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ZONE AND DOUBLE ZONE DIAGRAMS IN ABSTRACT SPACES

BY

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Abstract. A zone diagram of order n is a relatively new concept which was first defined and studied by T. Asano, J. Matoušek and T. Tokuyama. It can be interpreted as a state of equilibrium between n mutually hostile kingdoms. Formally, it is a fixed point of a certain mapping. These authors considered the Euclidean plane with finitely many singleton-sites and proved the existence and uniqueness of zone diagrams there. In the present paper we generalize this concept in various ways. We consider general sites in m-spaces (a simple generalization of metric spaces) and prove several existence and (non)uniqueness results in this setting. In contrast with previous works, our (rather simple) proofs are based on purely order-theoretic arguments. Many explicit examples are given, and some of them illustrate new phenomena which occur in the general case. We also re-interpret zone diagrams as a stable configuration in a certain combinatorial game, and provide an algorithm for finding this configuration in a particular case.

1. Introduction and notation. A zone diagram of order n is a relatively new concept which can be interpreted as a state of equilibrium between n mutually hostile kingdoms. More formally, let (X,d) be a metric space, and suppose $P=(P_k)_{k\in K}$ is a given tuple of nonempty sets in X. A *zone diagram* with respect to P is a tuple $R=(R_k)_{k\in K}$ of nonempty sets such that each R_k is the set of all $x\in X$ which are closer to P_k than to $\bigcup_{j\neq k} R_j$. In other words, R is a fixed point of a certain mapping (called the Dom mapping). Neither its existence nor its uniqueness are obvious *a priori*.

The concept of a zone diagram was first defined and studied by T. Asano, J. Matoušek and T. Tokuyama [1, 2], in the case where X was the Euclidean plane, K was finite, each P_k was a single point and all these points were different. They proved the existence and uniqueness of a zone diagram in this case. Their proofs rely heavily on this specific setting.

In our paper we generalize this concept in various ways. As we have already mentioned, we consider general tuples of sets $P=(P_k)_{k\in K}$, and general metric spaces. In fact, we consider a more general setting (m-spaces; see Section 3). One of the advantages of this generalization, besides its leading

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to general results, is that it yields a better understanding of the concept, and enables us to give plenty of explicit examples of zone diagrams, a task which is quite hard in the case of singleton-site zone diagrams in the Euclidean plane. These examples illustrate some new phenomena which occur in the general case. Moreover, this generalization also opens up new possibilities for applying this concept in other parts of mathematics and elsewhere.

Exact definitions, as well as several examples, are given in Sections 2 and 3. In Section 4 we re-interpret the concept of a zone diagram as a stable configuration in a certain combinatorial game.

Our main existence results are Theorem 5.6, which shows the existence of a zone diagram of order 2 in any m-space, and Theorem 5.5, which shows the existence of a double zone diagram (a fixed point of the second iteration Dom^2) of any order. Our method, which is different from the methods described in [1] and [2], is based on the Knaster–Tarski fixed point theorem for monotone (increasing) mappings. [One, in fact two, of the arguments in [2] do make use of a fixed point theorem (the Schauder fixed point theorem), but for continuous mappings rather than monotone ones.] It can be seen that our (rather simple) proofs have a purely order-theoretic character; there is no need to take into account any other considerations (algebraic, topological, analytical, etc.). As a corollary we obtain the existence of a trisector in any Hilbert space, and the proof can be considered "conceptual"; see Remark 5.7 in Section 5.

In Section 6 we discuss the uniqueness question. In general, there can be several zone diagrams, but we present several necessary and sufficient conditions for uniqueness. In Section 7 we describe a simple algorithm for constructing a zone diagram of order 2 and a double zone diagram of any order in the case where X is a finite set. We conclude the paper by formulating some interesting open problems.

We end this introduction with a couple of words about notation. Throughout the text we will make use of tuples, the components of which are sets. Every operation or relation between such tuples, or on a single tuple, is done componentwise. Hence, for example, if $K \neq \emptyset$ is a set of indices, and if $R = (R_k)_{k\in K}$ and $S = (S_k)_{k\in K}$ are two tuples of sets, then $R \cap S = (R_k \cap S_k)_{k\in K}$, $\overline{R} = (\overline{R}_k)_{k\in K}$, and $R \subseteq S$ means $R_k \subseteq S_k$ for each $k \in K$. The tuple $(X)_{k\in K}$ has all the components equal to X. Given a set X, we denote by $\mathcal{P}^*(X)$ the set of all nonempty subsets of X, and by $|X|$ the cardinal number of X.

2. Definitions and examples. Zone diagrams can be naturally defined in any metric space. In Section 5 we will prove a theorem which ensures the existence of zone diagrams of order $n = 2$ in any metric space. Surprisingly, the proof can be carried over to a more general setting, which we call m-spaces. However, since the latter concept seems to be new, and since the concept of a zone diagram is best understood in the context of metric spaces, we discuss it first in this context. The corresponding generalization will be carried out in Section 3.

Definition 2.1. Let (X, d) be a metric space. For any $P, A \in \mathcal{P}^*(X)$, the *dominance region* $\mathrm{dom}(P, A)$ of P with respect to A is the set of all $x \in X$ which are closer to P than to A, i.e.,

$$\mathrm{dom}(P, A) = \{x \in X : d(x, P) \leq d(x, A)\}.$$

The function $\mathrm{dom} : \mathcal{P}^*(X) \times \mathcal{P}^*(X) \to \mathcal{P}^*(X)$ is called the *dom mapping.*

For example, if a and p are two different points in a Hilbert space X, and if $P = \{p\}$ and $A = \{a\}$, then $\mathrm{dom}(P, A)$ is the half-space containing P determined by the hyperplane passing through the middle of the line segment $[p, a]$ and perpendicular to it; $\mathrm{dom}(A, P)$ is the other half-space.

Definition 2.2. Let (X, d) be a metric space and let K be a set of at least two elements (indices), possibly infinite. Given a tuple $(P_k)_{k\in K}$ of nonempty subsets $P_k \subseteq X$, a *zone diagram of order* $n = |K|$ with respect to that tuple is a tuple $R = (R_k)_{k\in K}$ of nonempty subsets $R_k \subseteq X$ such that

$$R_k = \mathrm{dom}\Big(P_k, \bigcup_{j\neq k} R_j\Big) \qquad \forall k \in K.$$

In other words, if we define $X_k = \mathcal{P}^*(X)$, then a zone diagram is a fixed point of the mapping $\mathrm{Dom} : \bigtimes_{k\in K} X_k \to \bigtimes_{k\in K} X_k$ defined by

$$\mathrm{Dom}(R) = \Big(\mathrm{dom}\Big(P_k, \bigcup_{j\neq k} R_j\Big)\Big)_{k\in K}. \tag{1}$$

If the second iteration $\mathrm{Dom}^2 = \mathrm{Dom} \circ \mathrm{Dom}$ has a fixed point R, we say that R is a *double zone diagram* in X.

