp. A-modules). Note that the sum is finite, because the field K is finite and the comodules (resp. modules) N, M, X are finite-dimensional.

It is easy to check that $\mathcal{H}(C)$ (resp. $\mathcal{H}(A)$) is an associative $\mathbb{C}$-algebra with the identity element u_0, called the *Ringel–Hall algebra* of C (resp. of A) (see [16, Proposition 1]).

Below we assume that a surjective homomorphism $f : A \to B$ of algebras has idempotent kernel (resp. an injective homomorphism $f : D \to C$ of coalgebras is of coidempotent type). Note that it is enough to assume instead that the subcategory $\Phi_f(\mathrm{mod}(B))$ of $\mathrm{mod}(A)$ (resp. $\Psi_f(D\text{-comod})$ of C-comod) is closed under extensions.

LEMMA 3.2. *Let K be a finite field.*

(a) *Let A, B be finite dimensional K-algebras and let $f : A \to B$ be a surjective homomorphism with idempotent kernel. The homomorphism f induces an injective algebra homomorphism*

$$\widehat{f} : \mathcal{H}(B) \to \mathcal{H}(A)$$

given by $u_{[M_B]} \mapsto u_{[M_A]}$ for all isomorphism classes $[M_B]$ of finite-dimensional B-modules, where any B-module M_B has the A-module structure M_A induced by f.

(b) *Let C, D be K-coalgebras and let $f : D \hookrightarrow C$ be an inclusion of coidempotent type. The inclusion f induces an injective algebra homomorphism*

$$\widehat{f} : \mathcal{H}(D) \to \mathcal{H}(C)$$

given by $u_{[_DM]} \mapsto u_{[_CM]}$ for all isomorphism classes $[_DM]$ of finite-dimensional D-comodules, where any D-comodule $_DM$ has the C-comodule structure $_CM$ induced by f.

(c) *Let C be a finite-dimensional K-coalgebra and let X, Y, Z be left C-comodules. Then $F^Y_{Z,X}(C) = F^Y_{Z,X}(C^*)$, under the identification $C\text{-comod} \cong \mathrm{mod}(C^*)$ (see (2.1)).*

Proof. (a) Let X, Y, Z be B-modules. It is straightforward to prove the following:

- If $U \subseteq Y_A$ is a submodule of the A-module Y_A such that $U \cong X_A$ and $Y_A/U \cong Z_A$, then U is a submodule of the B-module Y_B such that $U \cong X_B$ and $Y_B/U \cong Z_B$.
- If $U \subseteq Y_B$ is a submodule such that $U \cong X_B$ and $Y_B/U \cong Z_B$, then U is a submodule of Y_A such that $U \cong X_A$ and $Y_A/U \cong Z_A$.

Therefore $F^Y_{Z,X}(A) = F^Y_{Z,X}(B)$ and the map $\widehat{f} : \mathcal{H}(B) \to \mathcal{H}(A)$ is an injective algebra homomorphism, because (by Lemma 2.7) the subcategory $\Phi_f(\mathrm{mod}(B))$ is closed under extensions.

(b) Let X, Y, Z be D-comodules. It is straightforward to prove the following.

- If $U \subseteq {}_CY$ is a subcomodule such that $U \cong {}_CX$ and ${}_CY/U \cong {}_CZ$, then U is a subcomodule of ${}_DY$ such that $U \cong {}_DX$ and ${}_DY/U \cong {}_DZ$.
- If $U \subseteq {}_DY$ is a subcomodule such that $U \cong {}_DX$ and ${}_DY/U \cong {}_DZ$, then U is a subcomodule of ${}_CY$ such that $U \cong {}_CX$ and ${}_CY/U \cong {}_CZ$.

Therefore $F^Y_{Z,X}(D) = F^Y_{Z,X}(C)$ and the map $\widehat{f} : \mathcal{H}(D) \to \mathcal{H}(C)$ is an injective homomorphism of algebras, because (by Lemma 2.7) the subcategory $\Psi_f(D\text{-comod})$ is closed under extensions.

(c) follows easily, because the functor C-comod $\cong \mathrm{mod}(C^*)$, given in (2.1), is exact. ■

Let A, B be finite-dimensional algebras and $f : A \to B$ be a surjective homomorphism with idempotent kernel. By Lemma 3.2(a), we identify the algebra $\mathcal{H}(B)$ with the subalgebra $\widehat{f}(\mathcal{H}(B))$ of $\mathcal{H}(A)$. Similarly, if C, D are coalgebras and $f : D \to C$ is an inclusion of coidempotent type, we identify $\mathcal{H}(D)$ with $\widehat{f}(\mathcal{H}(D))$.

LEMMA 3.3. *If C is a finite-dimensional K-coalgebra, then the equivalence* (2.1) *induces an algebra isomorphism $\mathcal{H}(C) \simeq \mathcal{H}(C^*)$.*

Proof. Let M be a finite-dimensional C-comodule and f be a homomorphism of C-comodules. By (2.1), the mappings $M \mapsto M$, $f \mapsto f$ give an equivalence of categories C-comod $\cong \mathrm{mod}(C^*)$. Applying Lemma 3.2(c), it is straightforward to check that $u_{[M]} \mapsto u_{[M]}$ gives an isomorphism $\mathcal{H}(C) \simeq \mathcal{H}(C^*)$. ■

LEMMA 3.4. *Let $C^{(0)} \subseteq C^{(1)} \subseteq \cdots$ be an infinite chain of finite-dimensional K-subcoalgebras of C such that, for any $i \geq 0$, the inclusion $C^{(i)} \hookrightarrow C^{(i+1)}$ is of coidempotent type. If*

$$C = \bigcup C^{(i)},$$

then the induced chain $\mathcal{H}(C^{(0)}) \subseteq \mathcal{H}(C^{(1)}) \subseteq \cdots$ induces an algebra isomorphism $\mathcal{H}(C) = \bigcup \mathcal{H}(C^{(i)})$.

Proof. By Lemma 3.2, the inclusions

$$\mathcal{H}(C^{(0)}) \subseteq \mathcal{H}(C^{(1)}) \subseteq \cdots$$

are given by $u_{[M]} \mapsto u_{[M]}$ for all isomorphism classes of finite-dimensional $C^{(i)}$-comodules. Moreover, $\mathcal{H}(C^{(i)}) \subseteq \mathcal{H}(C)$ for any $i = 0, 1, \ldots$. Therefore, $\bigcup \mathcal{H}(C^{(i)}) \subseteq \mathcal{H}(C)$.