If we interpret each R_k as an ancient kingdom, and each P_k as a site or a collection of sites in R_k (cities, army camps, islands, etc.), then a zone diagram is a configuration in which each kingdom R_k consists of all the points $x \in X$ which are closer to P_k than to the other kingdoms. This can be regarded as a state of equilibrium between the kingdoms in the following sense. Suppose the kingdoms are mutually hostile. In particular, each kingdom has to defend its borders against attacks from the other kingdoms. Due to various considerations (resources, field conditions, etc.), the defending army is usually situated only in (part of) the sites P_k (unless the kingdom moves forces to attack another kingdom), and each P_k remains unchanged. Hence, if $(R_k)_{k\in K}$ is a zone diagram, then each point in each kingdom can be defended at least as fast as it takes to attack it from any other kingdom, and no kingdom can enlarge its territory without violating this condition. (But see Examples 2.3 and 2.5 for some non-realistic counterexamples.)

A double zone diagram R is a different state of equilibrium between the kingdoms: now each kingdom R_k consists of all points $x \in X$ which are closer to P_k than to the union $\bigcup_{j\neq k}(\mathrm{Dom}\, R)_j$. We note that any zone diagram R is obviously a double zone diagram since $\mathrm{Dom}^2\, R = \mathrm{Dom}(\mathrm{Dom}\, R) = \mathrm{Dom}\, R = R$, but the converse is not necessarily true as the following example shows.

EXAMPLE 2.3. Let $X = \{-1, 0, 1\}$ be a subset of $\mathbb{R}$ with the usual metric, and let $P_1 = \{-1\}$, , $P_2 = \{1\}$. If we let $R_1 = P_1$ and $R_2 = \{0, 1\}$, then -1 is the only point in X closer to P_1 than to R_2, and $0, 1$ are all the points in X closer to P_2 than to R_1, so

$$\mathrm{dom}(P_1, R_2) = \{-1\} = R_1 \quad \text{and} \quad \mathrm{dom}(P_2, R_1) = \{0, 1\} = R_2,$$

i.e., $R = (R_1, R_2)$ is a zone diagram in X. Similarly, $S = (\{-1, 0\}, \{1\})$ is a different zone diagram in X, and this shows that uniqueness does not hold in general. However, if we replace the point 0 with a point $a \in (0, 1)$, then the modified R is still a zone diagram, and it is unique. Indeed, suppose $Z = (Z_1, Z_2)$ is another zone diagram. Obviously, $P_1 \subseteq \mathrm{dom}(P_1, Z_2)$ and $P_2 \subseteq \mathrm{dom}(P_2, Z_1)$, so any $x \neq -1$ is closer to P_2, and hence to Z_2, than to Z_1. Thus Z_1 must be P_1, but then $Z_2 = \mathrm{dom}(P_2, Z_1) = \{a, 1\}$, so $Z = R$.

Already in the original example, where $X = \{-1, 0, 1\}$, it can be seen that a double zone diagram is not necessarily a zone diagram, since

$$\begin{aligned} \mathrm{dom}^2(P_1, P_2) &= (\mathrm{dom}(P_1, \mathrm{dom}(P_2, P_1)), \mathrm{dom}(P_2, \mathrm{dom}(P_1, P_2))) \\ &= (\mathrm{dom}(\{-1\}, \{0, 1\}), \mathrm{dom}(\{1\}, \{-1, 0\})) \\ &= (\{-1\}, \{1\}) = (P_1, P_2), \end{aligned}$$

but (P_1, P_2) is not a zone diagram because $P_1 \neq \{-1, 0\} = \mathrm{dom}(P_1, P_2)$. However, if we replace the point 0 with a point $a \in (0, 1)$, then the double zone diagram and the zone diagram coincide.

EXAMPLE 2.4. Let $X = \mathbb{R}^2$ with the max norm $|x| = |(x_1, x_2)| = \max\{|x_1|, |x_2|\}$, and let $P_1 = \{(0, 3)\}$, $P_2 = \{(0, -3)\}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(x_1) = \begin{cases} -x_1 - 1, & x_1 \leq -2, \\ 1, & x_1 \in [-2, 2], \\ x_1 - 1, & x_1 \geq 2. \end{cases}$$

If R_1 and R_2 are the domains above and below the graphs of f and $-f$ respectively, i.e., $R_1 = \{(x_1, x_2) : x_2 \geq f(x_1)\}$ and $R_2 = \{(x_1, x_2) : x_2 \leq -f(x_1)\}$, then $R = (R_1, R_2)$ is a zone diagram in X; see Figure 1.

Indeed, if $x = (x_1, x_2)$ is in R_1, then there are three possibilities: $x_1 \leq -2$, $x_1 \in [-2, 2]$ and $x_1 \geq 2$. The third case is treated in the same way as the first, so it suffices to consider the first two cases. In the first case, an elementary calculation shows that $d(x, R_2) = d(x, (-2, -1))$. Hence $d(x, R_2) \geq d(x, P_1)$, because $d(x, (-2, -1)) \geq x_2 + 1 \geq \max\{-x_1, |x_2 - 3|\} = d(x, P_1)$. In the

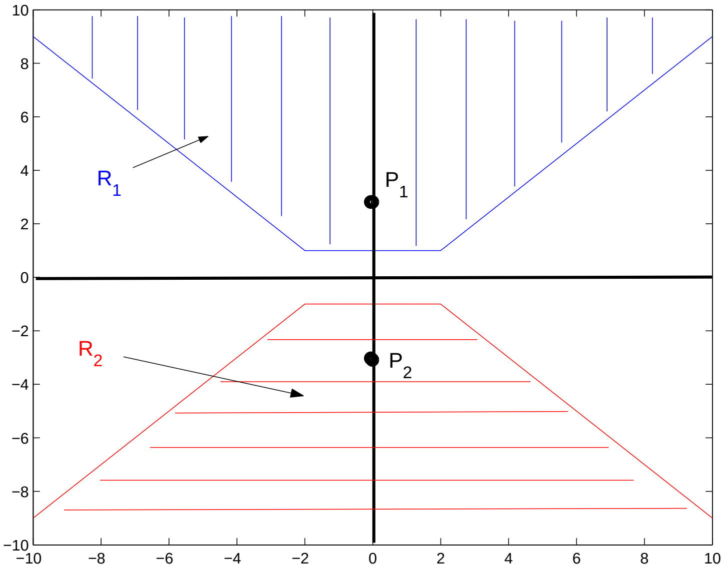

Fig. 1. Example 2.4

second case, either $x_2 \in [1, 5]$ and then $d(x, P_1) \le 2 \le x_2 + 1 = d(x, R_2)$, or $x_2 > 5$ and then $d(x, P_1) = x_2 - 3 < x_2 + 1 = d(x, R_2)$. Thus, in every case $d(x, P_1) \le d(x, R_2)$, i.e., $R_1 \subseteq \mathrm{dom}(P_1, R_2)$.

On the other hand, suppose $d(x, P_1) \le d(x, R_2)$ and assume to the contrary that $x \notin R_1$. Then $x_2 \ge 0$, for otherwise $d(x, R_2) \le d(x, R_1) < d(x, P_1)$. In addition, $|x_1| \le 2$, because otherwise, if, for example, $x_1 < -2$, then using the fact that $x \notin R_1$ implies $x_2 < -x_1 - 1$, we arrive at the inequality $d(x, R_2) \le d(x, (-2, -1)) = \max\{-x_1 - 2, x_2 + 1\} < |x_1| \le d(x, P_1)$. So $|x_1| \le 2$, but then $d(x, R_2) \le 2$, and since $x \notin R_1$, we have, in fact, $d(x, R_2) < 2 < d(x, P_1)$, a contradiction. Therefore $\mathrm{dom}(P_1, R_2) \subseteq R_1$ and we get equality. In the same way, $R_2 = \mathrm{dom}(P_2, R_1)$.

A reader familiar with the concept of a trisector (see [1]), may have already noticed that the boundaries of R_1 and R_2 (denoted by C_1 and C_2, respectively) represent the components of a trisector, i.e., they satisfy the equations $C_1 = \{x \in X : d(x, P_1) = d(x, C_2)\}$ and $C_2 = \{x \in X : d(x, P_2) = d(x, C_1)\}$. The sets C_1 and C_2 are indeed the graphs of convex/concave functions, but in contrast with the Euclidean case ([1, Theorem 2] and the discussion following it), these functions have a simple form and they are not analytic.

In our next example the two sites P_1 and P_2 have a nonempty intersection.

Example 2.5. $X = \mathbb{R}^2$ with the Euclidean norm, $P_1 = \mathbb{Q} \times \{0\}$ and $P_2 = (\mathbb{N} \cup (\mathbb{R} \setminus \mathbb{Q})) \times \{0\}$. At first sight it seems that either a zone diagram may not exist at all, or that it may be pathological if it does exist. Nevertheless, a simple check shows that $(\mathbb{R}\times\{0\},\mathbb{R}^2)$ is a zone diagram, as also is $(\mathbb{R}^2, \mathbb{R}\times\{0\})$. It is interesting to note that if we consider the infinite family $P_x=\{(x,0)\}$, $x\in\mathbb{R}$, then it is not clear at all whether there exists a zone diagram $R=(R_x)_x$. If it does, then it is probably pathological.

The above example can be generalized: if (X,d) is any metric space, and if the tuple $P=(P_k)_{k\in K}$ has the property that $\overline{P}_j=\overline{P}_k$ for all $k,j\in K$, then for any $i\in K$ and any tuple $R=(R_k)_{k\in K}$ with the property that $R_k=\overline{P}_k$ for all $k\neq i$ and $R_i=X$, the tuple R is a zone diagram in X. Hence, some restrictions on P have to be imposed in order to obtain uniqueness. (For instance, $\inf\{d(\overline{P}_k,\overline{P}_j): j\neq k\}>0$ for all $k\in K$ is necessary; it is not clear, however, when this condition is sufficient; see Example 2.3.)

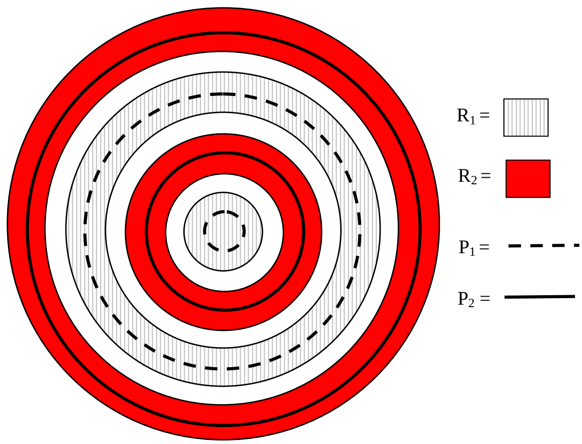

Fig. 2. Example 2.6

Example 2.6. Let $(X,|\cdot|)$ be any normed space, and let

$$P_1=\bigcup_{k=0}^{\infty}\{x\in X: |x|=6k+1\},\quad P_2=\bigcup_{k=0}^{\infty}\{x\in X: |x|=6k+4\}.$$

It can easily be checked that $R=(R_1,R_2)$ is a zone diagram in X, where

$$R_1=\bigcup_{k=0}^{\infty}\{x\in X: 6k\leq|x|\leq 6k+2\},$$
$$R_2=\bigcup_{k=0}^{\infty}\{x\in X: 6k+3\leq|x|\leq 6k+5\}.$$

See Figure 2. The zone diagram in this case is unique; see Section 6. Suppose now that we modify this example by letting $P_k=\{x\in X: |x|=3k+1\}$, $k\in\mathbb{N}\cup\{0\}$. The resulting zone diagram is $R=(R_k)_{k=0}^{\infty}$, where $R_k=\{x\in X:$

$3k \leq |x| \leq 3k+2\}$. We obtain the same array of rings as before, but now each ring represents one and only one R_k.

3. Generalization to m-spaces

DEFINITION 3.1. An *m-space* is a pair (X, d) of a nonempty set X and a function $d : X^2 \to [-\infty, \infty]$ with the property that

$$d(x, x) \leq d(x, y) \quad \forall x, y \in X. \tag{2}$$

We call d the *distance function*, although it is usually not a true distance since it is not assumed to be either symmetric, positive or to satisfy the triangle inequality (take, for example, $X = \mathbb{R}$ and $d(x, y) = \max\{x+1, y\}$). Given $x \in X$ and $A \in \mathcal{P}^*(X)$, we define

$$d(x, A) = \inf\{d(x, y) : y \in A\},$$

and call it the *distance* between the point x and the set A. All the relevant concepts (the dom mapping, zone diagrams, etc.) are defined exactly as in the metric case, but now, however, the interpretations are less clear.