To prove the opposite inclusion, let M be a finite-dimensional C-comodule and $u_{[M]} \in \mathcal{H}(C)$. By [24, Theorem 4.3(c)], there exists a finite-dimensional subcoalgebra D of C such that M lies in D-comod $\subseteq C$-comod. Note that there exists i such that $D \subseteq C^{(i)}$, because $C = \bigcup C^{(i)}$. Therefore the D-comodule M has the induced $C^{(i)}$-comodule structure, $u_{[M]} \in \mathcal{H}(C^{(i)})$ and our claim follows. ■

4. Species and cotensor coalgebras. Following Gabriel [8], we define a *species* to be a system

$$\mathcal{M} = (F_i, {}_iM_j)_{i,j \in I_{\mathcal{M}}},$$

where F_i is a division ring for each $i \in I_{\mathcal{M}}$, and ${}_iM_j$ is an F_i-F_j-bimodule for any $i, j \in I_{\mathcal{M}}$. A species $\mathcal{M} = (F_i, {}_iM_j)_{i,j \in I_{\mathcal{M}}}$ is called a *K-species* (resp. *locally finite-dimensional K-species*) if

- for each $i \in I_{\mathcal{M}}$, F_i is a division K-algebra (resp. a finite-dimensional division K-algebra),
- for any $i, j \in I_{\mathcal{M}}$, the F_i-F_j-bimodule ${}_iM_j$ is a K-vector space (resp. a finite-dimensional K-vector space),
- K acts centrally on each F_i and on each ${}_iM_j$.

Following [27], a species $\mathcal{M}$ is said to be *locally finite* if every F_i-F_j-bimodule ${}_iM_j$ is a directed union of finite-dimensional F_i-F_j-bimodules.

Following [6], with any species $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_{\mathcal{M}}}$ we associate a valued quiver $(Q^{\mathcal{M}}, \mathbf{d}^{\mathcal{M}})$ as follows. The vertices of $Q^{\mathcal{M}}$ are the elements of $I_{\mathcal{M}}$ and, for any $i, j \in I_{\mathcal{M}}$ such that ${}_iM_j \neq 0$, there exists a unique valued arrow

$$i \xrightarrow{(d'_{ij}, d''_{ij})} j$$

in $(Q^{\mathcal{M}}, \mathbf{d}^{\mathcal{M}})$, where $d'_{ij} = \dim ({}_iM_j)_{F_j}$ and $d''_{ij} = \dim {}_{F_i}({}_iM_j)$.

Let $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in I_{\mathcal{M}}}$ be a K-species. A *right linear representation of* $\mathcal{M}$ is a system $X = (X_i, \varphi_{ij})_{i,j\in I_{\mathcal{M}}}$, where X_i is an F_i-vector space, $\varphi_{ij} : X_i \otimes {}_iM_j \to X_j$ is an F_j-linear map for any i, j in $I_{\mathcal{M}}$ such that ${}_iM_j \neq 0$, and $X_i \otimes {}_iM_j$ means $X_i \otimes_{F_i} {}_iM_j$.

A *morphism* $f : X \to X'$ of representations of $\mathcal{M}$ is a system $f = (f_i)_{i\in I_{\mathcal{M}}}$ of F_i-linear maps $f_i : X_i \to X'_i$, $i \in I_{\mathcal{M}}$, such that $\varphi'_{ij}(f_i \otimes \mathrm{id}) = f_j\varphi_{ij}$ for all $i, j \in I_{\mathcal{M}}$.

A representation X of $\mathcal{M}$ is said to be *of finite length* if X has a finite composition series (see [2], [11] and [23, Chapter 14]). We denote by $\mathrm{Rep}(\mathcal{M})$ the Grothendieck category of all linear representations of $\mathcal{M}$, and by $\mathrm{rep}(\mathcal{M}) \supseteq \mathrm{rep}^{\ell f}(\mathcal{M})$ the full subcategories of $\mathrm{Rep}(\mathcal{M})$ formed by finitely generated objects and by finitely generated representations of finite length, respectively. Moreover, we denote by $\mathrm{Rep}^{\ell f}(\mathcal{M})$ the full Grothendieck subcategory of $\mathrm{Rep}(\mathcal{M})$ formed by *locally finite-dimensional* representations, that is, directed unions of representations of finite length.

Given $X = (X_i, \varphi_{ij})_{i,j\in I_{\mathcal{M}}}$ and $i_0, \ldots, i_n \in I_{\mathcal{M}}$, we denote by

$$\varphi_{i_0\ldots i_m} : X_{i_0} \otimes {}_{i_0}M_{i_1} \otimes \cdots \otimes {}_{i_{m-1}}M_{i_m} \to X_{i_m}$$

the composed K-linear map

$$X_{i_0} \otimes {}_{i_0}M_{i_1} \otimes \cdots \otimes {}_{i_{m-1}}M_{i_m} \xrightarrow{\varphi_{i_0i_1}\otimes\mathrm{id}} X_{i_1} \otimes {}_{i_1}M_{i_2} \otimes \cdots \otimes {}_{i_{m-1}}M_{i_m} \xrightarrow{\varphi_{i_1i_2}\otimes\mathrm{id}} \ldots$$
$$\ldots \xrightarrow{\varphi_{i_{m-2}i_{m-1}}\otimes\mathrm{id}} X_{i_{m-1}} \otimes {}_{i_{m-1}}M_{i_m} \xrightarrow{\varphi_{i_{m-1}i_m}\otimes\mathrm{id}} X_{i_m},$$

where $- \otimes {}_iM_j = - \otimes_{F_i} {}_iM_j$.

A representation X of $\mathcal{M}$ is said to be *nilpotent* if there exists an integer $m \geq 1$ such that $\varphi_{i_0\ldots i_m} = 0$ for any path $i_0 \to i_1 \to \cdots \to i_m$ in $Q^{\mathcal{M}}$ of length m.

Following [11], a representation X of $\mathcal{M}$ is said to be *locally nilpotent* if, for each $i_0 \in I_{\mathcal{M}}$ and each $x_0 \in X_{i_0}$, there exists an integer $m \geq 1$ such that $\varphi_{i_0\ldots i_m}$ vanishes on the F_{i_m}-subspace $x_0 F_{i_0}\otimes{}_{i_0}M_{i_1}\otimes\ldots\otimes{}_{i_{m-1}}M_{i_m}$ of $X_{i_0}\otimes{}_{i_0}M_{i_1}\otimes\ldots\otimes{}_{i_{m-1}}M_{i_m}$ for any path $i_0\to i_1\to\cdots\to i_m$ in $Q^{\mathcal{M}}$.