We see that the only constraints on d are condition (2), which implies that the natural property $P \subseteq \mathrm{dom}(P, A)$ will be satisfied, and that its range is $[-\infty, \infty]$. These requirements alone suffice for ensuring the existence of zone diagrams of order 2, and actually the concept of an m-space has its origin in an examination of an earlier proof we had of the existence of a zone diagram in the case $n = |K| = 2$. We also remark that instead of taking $[-\infty, \infty]$ as the range of d, one can take any totally ordered set which has the greatest lower bound property, but we will confine ourselves to the above definition.

We now provide several examples in order to illustrate the concepts of m-spaces and zone diagrams in this general setting.

EXAMPLE 3.2. Let X be any nonempty set and let $a < b$ be real numbers. Suppose $d : X^2 \to \mathbb{R}$ is defined by $d(x, x) = a < b = d(x, y)$ for all $x \neq y$. The function d is usually not a metric. If $(P_k)_{k\in K}$ is any tuple of nonempty and pairwise disjoint subsets of X, then any tuple $(R_k)_{k\in K}$ of nonempty and pairwise disjoint subsets of X for which $P_k \subseteq R_k$ and $\bigcup_{k\in K} R_k = X$ is a zone diagram in X of order $|K|$.

EXAMPLE 3.3. Let $X = \bigcup_{i=1}^{3} X_i$ where $X_1 = \mathbb{R} \times \{0\}$, $X_2 = \{-1\} \times \mathbb{R}$, $X_3 = \{1\} \times \mathbb{R}$. Define $d : X^2 \to [-\infty, \infty]$ by $d(x, y) = |x - y|$ if $x, y \in X_i$ for some i, and $d(x, y) = \infty$ otherwise. The function d completely isolates each component X_i of X in the sense that a point $x \in X$ "feels" (or "is affected by") only points from the component to which it belongs. Let $P_1 = \{(-1, 0)\}$, $P_2 = \{(-1, 3)\}$, $P_3 = \{(1, 4)\}$. Then $R = (R_k)_{k=1}^{3}$ is a zone diagram in X,

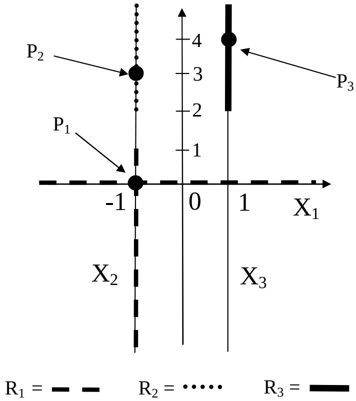

Fig. 3. Example 3.3

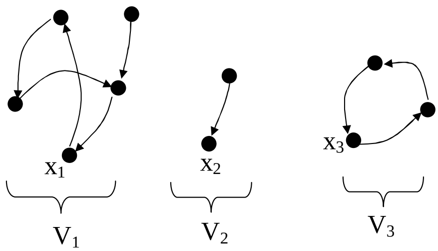

Fig. 4. Example 3.4

where $R_1 = (\mathbb{R} \times \{0\}) \cup (\{-1\} \times (-\infty, 1])$, $R_2 = \{-1\} \times [2, \infty)$, $R_3 = \{1\} \times [2, \infty)$. See Figure 3.

EXAMPLE 3.4. Let $G = (V, E)$ be a directed graph, and suppose $d : V^2 \to [0, \infty]$ assigns to $(x, y) \in V^2$ the length of the minimal finite directed path starting from x and ending in y (with $d(x, x) = 0$), including ∞ if there is no such finite path. Let G be the graph in Figure 4, and let $P_1 = \{x_1, x_2\}$ and $P_2 = \{x_2, x_3\}$. Then $R = (V_1 \cup \{x_2\}, V_2 \cup V_3)$ and $S = (V_1 \cup V_2, \{x_2\} \cup V_3)$ are zone diagrams in V.

4. A combinatorial interpretation. In this section (which is not needed for later sections) we describe a second interpretation of the concept of a zone diagram as a certain combinatorial game of one player.

The player is given a set of points X, a metric $d : X^2 \to [0, \infty)$ and a tuple $(P_k)_{k \in K}$ of nonempty subsets of X. For simplicity we assume that these sets are pairwise disjoint, and that X and K are finite. The set K is interpreted as a set of different colors, and each point $x \in X$ can be colored by one of the colors in K, or by an additional neutral color. In the initial position each set P_k is colored by the color k, and all other points are colored by the neutral color. Let $R_k := P_k$ and $Q := X \setminus \bigcup_{k \in K} P_k$.

Now the game starts. The player chooses a point $x \in Q$ and checks its position. By this we mean that the player checks whether x belongs to one of the sets $\mathrm{dom}(P_k, \bigcup_{j \neq k} R_j)$. If x does not belong to any $\mathrm{dom}(P_k, \bigcup_{j \neq k} R_j)$, then x is colored by the neutral color. If it does, then it may happen that x belongs to several different such sets, corresponding to a subset $K' \subseteq K$. In this case, if the current color of x is in K', then the player does not change this color. Otherwise, the player arbitrarily picks a color $k \in K'$, colors x with it, and adds x to R_k, i.e., the player defines $R_k := R_k \cup \{x\}$. If the

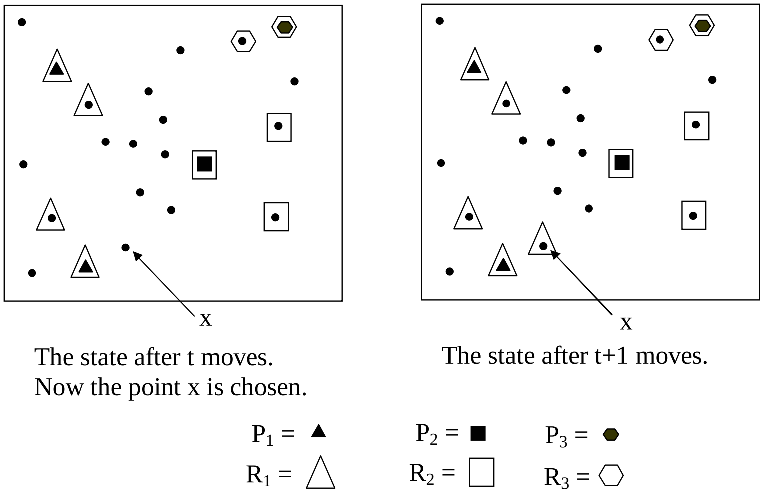

Fig. 5. Illustration of the game

color of x was changed, this means that before the change either x had the neutral color or it was in some R_j, but now it does not belong to those R_j any more. In this case the player removes x from them, i.e., the player defines $R_j := R_j \setminus \{x\}$ for all such j. The game continues in the same manner with the other points in Q; see Figure 5 in which (X, d) is a finite subset of the Euclidean plane and $|K| = 3$.

The goal is to reach a stable configuration $(R_k)_{k\in K}$, i.e., a configuration in which nothing has to be done: for each $x \in Q$, the player does not need to color it by a new color after its position is checked. A stable configuration is exactly a zone diagram, because it means that both $\mathrm{dom}(P_k, \bigcup_{j\neq k} R_j) \subseteq R_k$ and $R_k \subseteq \mathrm{dom}(P_k, \bigcup_{j\neq k} R_j)$ for each k. (If $x \in \mathrm{dom}(P_k, \bigcup_{j\neq k} R_j) \setminus R_k$ for a subset of indices $K' \subseteq K$, then its color is not in K', but it has to be changed to some $k \in K'$, and if $x \in R_k \setminus \mathrm{dom}(P_k, \bigcup_{j\neq k} R_j)$, then either $x \in \mathrm{dom}(P_i, \bigcup_{j\neq i} R_j)$ for some $i \in K' \subseteq K \setminus \{k\}$ and then its color has to be changed to some $i \in K'$, or $x \notin \mathrm{dom}(P_i, \bigcup_{j\neq i} R_j)$ for all i and then its color should be changed from k to the neutral color.)

It is definitely not clear in advance that a stable configuration can be obtained, because of the dynamical character of the game: when changing one R_k, adding/removing a point to/from it may affect the other R_i, since now $\mathrm{dom}(P_i, \bigcup_{j\neq i} R_j)$ becomes smaller/larger. Hence, even if a point x has already been colored, now it is possible that its color is not correct any more and it will have to be updated. Hence, passing over all of Q once will usually not suffice and it is not clear at all that even infinitely many passes will indeed suffice.

Theorem 5.6 ensures the existence of a stable configuration in the case where $|K| = 2$ and X, P_1, P_2 are arbitrary. Using the algorithm described in Section 7, such a configuration can be explicitly constructed. We note, however, that the algorithm does not show whether the player has a winning strategy, i.e., if and how the player can obtain a stable configuration according to the rules of the game.

5. Existence. Our main tool for establishing the existence of a zone diagram of order 2, and of a double zone diagram of any order, in any m-space (no matter how bizarre the space (X, d) or the sets P_k are) is the Knaster–Tarski fixed point theorem [4, 6] which will be stated below. We need two definitions before formulating it.

Definition 5.1. Let Y be a nonempty set and suppose $\leq$ is a partial order on Y. A mapping $g : Y \to Y$ is called *monotone* (or *isotone*, or *increasing*) if for any A and B in Y the condition $A \leq B$ implies $g(A) \leq g(B)$. The mapping is called *antimonotone* if $A \leq B$ implies $g(B) \leq g(A)$.

Definition 5.2. Let Y be a nonempty set and suppose $\leq$ is a partial order on Y. The pair $(Y, \leq)$ is called a *complete lattice* if any subset Z of Y has a least upper bound $\bigvee Z \in Y$ and a greatest lower bound $\bigwedge Z \in Y$.

Theorem 5.3 (Knaster–Tarski, [6, Theorem 1]). *Let $(Y, \leq)$ be a complete lattice and let $g : Y \to Y$ be monotone. Then the set F of fixed points of g is nonempty and $(F, \leq)$ is a complete lattice. In particular, g has two fixed points m and M with the property that $m \leq \mu \leq M$ for any $\mu \in F$. Moreover, $m = \bigwedge\{y \in Y : g(y) \leq y\}$ and $M = \bigvee\{y \in Y : y \leq g(y)\}$.*

Both of the sets $\{y \in Y : g(y) \leq y\}$ and $\{y \in Y : y \leq g(y)\}$ are nonempty because $\bigvee Y$ belongs to the first and $\bigwedge Y$ to the second. It is interesting to note that the proof of Theorem 5.3 is elementary and is not based on the axiom of choice. In order to apply Theorem 5.3 we need the following lemma. We note that its part (a) generalizes [1, Lemma 3(ii)].

Lemma 5.4. *Let (X, d) be an m-space. Partially order $\mathcal{P}^*(X)$ by inclusion and let $(P_k)_{k\in K}$ be a tuple of nonempty subsets in X.*

(a) *Given $P \in \mathcal{P}^*(X)$, the mapping $A \mapsto \mathrm{dom}(P, A)$ is antimonotone.*
(b) *The Dom mapping is antimonotone with respect to componentwise inclusion. The mapping Dom^2 is monotone.*
(c) *Given $P \in \mathcal{P}^*(X)$, the mapping $A \mapsto \mathrm{dom}(A, P)$ is monotone.*
(d) *$P \subseteq \mathrm{dom}(P, A) \subseteq X$ for all $A, P \in \mathcal{P}^*(X)$.*
(e) *Let $Y_k = \{A \in \mathcal{P}^*(X) : P_k \subseteq A\}$ and $Y = \bigtimes_{k\in K} Y_k$. Then Dom maps Y into Y, i.e., $\mathrm{Dom}(W) \in Y$ for any $W \in Y$.*

(f) *Let Y be as above and define on Y the natural partial order $\subseteq$ as follows*:

$$(A_k)_{k\in K} \subseteq (B_k)_{k\in K} \;\Leftrightarrow\; A_k \subseteq B_k \text{ for each } k \in K.$$

Then $(Y, \subseteq)$ is a complete lattice.

Proof. (a) If $A \subseteq B$, then $d(x,B) = \inf\{d(x,y) : y \in B\} \le \inf\{d(x,y) : y \in A\} = d(x,A)$, so $x \in \mathrm{dom}(P,B)$ implies $d(x,P) \le d(x,B) \le d(x,A)$, i.e., $x \in \mathrm{dom}(P,A)$.

(b) If $R \subseteq S$, then $\bigcup_{j\neq k} R_j \subseteq \bigcup_{j\neq k} S_j$ for all $k \in K$, so $\mathrm{dom}(P_k, \bigcup_{j\neq k} S_j) \subseteq \mathrm{dom}(P_k, \bigcup_{j\neq i} R_j)$ by part (a), and hence $\mathrm{Dom}(S) \subseteq \mathrm{Dom}(R)$. The second assertion follows from the first, since a composition of two antimonotone mappings is monotone.