We denote by $\mathrm{nilrep}^{\ell f}(\mathcal{M})$ the full subcategory of $\mathrm{rep}^{\ell f}(\mathcal{M})$ formed by all nilpotent representations of finite length, and by $\mathrm{Rep}^{\ell n\ell f}(\mathcal{M})$ the full subcategory of $\mathrm{Rep}^{\ell f}(\mathcal{M})$ formed by all *locally nilpotent representations.*

Let K be any field and C be a basic K-coalgebra with a fixed decomposition (2.3). Following [11, Definition 4.3], we define the left Ext-*species* of C to be the K-species

$$_C\mathcal{E}xt=(F_j,{}_iE_j)_{i,j\in I_C},$$

where $F_j=\mathrm{End}_C S(j)$ and ${}_iE_j=\mathrm{Ext}^1_C(S(j),S(i))$, viewed as an F_i-F_j-bimodule in the obvious way.

Denote by ${}_C\mathcal{E}xt^{\#}$ the #-dual to ${}_C\mathcal{E}xt$, i.e.

$$_C\mathcal{E}xt^{\#}=(F_j,{}_jE_i^{\#})_{i,j\in I_C},$$

where ${}_jE_i^{\#}=\mathrm{Hom}_{F_j}({}_iE_j,F_j)$ is viewed as an F_j-F_i-bimodule in the obvious way.

Let $\mathcal{M}=(F_i,{}_iM_j)_{i,j\in\mathcal{I}_{\mathcal{M}}}$ be a K-species. Put $F=\bigoplus_{i\in I_{\mathcal{M}}}F_i$ (direct sum of division rings, viewed as a ring with local units), and let $M=\bigoplus_{i,j\in I_{\mathcal{M}}}{}_iM_j$ be viewed as a unitary F-F-bimodule in the obvious way. Denote by

$$T(\mathcal{M})=F\oplus M\oplus M^{\otimes^2}\oplus M^{\otimes^3}\oplus\cdots\oplus M^{\otimes^m}\oplus\cdots$$

the tensor K-algebra of $\mathcal{M}$, where $M^{\otimes^m}=M\otimes_F\cdots\otimes_F M$ (m times).

We recall that the *cotensor coalgebra* of $\mathcal{M}$ is the positively graded K-vector space

$$T^{\Box}(\mathcal{M})=\bigoplus_{n=0}^{\infty}M^{\Box^n}=F\oplus M\oplus M\,\Box\,M\oplus\cdots\oplus M^{\Box^n}\oplus\cdots,$$

where $M^{\Box^0}=F$, $M^{\Box^1}=M$ and $M^{\Box^n}=M\,\Box\cdots\Box\,M$ (n times) for $n\geq 2$, equipped with the K-coalgebra structure $\varepsilon: T^{\Box}(\mathcal{M})\to K$, $\Delta: T^{\Box}(\mathcal{M})\to T^{\Box}(\mathcal{M})\otimes T^{\Box}(\mathcal{M})$ defined as follows. Given a local unit $e_a\in F$ at a, we put $\Delta(e_a)=e_a\otimes e_a$ and $\varepsilon(e_a)=1$. For $s\geq 1$ and any element ${}_a\overline{m}_b\in M^{\Box^s}$ of the form

$${}_a\overline{m}_b={}_am_{j_1}\otimes{}_{j_1}m_{j_2}\otimes\cdots\otimes{}_{j_{s-1}}m_b\in{}_aM_{j_1}\otimes{}_{j_1}M_{j_2}\otimes\ldots\otimes{}_{j_{s-1}}M_b,$$

we set

$$\Delta({}_a\overline{m}_b)=e_a\otimes{}_a\overline{m}_b+{}_a\overline{m}_b\otimes e_b+\sum_{r=1}^{s-1}({}_a\overline{m}_{j_r})\otimes({}_{j_r}\overline{m}_b)\quad\text{and}\quad\varepsilon({}_a\overline{m}_b)=0,$$

where ${}_a\overline{m}_{j_r}={}_am_{j_1}\otimes\cdots\otimes{}_{j_{r-1}}m_{j_r}$ and ${}_{j_r}\overline{m}_b={}_{j_r}m_{j_{r+1}}\otimes\cdots\otimes{}_{j_{s-1}}m_b$ (see [3], [27, (5.4)]).

LEMMA 4.1. *Let* $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in\mathcal{I}_\mathcal{M}}$ *be a* K*-species*, $\mathcal{I}_{\mathcal{M}^{(x)}} \subseteq \mathcal{I}_\mathcal{M}$, *and let* $\mathcal{M}^{(x)} = (F_i, {}_iM_j)_{i,j\in\mathcal{I}_{\mathcal{M}^{(x)}}}$ *be a subspecies of* $\mathcal{M}$ *such that the valued quiver* $(Q^{\mathcal{M}^{(x)}}, \mathbf{d}^{(x)})$ *is a convex valued subquiver of* $(Q^\mathcal{M}, \mathbf{d})$. *Then the full subcategories* $\mathrm{Rep}(\mathcal{M}^{(x)})$ *and* $\mathrm{rep}(\mathcal{M}^{(x)})$ *of* $\mathrm{Rep}(\mathcal{M})$ *are closed under extensions.*

Proof. Let X, Y be representations in $\mathrm{Rep}(\mathcal{M}^{(x)})$ (resp. in $\mathrm{rep}(\mathcal{M}^{(x)})$) and let Z be an extension of X by Y in $\mathrm{Rep}(\mathcal{M})$. Since $(Q^{\mathcal{M}^{(x)}}, \mathbf{d}^{(x)})$ is a convex valued subquiver of $(Q^\mathcal{M}, \mathbf{d})$, it is easy to see that Z belongs to $\mathrm{Rep}(\mathcal{M}^{(x)})$ (resp. to $\mathrm{rep}(\mathcal{M}^{(x)})$). Therefore the subcategories $\mathrm{Rep}(\mathcal{M}^{(x)})$ and $\mathrm{rep}(\mathcal{M}^{(x)})$ of $\mathrm{Rep}(\mathcal{M})$ are closed under extensions. ∎

The following theorem collects basic facts which connect representations of species and comodules.