(c) Suppose $A \subseteq B$. Then $d(x,B) \le d(x,A)$ for all $x \in X$. Hence, if $x \in \mathrm{dom}(A,P)$, then $d(x,B) \le d(x,A) \le d(x,P)$, i.e., $x \in \mathrm{dom}(B,P)$.

(d) The second inclusion is obvious and the first one follows immediately from $d(x,P) \le d(x,x) \le d(x,y)$ for all $x \in P$ and $y \in A$ by (2).

(e) The kth component of $\mathrm{Dom}(W)$ contains P_k by part (d).

(f) Let $J \neq \emptyset$ be a subset of Y. Then the componentwise intersection $\bigcap_{R\in J} R$ is in Y, and it is the greatest lower bound of J, and $\bigcup_{R\in J} R$ is the least upper bound of J. In addition, $\bigwedge \emptyset = (X)_{k\in K}$ and $\bigvee \emptyset = (P_k)_{k\in K}$. ∎

Theorem 5.5. *Let (X,d) be an m-space and suppose that $(P_k)_{k\in K}$ is a tuple of nonempty subsets of X. Then there is a double zone diagram of order $|K|$ in X with respect to $(P_k)_{k\in K}$. Furthermore, there are double zone diagrams m and M with the property that $m \subseteq \mu \subseteq M$ for any other double zone diagram μ.*

Proof. Because of Lemma 5.4, the conditions of Theorem 5.3 are satisfied with $(Y, \subseteq)$ and $g = \mathrm{Dom}^2$. Hence Dom^2 has two (not necessarily different) fixed points m and M which are double zone diagrams by definition, and they have the required property. ∎

Theorem 5.6. *Let (X,d) be an m-space and let $P_1, P_2 \in \mathcal{P}^*(X)$. Then there exists a zone diagram of order 2 in X with respect to (P_1, P_2).*

Proof. Let $S = (S_1, S_2)$ be a fixed point of Dom^2; its existence is ensured by Theorem 5.5. We have

$$(S_1,S_2) = \mathrm{Dom}^2(S_1,S_2) = (\mathrm{dom}(P_1, \mathrm{dom}(P_2,S_1)), \mathrm{dom}(P_2, \mathrm{dom}(P_1,S_2))).$$

Let $R_1 := S_1$, $R_2 := \mathrm{dom}(P_2, R_1)$. Then $R_1 = S_1 = \mathrm{dom}(P_1, \mathrm{dom}(P_2, S_1)) = \mathrm{dom}(P_1, R_2)$, and hence $R = (R_1, R_2)$ is a zone diagram in X. ∎

Remark 5.7. By a simple argument (repeating word by word the proof of [1, Lemma 3(i),(iii)] and using the fact that the distance between a point and a nonempty, closed and convex subset of a Hilbert space is attained), it can be shown that if p_k, $k \in K = \{1,2\}$, are two different points in a Hilbert space X and if (R_1, R_2) is a zone diagram in X, then the boundaries C_k of R_k represent the components of a trisector with respect to $P_k = \{p_k\}$, i.e., $C_k = \{x \in X : d(x, P_k) = d(x, C_j)\}$ for $k \neq j \in K$. This conclusion extends the existence part of [1, Theorem 1]. Actually, by different arguments this fact can be generalized to other spaces and to more general sets P_k. This issue will be treated in another paper which is in preparation.

6. Uniqueness. As the examples given in Sections 2 and 3 show, a zone diagram is not unique in general. In spite of this, it is possible to formulate several necessary and sufficient conditions for uniqueness. We first state and prove a general uniqueness theorem for antimonotone mappings.

THEOREM 6.1. *Let $(Y, \leq)$ be a partially ordered set, and let $T : Y \to Y$ be antimonotone. If T^2 has fixed points m and M with the property that $m \leq \mu \leq M$ for any other fixed point μ of T^2, then the following conditions are equivalent and each of them suffices for T to have exactly one fixed point.*

(a) *$m = M$.*
(b) *T^2 has a unique fixed point.*
(c) *Any fixed point of T^2 is a fixed point of T.*
(d) *The fixed point sets of T and T^2 coincide.*
(e) *Either m or M is a fixed point of T.*

If, in addition, $(Y, \leq)$ is a complete lattice, then all these conditions are equivalent to the following one:

(f) *$A \leq B$ for any $A, B \in Y$ which satisfy $T^2(B) \leq B$ and $A \leq T^2(A)$.*

Proof. (a)$\Rightarrow$(b): If $\mu = T^2\mu$, then by assumption $m \leq \mu \leq M = m$, i.e., $\mu = m = M$ is the unique fixed point of T^2.

(b)$\Rightarrow$(c): If $\mu = T^2\mu$, then $T\mu = T^3\mu = T^2(T\mu)$, i.e., $T\mu$ is a fixed point of T^2, so by uniqueness, $T\mu = \mu$.

(c)$\Rightarrow$(d): One inclusion holds by assumption, and the other holds in general, since if $\mu = T\mu$, then $\mu = T\mu = T^2\mu$, so μ is also a fixed point of T^2.

(d)$\Rightarrow$(e): Obvious.

(e)$\Rightarrow$(a): Suppose, for example, that $TM = M$. Since $m \leq M$, the antimonotonicity of T implies that $Tm \geq TM = M$. Since Tm is a fixed point of T^2, it follows that $Tm = M$. Hence $m = T^2m = TM = M$, that is, $m = M$.

(b),(d) $\Rightarrow$ "T has a unique fixed point": obvious.

(f)$\Rightarrow$(a): Since $M \leq T^2M$ and $T^2m \leq m$, it follows that $M \leq m$; but $m \leq M$ and hence there is equality.

(a)$\Rightarrow$(f): Suppose $A, B \in Y$ satisfy $T^2(B) \leq B$ and $A \leq T^2(A)$. Since the pair $(Y, \leq)$ is a complete lattice, the Knaster–Tarski fixed point theorem implies that $m = \bigwedge\{y \in Y : T^2(y) \le y\}$ and $M = \bigvee\{y \in Y : y \le T^2(y)\}$. Hence $A \le M = m \le B$. ∎

Since by Theorem 5.5 the Dom mapping satisfies the conditions of Theorem 6.1, we obtain several equivalent sufficient conditions for the uniqueness of zone diagrams. This again shows the importance of the concept of a double zone diagram. In fact, because of the special structure of the Dom mapping we can get a stronger result which will be formulated as a special corollary below. Before formulating this corollary, we note that, in fact, the implication (b) $\Rightarrow$ "T has a unique fixed point" is true without any assumption on T and X and without the partial order.