THEOREM 4.2. *Let* K *be an arbitrary field and let* C *be a basic indecomposable hereditary* K*-coalgebra whose left Gabriel valued quiver* $({}_CQ, {}_C\mathbf{d})$ *is a valued tree and contains no infinite path of the form*

$$\bullet \longleftarrow \bullet \longleftarrow \cdots \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \cdots .$$

(a) *There exists an equivalence of* K*-categories*

$$C\text{-Comod} \cong \mathrm{Rep}^{\ell n\ell f}(\mathcal{M}) \cong T^\square(\mathcal{M})\text{-Comod},$$

$$C\text{-comod} \cong \mathrm{nilrep}^{\ell f}(\mathcal{M}) \cong T^\square(\mathcal{M})\text{-comod},$$

where $\mathcal{M} = {}_C\mathcal{E}xt^\#$. *If*, *in addition*, *the valued quiver* $({}_CQ, {}_C\mathbf{d})$ *is finite*, *then* $\mathrm{nilrep}^{\ell f}(\mathcal{M}) = \mathrm{rep}(\mathcal{M})$ *and* C*-comod* $\cong \mathrm{rep}(\mathcal{M})$.

(b) *The valued quiver* $({}_CQ, {}_C\mathbf{d})$ *coincides with* $(Q^\mathcal{M}, \mathbf{d}^\mathcal{M})$.

(c) *If in addition the species* $\mathcal{M}$ *is locally finite*, *then there is a coalgebra isomorphism* $C \cong T^\square(\mathcal{M})$.

Proof. For the proof of (a) the reader is referred to [11, Proposition 4.16]. The proof of (b) can be found in [11, Proposition 4.10] and [27, Proposition 5.8]. Finally, (c) is proved in [27, Theorem 5.15]. ∎

REMARK 4.3. Let C be a K-coalgebra satisfying the conditions of Theorem 4.2. If $\mathcal{M} = (F_i, {}_iM_j)_{i,j\in\mathcal{I}_\mathcal{M}} = {}_C\mathcal{E}xt^\#$ is locally finite, then by Theorem 4.2(c), $C \cong T^\square(\mathcal{M})$. Let $\mathcal{I}_{\mathcal{M}^{(x)}} \subseteq \mathcal{I}_\mathcal{M}$ be a finite subset. Consider the subcoalgebra $C^{(x)} = T^\square(\mathcal{M}^{(x)})$ of C, where $\mathcal{M}^{(x)} = (F_i, {}_iM_j)_{i,j\in\mathcal{I}_{\mathcal{M}^{(x)}}}$ is a subspecies of $\mathcal{M}$. Assume that the valued quiver $(Q^{\mathcal{M}^{(x)}}, \mathbf{d}^{\mathcal{M}^{(x)}})$ is a convex valued subquiver of $(Q^\mathcal{M}, \mathbf{d}^\mathcal{M})$. It is straightforward to check that the inclusion $C^{(x)} \subseteq C$ is of coidempotent type. If $(Q^{\mathcal{M}^{(x)}}, \mathbf{d}^{\mathcal{M}^{(x)}})$ is not a convex valued subquiver of $(Q^\mathcal{M}, \mathbf{d}^\mathcal{M})$, then the inclusion $C^{(x)} \subseteq C$ is not of coidempotent type in general (cf. Example 2.6 and Remark 2.8).

5. Ringel–Hall algebras of hereditary pure semisimple coalgebras. Let K be a finite field and let C be a K-coalgebra. Set $u_i = u_{[S(i)]} \in \mathcal{H}(C)$ for all simple C-comodules $S(i)$.

LEMMA 5.1. *Let K be a finite field and C be a basic indecomposable hereditary pure semisimple K-coalgebra.*

(a) *There exists a family of finite-dimensional hereditary coalgebras $C^{(x)}$ of finite representation type such that $\mathcal{H}(C) = \bigcup \mathcal{H}(C^{(x)})$ is a directed union of algebras.*

(b) *$\mathcal{H}(C)$ is generated* (*as an algebra*) *by the set* $\{u_i;\, i \in I_C\}$.

Proof. It follows from [6], [7], [11, Theorem 4.14] and [13] that $({}_CQ, {}_C\mathbf{d})$ is one of the valued quivers in Tables 1.1 and 1.2. Therefore $({}_CQ, {}_C\mathbf{d})$ is locally finite and ${}_CQ$ does not contain infinite chains of the form $\bullet \leftarrow \bullet \leftarrow \cdots \leftarrow \bullet \leftarrow \bullet \leftarrow \cdots$. Hence, by Theorem 4.2, there is a coalgebra isomorphism $C \cong T^{\square}(\mathcal{M})$, where $\mathcal{M} = {}_C\mathcal{E}xt^{\#}$. Moreover, [27, Proposition 5.8] shows that the coalgebra $T^{\square}(\mathcal{M})$ is a directed union of the finite-dimensional subcoalgebras $C^{(x)} = T^{\square}(\mathcal{M}^{(x)})$, where $\mathcal{M}^{(x)}$ runs through finite K-subspecies of $\mathcal{M}$. We may choose the species $\mathcal{M}^{(x)}$ in such a way that $(Q^{\mathcal{M}^{(x)}}, \mathbf{d}^{\mathcal{M}^{(x)}})$ is a finite convex valued subquiver of $(Q^{\mathcal{M}}, \mathbf{d}^{\mathcal{M}})$. By Lemma 4.1, in this case the subcategory $C^{(x)}$-comod $\cong \operatorname{rep}(\mathcal{M}^{(x)})$ of C-comod $\cong \operatorname{nilrep}^{\ell f}(\mathcal{M})$ is closed under extensions. Since $\mathcal{M}^{(x)}$ is finite and $(Q^{\mathcal{M}^{(x)}}, \mathbf{d}^{\mathcal{M}^{(x)}}) = ({}_{C^{(x)}}Q, {}_{C^{(x)}}\mathbf{d})$ is a finite convex valued subquiver of $({}_CQ, {}_C\mathbf{d})$, it follows that $({}_{C^{(x)}}Q, {}_{C^{(x)}}\mathbf{d})$ is one of the valued quivers in Table 1.1. Moreover, [27, Lemma 5.5] shows that $T^{\square}(\mathcal{M}^{(x)})$ is a finite-dimensional coalgebra and $(T^{\square}(\mathcal{M}^{(x)}))^* \cong T(\mathcal{M}^{(x)})$. It is easy to see that $T(\mathcal{M}^{(x)})$ is a basic indecomposable finite-dimensional algebra. By [6], $T(\mathcal{M}^{(x)})$ is hereditary and of finite representation type. Therefore $C = \bigcup C^{(x)}$, where $C^{(x)}$ are finite-dimensional hereditary K-coalgebras and $(C^{(x)})^* \cong T(\mathcal{M}^{(x)})$ are finite-dimensional hereditary K-algebras of finite representation type. By Remark 4.3, $C^{(x)} \hookrightarrow C$ is of coidempotent type. Then, by Lemmata 3.3, 3.4 and 4.1, we have

$$\mathcal{H}(C) = \bigcup \mathcal{H}(C^{(x)}) = \bigcup \mathcal{H}((C^{(x)})^*),$$

and (a) is proved. Moreover, it follows from [17, Proposition 6] that the algebra $\mathcal{H}((C^{(x)})^*)$ is generated by $\{u_i;\, i \in I_{C^{(x)}}\}$. Hence the lemma follows, because $I_C = \bigcup_x I_{C^{(x)}}$. ∎

To investigate Ringel–Hall algebras and Lie algebras of hereditary pure semisimple coalgebras, we first associate a Ringel–Hall algebra with a valued quiver.

With any triple $(Q, \mathbf{d}, f)$, where $(Q, \mathbf{d})$ is an acyclic valued quiver and $f = (f_i)$ are positive integers satisfying

(5.2) $$d'_{ij} f_j = d''_{ij} f_i,$$

we associate an integral quadratic form $q_{(Q,\mathbf{d},f)} : \mathbb{Z}^{(Q_0)} \to \mathbb{Z}$ defined by

(5.3) $$q_{(Q,\mathbf{d},f)}(x) = \sum_i f_i x_i^2 - \sum_{i,j} d'_{ij} f_j x_i x_j$$

(see [6, p. 7]). Denote by

$$\mathcal{R}^+_{(Q,\mathbf{d},f)} = \{v \in \mathbb{N}^{(Q_0)};\ q_{(Q,\mathbf{d},f)}(v) = f_i \text{ for some } i\}$$

the set of all positive roots of $q_{(Q,\mathbf{d},f)}$.

LEMMA 5.4. *Let $(Q, \mathbf{d})$ be a connected acyclic valued quiver and $f = (f_i)$, $g = (g_i)$ be integers satisfying* (5.2)*. Then*

$$\mathcal{R}^+_{(Q,\mathbf{d},f)} = \mathcal{R}^+_{(Q,\mathbf{d},g)}.$$

Proof. Let $s, t \in Q_0$ be such that there exists a valued arrow $i \xrightarrow{(d'_{st}, d''_{st})} j$. Note that $d'_{st} f_t = d''_{st} f_s$ and $d'_{st} g_t = d''_{st} g_s$. Therefore,

$$g_t = g_s \cdot \frac{d''_{st}}{d'_{st}} = g_s \cdot \frac{f_t}{f_s} = f_t \cdot \frac{g_s}{f_s}$$

and

$$g_s = g_t \cdot \frac{d'_{st}}{d''_{st}} = g_t \cdot \frac{f_s}{f_t} = f_s \cdot \frac{g_t}{f_t}.$$

Fix $s \in Q_0$. We claim that $g_j = f_j \cdot \frac{g_s}{f_s}$ for all $j \in Q_0$. Indeed, for any $j \in Q_0$, there exists an unoriented path $s = j_0, j_1, \ldots, j_m = j$, because Q is connected. We prove our claim by induction on m. For $m = 0$ we obviously have $g_s = f_s \cdot \frac{g_s}{f_s}$. By the above remarks, $g_m = f_m \cdot \frac{g_{m-1}}{f_{m-1}}$ and by induction

$$g_m = f_m \cdot f_{m-1} \cdot \frac{g_s}{f_s} \cdot \frac{1}{f_{m-1}} = f_m \cdot \frac{g_s}{f_s},$$

proving the claim. Therefore $q_{(Q,\mathbf{d},g)}(x) = \frac{g_s}{f_s} \cdot q_{(Q,\mathbf{d},f)}(x)$ and it is easy to see that $q_{(Q,\mathbf{d},f)}(v) = f_i$ if and only if $q_{(Q,\mathbf{d},g)}(v) = \frac{g_s}{f_s} \cdot f_i = g_i$. This finishes the proof. ∎

By Lemma 5.4, with any acyclic valued quiver $(Q, \mathbf{d})$ we associate the set

(5.5) $$\mathcal{R}^+_{(Q,\mathbf{d})}$$

of positive roots of any quadratic form $q_{(Q,\mathbf{d},f)}$. Let $(\overline{Q}, \overline{\mathbf{d}})$ be a convex valued subquiver of $(Q, \mathbf{d})$. Note that any $x \in \mathcal{R}^+_{(\overline{Q},\overline{\mathbf{d}})}$ may be viewed as an element of $\mathcal{R}^+_{(Q,\mathbf{d})}$. Indeed, it is enough to extend x by zeros. Below we develop this observation and we view $\mathcal{R}^+_{(\overline{Q},\overline{\mathbf{d}})}$ as a subset of $\mathcal{R}^+_{(Q,\mathbf{d})}$.

Let C be an Ext-finite hereditary coalgebra with a fixed decomposition (2.3). Denote by

$$\mathcal{R}_C^+ = \{v \in \mathbb{N}^{(Q_0)};\ q_C(v) = \mathbf{s}_i^0 = \dim_K F_i \text{ for some } i\}$$

the set of all positive roots of q_C.

Let $(Q, \mathbf{d})$ be any of the valued quivers in Tables 1.1 and 1.2. Note that there exist positive integers f_i satisfying (5.2).