COROLLARY 6.2. *Let (X,d) be an m-space, $P = (P_k)_{k\in K}$ a tuple of nonempty sets in X, and let $T = \mathrm{Dom}$. Then each one of the six conditions in Theorem* 6.1 *suffices for X to have exactly one zone diagram with respect to P. If, in addition, $K = \{1,2\}$, then these conditions are also necessary.*

Proof. In view of Theorem 6.1 and the above discussion, only the last assertion remains to be proven. So suppose that $K = \{1,2\}$ and that T has a unique fixed point. We will show that part (c) of Theorem 6.1 holds. To this end, let $Z = (Z_1, Z_2)$ be any fixed point of T^2. Then

$$Z_1 = \mathrm{dom}(P_1, \mathrm{dom}(P_2, Z_1)) = g(Z_1),$$
$$Z_2 = \mathrm{dom}(P_2, \mathrm{dom}(P_1, Z_2)) = h(Z_2),$$

where $T_k(A) = \mathrm{dom}(P_k, A)$, $k = 1,2$, $g = T_1 \circ T_2$ and $h = T_2 \circ T_1$. Let $Y_1 = \{A \in \mathcal{P}^*(X) : P_1 \subseteq A\}$. By Lemma 5.4, g is a monotone mapping which maps Y_1 into itself and Y_1 is a complete lattice. Therefore Theorem 5.3 implies that g has a least and a greatest fixed points m_1 and M_1, respectively. By defining $m_2 := T_2 m_1$ and $M_2 := T_2 M_1$, we find that (m_1, m_2) and (M_1, M_2) are fixed points of T (since, for instance, $m_1 = T_1(T_2 m_1) = T_1 m_2$), so by uniqueness $m_1 = M_1$. This implies that g has a unique fixed point in Y_1, so $Z_1 = m_1 = M_1$. In the same way, h has a unique fixed point, and since both Z_2 and m_2 are fixed points of h, they must coincide, i.e., $Z = (Z_1, Z_2) = (m_1, m_2)$ is a fixed point of T. ∎

To illustrate an application of this corollary, let $X = [-3,3]$ with $d(x,y) = |x-y|$, and let $P = (P_1, P_2) = (\{-3\}, \{3\})$. For $t \in \mathbb{N}$ define $R^t = \mathrm{Dom}^t(P)$. A short calculation shows that $\mathrm{Dom}^2(X,X) = \mathrm{Dom}(P) = ([-3,0],[0,3])$. More generally, we obtain $\mathrm{Dom}^t(P) = ([-3,a_t],[b_t,3])$, where $a_0 = -3, b_0 = 3$ and $a_{t+1} = (b_t - 3)/2$, $b_{t+1} = (a_t + 3)/2$. By elementary considerations, $\{a_{2t}\}$ increases to -1, $\{b_{2t}\}$ decreases to 1, $\{a_{2t+1}\}$ decreases to -1 and $\{b_{2t+1}\}$ increases to 1. Hence $\bigcup_{t=0}^{\infty} \mathrm{Dom}^{2t}(P)$ increases to $([-3,-1),(1,3])$, and $\bigcap_{t=0}^{\infty} \mathrm{Dom}^{2t+1}(P)$ decreases to $([-3,-1],[1,3])$. Since by Lemma 5.4(e), a double zone diagram $R = (R_1, R_2)$ satisfies $P \subseteq R \subseteq X$, by repeated iterations we obtain $\bigcup_{t=0}^{\infty} \mathrm{Dom}^{2t}(P) \subseteq R \subseteq \bigcap_{t=0}^{\infty} \mathrm{Dom}^{2t}(X,X) = \bigcap_{t=0}^{\infty} \mathrm{Dom}^{2t+1}(P)$, and using the fact that R_1, R_2 are obviously closed we deduce that $R = ([-3,-1],[1,3])$. This indeed proves the uniqueness of the double zone diagram, and hence, by Corollary 6.2, of the zone diagram. Similar considerations show the uniqueness of the zone diagram in Example 2.6.

7. The finite case. In this short section we describe a practical way of constructing a double zone diagram of arbitrary order and a zone diagram of order 2 in the particular case where X is a finite set. In both cases the construction is by iteration, and we give some estimates on the number of iterations required.

THEOREM 7.1. *Let (X,d) be a finite m-space and let $(P_1,P_2) \in \mathcal{P}^*(X) \times \mathcal{P}^*(X)$. Let $g_1 = T_1 \circ T_2$ and $g_2 = T_2 \circ T_1$, where $T_k(A) = \mathrm{dom}(P_k, A)$. Then there are nonnegative integers $n_1, N_1 \leq |X| - |P_1|$ and $n_2, N_2 \leq |X| - |P_2|$ such that*

$$R = (g_1^{n_1}(P_1), T_2(g_1^{n_1}(P_1))), \qquad S = (g_1^{N_1}(X), T_2(g_1^{N_1}(X))),$$
$$Z = (T_1(g_2^{n_2}(P_2)), g_2^{n_2}(P_2)), \qquad W = (T_1(g_2^{N_2}(X)), g_2^{N_2}(X))$$

are all zone diagrams in X.

Proof. We will show this for the first case. The other cases are proved similarly. Since g_1 is monotone and $P_1 \subseteq g_1(P_1)$, it follows that the sequence $a_t = g_1^t(P_1)$, $t \in \mathbb{N} \cup \{0\}$, is increasing. Since $\mathcal{P}^*(X)$ is finite, the sequence becomes constant starting from some index n_1. One can give the following linear estimate instead of the "default" exponential one by observing that the sequence $b_t = |X| - |g_1^t(P_1)|$ is a decreasing sequence of nonnegative integers, and hence also becomes constant starting from $n_1 \leq b_0$. At this point $\{a_t\}_t$ becomes constant. Let $R_1 := a_{n_1}$. Then $R_1 = a_{n_1} = a_{n_1+1} = g_1(R_1)$, and hence R_1 is a fixed point of g_1, i.e., $R_1 = \mathrm{dom}(P_1, \mathrm{dom}(P_2, R_1))$. Letting $R_2 := T_2(R_1) = \mathrm{dom}(P_2, R_1)$, we find that (R_1, R_2) is a zone diagram in X. ■

THEOREM 7.2. *Let (X,d) be a finite m-space and let $P = (P_k)_{k\in K}$, K finite, be a given tuple of nonempty sets in X. Then there are nonnegative integers $n_1, N_1 \leq \sum_{k\in K}(|X| - |P_k|)$ such that*

$$R = \mathrm{Dom}^{2n_1}(P), \qquad S = \mathrm{Dom}^{2N_1}((X)_{k\in K})$$

are double zone diagrams in X.