LEMMA 5.6. *Let C be a hereditary coalgebra such that $({}_CQ, {}_C\mathbf{d})$ is one of the valued quivers in Tables* 1.1 *and* 1.2. *Then $\mathcal{R}_C^+ = \mathcal{R}_{({}_CQ, {}_C\mathbf{d})}$.*

Proof. Set $f_i = \dim_K F_i$ and note that

$$\dim_K \mathrm{Ext}_C^1(S(i), S(j)) = \dim_{F_j} \mathrm{Ext}_C^1(S(i), S(j)) \cdot \dim_K F_i = {}_Cd'_{ij} \cdot f_j.$$

Therefore $q_C = q_{({}_CQ, {}_C\mathbf{d}, f)}$ and the lemma easily follows. ∎

Let C be a basic pure semisimple hereditary K-coalgebra. The map

$$\mathbf{lgth} : C\text{-comod} \to \mathbb{Z}^{(I_C)}$$

defines a bijection between the set of isomorphism classes of indecomposable left C-comodules and the set $\mathcal{R}_C^+$ of positive roots of q_C, that is, vectors $v \in \mathbb{Z}^{(I_C)}$ such that $q_C(v) = \dim_K \mathrm{End}_C S(i) = \mathbf{s}_i^0$ for some $i \in I_C$ (see [6] and [11, Theorem 4.14]). Let $(Q, \mathbf{d})$ be any of the valued quivers in Tables 1.1 and 1.2. Let C be a hereditary K-coalgebra with left Gabriel valued quiver $(Q, \mathbf{d})$. Note that $\mathbf{s}_i^0 = \mathbf{s}_j^0$ for all $i, j \geq 3$ and denote by s_C the greatest common divisor of the integers $\{\mathbf{s}_i^0;\ i \in I_C\}$. As in the algebra case, we call s_C the *symmetrisation index* of the K-coalgebra C. For any $x \in \mathcal{R}_{(Q,\mathbf{d})}^+ = \mathcal{R}_C^+$, we fix an indecomposable C-comodule

$$M(C, x)$$

such that $\mathbf{lgth}\, M(C, x) = x$. Denote by $\mathcal{B}_{(Q,\mathbf{d})}$ the set of all functions $a : \mathcal{R}_{(Q,\mathbf{d})}^+ \to \mathbb{N}$ which have finite support, i.e. $a(x) \neq 0$ for only finitely many $x \in \mathcal{R}_{(Q,\mathbf{d})}^+$. With any $a \in \mathcal{B}_{(Q,\mathbf{d})}$, we associate the C-comodule

$$M(C, a) = \bigoplus_{x \in \mathcal{R}_{(Q,\mathbf{d})}^+} M(C, x)^{a(x)}.$$

This establishes a bijection between $\mathcal{B}_{(Q,\mathbf{d})}$ and the set of all isomorphism classes of finite-dimensional left C-comodules. If we identify any element $x \in \mathcal{R}_{(Q,\mathbf{d})}^+$ with the characteristic function $x : \mathcal{R}_{(Q,\mathbf{d})}^+ \to \mathbb{N}$, we get $\mathcal{R}_{(Q,\mathbf{d})}^+ \subseteq \mathcal{B}_{(Q,\mathbf{d})}$.

We can do the same kind of construction for any finite-dimensional hereditary K-algebra of finite representation type. By [5], any such algebra A is Morita equivalent to the tensor algebra $T(\mathcal{M})$ of a K-species such that $(Q^{\mathcal{M}}, \mathbf{d}^{\mathcal{M}})$ is one of the valued quivers in Table 1.1. Moreover, $\mathbf{lgth}$ gives a bijection between the set of isomorphism classes of finite-dimensional indecomposable right A-modules and the set $\mathcal{R}^+_{(Q,\mathbf{d})}$ (see [6] and [7]). We use the same notation $M(A,x)$ and $M(A,a)$ for A-modules, where $x \in \mathcal{R}^+_{(Q,\mathbf{d})}$ and $a \in \mathcal{B}_{(Q,\mathbf{d})}$ (see [16]). Moreover, we recall that there exists an equivalence of categories $\mathrm{mod}(A) \cong \mathrm{rep}(\mathcal{M})$ (see [6] and [7]).

Lemma 5.7. *Let $\mathcal{M}$ be a K-species such that $(Q^{\mathcal{M}}, \mathbf{d}^{\mathcal{M}})$ is one of the valued quivers in Table* 1.1. *If $x \in \mathcal{R}^+_{(Q^{\mathcal{M}},\mathbf{d}^{\mathcal{M}})}$, then $M(T(\mathcal{M}),x) \cong M(T^{\square}(\mathcal{M}),x)$ (in the category* $\mathrm{rep}(\mathcal{M})$*).*

Proof. Apply the previous discussion and Theorem 4.2. ∎

Proposition 5.8. *Let $(Q,\mathbf{d})$ be any of the valued quivers in Tables* 1.1 *and* 1.2 *and let $\mathcal{R}^+_{(Q,\mathbf{d})}$ be the set* (5.5). *For any functions $a,b,c : \mathcal{R}^+_{(Q,\mathbf{d})} \to \mathbb{N}$ from the set $\mathcal{B}_{(Q,\mathbf{d})}$ there exists a polynomial $\varphi^b_{ca} \in \mathbb{Z}[T]$ such that for any finite field K and for any hereditary K-coalgebra C with $({}_CQ, {}_C\mathbf{d}) = (Q,\mathbf{d})$ and symmetrisation index s_C, we have*

$$\varphi^b_{ca}(|K|^{s_C}) = F^{M(C,b)}_{M(C,c),M(C,a)},$$

where $|X|$ denotes the cardinality of a finite set X.

Proof. Let $a,b,c : \mathcal{R}^+_{(Q,\mathbf{d})} \to \mathbb{N}$ be functions from $\mathcal{B}_{(Q,\mathbf{d})}$. It is easy to see that there exists a finite convex valued subquiver $(\overline{Q},\overline{\mathbf{d}}) \subseteq (Q,\mathbf{d})$ such that the supports of a,b,c are contained in $(\overline{Q},\overline{\mathbf{d}})$. We may assume that $i \in \overline{Q}_0$ for all $i \leq 3$.

By [16, Theorem 1], there exists a polynomial $\varphi^b_{ca} \in \mathbb{Z}[T]$ such that for any finite field K and for any finite-dimensional hereditary K-algebra A with valued (Gabriel) quiver $(\overline{Q},\overline{\mathbf{d}})$ and symmetrisation index s_A, we have

$$\varphi^b_{ca}(|K|^{s_A}) = F^{M(A,b)}_{M(A,c),M(A,a)}.$$

Let K be a finite field and let C be a hereditary K-coalgebra such that $({}_CQ, {}_C\mathbf{d}) = (Q,\mathbf{d})$. By Theorem 4.2, there is a coalgebra isomorphism $C \cong T^{\square}(\mathcal{M})$, where $\mathcal{M} = {}_C\mathcal{E}xt^{\#}$. Let $\mathcal{M}^{(x)} = (F_i, {}_iM_j)_{i,j\in\overline{Q}_0}$ be a finite subspecies of $\mathcal{M}$ and let $C^{(x)} = T^{\square}(\mathcal{M}^{(x)})$. By Theorem 4.2, Lemma 4.1 and Remark 4.3, there is an injective homomorphism (of coidempotent type) of $C^{(x)}$ into C such that the subcategory $C^{(x)}$-comod of C-comod is closed under extensions, because $(\overline{Q},\overline{\mathbf{d}})$ is a convex valued subquiver of $(Q,\mathbf{d})$. Moreover, $s_C = s_{C^{(x)}}$. It follows that $({}_{C^{(x)}}Q, {}_{C^{(x)}}\mathbf{d}) = (\overline{Q},\overline{\mathbf{d}})$ is one of the valued quivers in Tables 1.1 and $(T^{\square}(\mathcal{M}^{(x)}))^* \cong T(\mathcal{M}^{(x)})$. The algebra $T(\mathcal{M}^{(x)})$ is of finite representation type, the valued quiver $(Q^{\mathcal{M}}, \mathbf{d}^{\mathcal{M}})$ associated with $T(\mathcal{M}^{(x)})$ equals $(\overline{Q},\overline{\mathbf{d}})$ (see [6, p. 5]) and there exists a bijection between $\mathcal{R}^+_{(\overline{Q},\overline{\mathbf{d}})}$ and the set of all isomorphism classes of indecomposable right finite-dimensional $T(\mathcal{M}^{(x)})$-modules (see [6]). For $A^{(x)} = T(\mathcal{M}^{(x)})$ we have $\varphi^b_{ca}(|K|^s) = F^{M(A^{(x)},b)}_{M(A^{(x)},c),M(A^{(x)},a)}$, where $s = s_{A^{(x)}}$. The equivalence (2.1) gives $s = s_{A^{(x)}} = s_{C^{(x)}}$. Finally, by Lemma 3.2, Lemma 5.7 and the remarks above, we get

$$\varphi^b_{ca}(|K|^{s_C}) = \varphi^b_{ca}(|K|^s) = F^{M(C^{(x)},b)}_{M(C^{(x)},c),M(C^{(x)},a)} = F^{M(C,b)}_{M(C,c),M(C,a)}. \ \blacksquare$$

Following an idea given in [16], we define a *generic Ringel–Hall algebra* as follows. Let $(Q, \mathbf{d})$ be any of the valued quivers in Tables 1.1 and 1.2. Let $\mathcal{H}((Q, \mathbf{d}), \mathbb{Z}[T])$ be the free $\mathbb{Z}[T]$-module with basis $\{u_a\}_{a\in\mathcal{B}_{(Q,\mathbf{d})}}$. We define a multiplication by

$$u_c u_a = \sum_b \varphi^b_{ca} u_b.$$

The sum in this formula is finite, because $\varphi^b_{ca} = 0$ unless

$$\mathbf{lgth}\, M(C, b) = \mathbf{lgth}\, M(C, a) + \mathbf{lgth}\, M(C, c),$$

where C is any hereditary coalgebra with left Gabriel valued quiver $(Q, \mathbf{d})$. The arguments in [16, Proposition 4] show that $\mathcal{H}((Q, \mathbf{d}), \mathbb{Z}[T])$ is an associative ring with the identity element u_0.

Moreover, we consider the degeneration of $\mathcal{H}((Q, \mathbf{d}), \mathbb{Z}[T])$ given by the specialisation of T to 1, and tensor this degeneration by $\mathbb{C}$ over $\mathbb{Z}$. More precisely, let $\mathcal{H}(Q, \mathbf{d})_1$ be the $\mathbb{C}$-vector space with basis $\{u_a\}_{a\in\mathcal{B}_{(Q,\mathbf{d})}}$ and multiplication

$$u_c u_a = \sum_b \varphi^b_{ca}(1) u_b.$$

Denote by $\mathcal{K}(Q, \mathbf{d})_1$ the $\mathbb{C}$-subspace of $\mathcal{H}(Q, \mathbf{d})_1$ with basis $\{u_a\}_{a\in\mathcal{R}^+_{(Q,\mathbf{d})}}$.

THEOREM 5.9. *Let $(Q, \mathbf{d})$ be any of the valued quivers in Tables* 1.1 *and* 1.2.

(a) *$\mathcal{H}(Q, \mathbf{d})_1$ is an associative $\mathbb{C}$-algebra with the identity element u_0.*

(b) *There exists a family of finite convex valued subquivers $(Q^{(x)}, \mathbf{d}^{(x)})$ of $(Q, \mathbf{d})$ with $\mathcal{H}(Q, \mathbf{d})_1 = \bigcup \mathcal{H}(Q^{(x)}, \mathbf{d}^{(x)})_1$ and $\mathcal{K}(Q, \mathbf{d})_1 = \bigcup \mathcal{K}(Q^{(x)}, \mathbf{d}^{(x)})_1$, where $\bigcup$ means the directed union of algebras and vector spaces*, *respectively.*

(c) *$\mathcal{K}(Q, \mathbf{d})_1$ is a Lie subalgebra of $\mathcal{H}(Q, \mathbf{d})_1$ (with the Lie bracket $[a, b] = ab - ba$) and $\mathcal{K}(Q, \mathbf{d})_1 = \bigcup \mathcal{K}(Q^{(x)}, \mathbf{d}^{(x)})_1$, where $\bigcup$ means the directed union of Lie algebras.*

(d) *$\mathcal{H}(Q, \mathbf{d})_1$ is the universal enveloping algebra of $\mathcal{K}(Q, \mathbf{d})_1$.*

(e) *The Lie algebra $\mathcal{K}(Q, \mathbf{d})_1$ is isomorphic to the positive part $\mathfrak{n}_+$ of the simple Lie algebra $\mathfrak{g}$ associated with $(Q, \mathbf{d})$ if $(Q, \mathbf{d})$ is one of the valued quivers in Table* 1.1, *or to the positive part $\mathfrak{n}_+$ of the infinite rank affine Lie algebra $\mathfrak{g}$ associated with $(Q, \mathbf{d})$ if $(Q, \mathbf{d})$ is one of the valued quivers in Table* 1.2 (*see* [10, 7.11]).

Proof. Let $(Q, \mathbf{d})$ be any of the valued quivers in Tables 1.1 and 1.2. Note that

$$\mathcal{H}(Q, \mathbf{d})_1 \cong (\mathcal{H}((Q, \mathbf{d}), \mathbb{Z}[T])/((T-1)u_0)) \otimes_{\mathbb{Z}} \mathbb{C},$$

where $((T-1)u_0)$ is the ideal of $\mathcal{H}((Q, \mathbf{d}), \mathbb{Z}[T])$ generated by $(T-1)u_0$. Therefore (a) follows.

(b) For any integer $x \geq 1$, we define $(Q^{(x)}, \mathbf{d}^{(x)})$ to be the convex valued subquiver of $(Q, \mathbf{d})$ such that $Q_0^{(x)} = \{i \in Q_0;\, i \leq x\}$. It is clear that $(Q^{(x)}, \mathbf{d}^{(x)})$ is finite for any x. It is easy to see that $\mathcal{R}^+_{(Q^{(x)}, \mathbf{d}^{(x)})} \subseteq \mathcal{R}^+_{(Q,\mathbf{d})}$ for any x, and $\mathcal{R}^+_{(Q,\mathbf{d})} = \bigcup_{x \geq 1} \mathcal{R}^+_{(Q^{(x)}, \mathbf{d}^{(x)})}$. Therefore $\mathcal{H}(Q, \mathbf{d})_1 = \bigcup \mathcal{H}(Q^{(x)}, \mathbf{d}^{(x)})_1$ and $\mathcal{K}(Q, \mathbf{d})_1 = \bigcup \mathcal{K}(Q^{(x)}, \mathbf{d}^{(x)})_1$, where $\bigcup$ means the directed union of algebras and vector spaces, respectively.

(c) Let $x, y \in \mathcal{R}^+_{(Q,\mathbf{d})}$ and let $(Q^{(x)}, \mathbf{d}^{(x)})$ be a finite convex valued subquiver of $(Q, \mathbf{d})$ such that $x, y \in \mathcal{R}^+_{(Q^{(x)}, \mathbf{d}^{(x)})}$. By [16, Theorem 2], we have $\varphi^a_{xy}(1) = \varphi^a_{yx}(1)$ for all $a \in \mathcal{B}_{(Q^{(x)}, \mathbf{d}^{(x)})} \setminus \mathcal{R}^+_{(Q^{(x)}, \mathbf{d}^{(x)})}$. Finally, $\mathcal{K}(Q^{(x)}, \mathbf{d}^{(x)})_1$ is a Lie subalgebra of $\mathcal{H}(Q^{(x)}, \mathbf{d}^{(x)})_1$ and $\mathcal{K}(Q, \mathbf{d})_1 = \bigcup \mathcal{K}(Q^{(x)}, \mathbf{d}^{(x)})_1$ is a Lie subalgebra of $\mathcal{H}(Q, \mathbf{d})_1$.

(d) It follows from [16, Proposition 5] that $\mathcal{H}(Q^{(x)}, \mathbf{d}^{(x)})_1$ is the universal enveloping algebra of $\mathcal{K}(Q^{(x)}, \mathbf{d}^{(x)})_1$ for any finite valued subquiver $(Q^{(x)}, \mathbf{d}^{(x)})$ of $(Q, \mathbf{d})$. Therefore $\mathcal{H}(Q, \mathbf{d})_1 = \bigcup \mathcal{H}(Q^{(x)}, \mathbf{d}^{(x)})_1$ is the universal enveloping algebra of $\mathcal{K}(Q, \mathbf{d})_1 = \bigcup \mathcal{K}(Q^{(x)}, \mathbf{d}^{(x)})_1$.

For valued quivers from Table 1.1 the statement (e) is proved in [18, Corollary 3]. Let $(Q, \mathbf{d})$ be any of the valued quivers in Table 1.2. Let $\mathfrak{g}$ be the infinite rank affine Lie algebra associated with $(Q, \mathbf{d})$ and let $\mathfrak{n}_+$ be the positive part of $\mathfrak{g}$ (see [10, 7.11]). By [10], the Lie algebra $\mathfrak{n}_+$ is isomorphic to the quotient Lie algebra

$$\mathrm{Lie}_{\mathbb{C}}\langle x_i;\, i \in Q_0\rangle/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by the relations

$$(\operatorname{ad} x_i)^{1+d'_{ij}} x_j = 0 \quad \text{ for all } i, j \in Q_0.$$

By [17, Proposition 2], these relations are satisfied in $\mathcal{K}(Q^{(x)}, \mathbf{d}^{(x)})_1$, hence also in $\mathcal{K}(Q, \mathbf{d})_1$. Note that $\mathcal{K}(Q, \mathbf{d})_1 = \bigoplus_{x \in \mathcal{R}^+_{(Q,\mathbf{d})}} \mathbb{C}u_x$ is the direct sum of one-dimensional vector spaces. By [10], so is $\mathfrak{n}_+ = \bigoplus_{\alpha \in Q^+} \mathbb{C}\alpha$. Moreover, by [10] and [11], we have $\mathcal{R}^+_{(Q,\mathbf{d})} = Q^+$ and there exists an epimorphism of graded Lie algebras

$$\mathfrak{n}_+ \to \mathcal{K}(Q, \mathbf{d})_1.$$

Therefore $\mathfrak{g} \cong \mathcal{K}(Q, \mathbf{d})_1$. ∎

The following corollary is an easy consequence of Theorem 5.9.

COROLLARY 5.10. *Let* $(Q, \mathbf{d})$ *be any of the valued quivers in Tables* 1.1 *and* 1.2.

(a) *There is an isomorphism of Lie algebras*

$$\mathcal{K}(Q, \mathbf{d})_1 \cong \mathrm{Lie}_{\mathbb{C}}\langle x_i;\, i \in Q_0\rangle/\mathcal{I},$$

where $\mathcal{I}$ *is the ideal generated by the relations*

$$(\mathrm{ad}\, x_i)^{1+d'_{ij}} x_j = 0 \quad \textit{for all } i, j \in Q_0.$$

(b) *There is an isomorphism of algebras*

$$\mathcal{H}(Q, \mathbf{d})_1 \cong \langle x_i;\, i \in Q_0\rangle/\mathcal{I},$$

where $\langle x_i\ ;\ i \in Q_0\rangle$ *is the free* $\mathbb{C}$*-algebra generated by the set* $\{x_i;\, i \in Q_0\}$ *and* $\mathcal{I}$ *is the ideal generated by the relations*

$$(\mathrm{ad}\, x_i)^{1+d'_{ij}} x_j = 0 \quad \textit{for all } i, j \in Q_0.$$

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