Proof. Take Y from Lemma 5.4(e). Then $\mathrm{Dom}^2(P), \mathrm{Dom}^2((X)_{k\in K}) \in Y$, so $P \subseteq \mathrm{Dom}^2(P)$ and $\mathrm{Dom}^2((X)_{k\in K}) \subseteq (X)_{k\in K}$. Since the mapping Dom^2 is monotone, the sequence $\{\mathrm{Dom}^{2t}(P)\}_{t=1}^{\infty}$ is increasing and the sequence $\{\mathrm{Dom}^{2t}((X)_{k\in K})\}_{t=1}^{\infty}$ is decreasing. Since Y is finite, these sequences are eventually constant, and this constant must be a fixed point of Dom^2. As for the estimate, by defining the sequence $a_t=\sum_{k\in K}|(\mathrm{Dom}^{2t}(P))_k|$, we see that $\{a_t\}_{t=0}^{\infty}$ is increasing and can take integer values between $\sum_{k\in K}|P_k|$ and $\sum_{k\in K}|X|$, and when it becomes constant, so does $\{\mathrm{Dom}^{2t}(P)\}_{t=1}^{\infty}$. ■

The above algorithm may be applied to the (approximate) construction of zone diagrams in normed spaces, by considering a large finite set of points (grid) there, and constructing the zone diagram as above. However, it should be checked in what sense the resulting zone diagram is close (perhaps with respect to the Hausdorff distance) to the real one(s). This algorithm can also be used for finding a stable configuration of the combinatorial game described in Section 4.

8. Concluding remarks and open problems. The most interesting open problem is whether existence holds for a general cardinal n. Our method of proof for the case $n = 2$ cannot be carried over to the general case. This can be clearly illustrated already in the case $n = 3$. By Theorem 5.5, we know that a double zone diagram $R = (R_1, R_2, R_3)$ exists, and by definition it satisfies the following system of equations:

$$\begin{aligned}
R_1 &= \mathrm{dom}(P_1, \mathrm{dom}(P_2, R_3 \cup R_1) \cup \mathrm{dom}(P_3, R_1 \cup R_2)),\\
R_2 &= \mathrm{dom}(P_2, \mathrm{dom}(P_3, R_1 \cup R_2) \cup \mathrm{dom}(P_1, R_2 \cup R_3)),\\
R_3 &= \mathrm{dom}(P_3, \mathrm{dom}(P_1, R_2 \cup R_3) \cup \mathrm{dom}(P_2, R_3 \cup R_1)).
\end{aligned}$$

Unfortunately, in contrast with the case $n = 2$, the above system is coupled, and it is not clear how to obtain a zone diagram $Z = (Z_1, Z_2, Z_3)$ from R. We will, however, describe several possible ways to construct a zone diagram and point out the difficulties we have encountered.

The first way is to use the double zone diagram R in a similar way to the proof of Theorem 5.6, by defining $Z_1 := R_1$, $Z_2 := R_2$ and $Z_3 := \mathrm{dom}(P_3, Z_1 \cup Z_2)$. However, it is not clear why (Z_1, Z_2, Z_3) is a zone diagram because it is not clear from the above system why $Z_1 = \mathrm{dom}(P_1, Z_2 \cup Z_3)$ and $Z_2 = \mathrm{dom}(P_2, Z_3 \cup Z_1)$.

The second way again uses the double zone diagram R. We define $Z_1 := R_1$, plug R_1 in the second and third equations, and then define Z_2, Z_3 as solutions of the resulting system:

$$\begin{aligned}
Z_2 &= \mathrm{dom}(P_2, \mathrm{dom}(P_3, R_1 \cup Z_2) \cup \mathrm{dom}(P_1, Z_2 \cup Z_3)),\\
Z_3 &= \mathrm{dom}(P_3, \mathrm{dom}(P_1, Z_2 \cup Z_3) \cup \mathrm{dom}(P_2, Z_3 \cup R_1)).
\end{aligned}$$

This system does have a solution because the mapping on the right hand side is monotone. However, it is not clear why (Z_1, Z_2, Z_3) will be a zone diagram. A similar phenomenon occurs if one takes $Z_1 := R_1$, $Z_2 := R_2$ and defines Z_3 as (one of) the solution(s) of the resulting equation.

The third way is to look at the system defining the zone diagram:

$$R_1 = \mathrm{dom}(P_1, R_2\cup, R_3), \quad R_2 = \mathrm{dom}(P_2, R_3\cup R_1), \quad R_3 = \mathrm{dom}(P_3, R_1\cup R_2).$$

(Now R is no longer the above double zone diagram.) We eliminate one of the unknowns, say R_1, and arrive at the system

$$\begin{aligned} R_2 &= \mathrm{dom}(P_2, R_3 \cup \mathrm{dom}(P_1, R_2 \cup R_3)),\\ R_3 &= \mathrm{dom}(P_3, R_2 \cup \mathrm{dom}(P_1, R_2 \cup R_3)). \end{aligned}$$

If we could show that the above system has a solution (R_2, R_3), then by defining $R_1 := \mathrm{dom}(P_1, R_2 \cup R_3)$ we would indeed see that $R = (R_1, R_2, R_3)$ is a zone diagram in X. Unfortunately, the mapping on the right hand side is no longer monotone, so it is not clear why there exists a solution.

Therefore, in order to prove existence for general n one has to adopt other strategies, or to somehow modify the above ones. At the moment we have several partial results in specific cases (a class of normed spaces) which are in preparation, but the general case of m-spaces is open, and even in the case where X is a finite metric space the situation is not clear (but we feel that the combinatorial interpretation may help here). Anyway, we conjecture that existence holds in any m-space, at least for finite n. The infinite case may be problematic as the remark in Example 2.5 shows, but we conjecture that existence holds in any metric space for any cardinal if the sets P_k are "nice" and "far enough" from each other.

The question of uniqueness is also interesting. We conjecture that uniqueness holds at least in finite-dimensional normed spaces, under the assumption that the sets P_k are "nice" and "far enough" from each other. It is also of interest to determine whether the second assertion in Corollary 6.2 holds for general n. Theorem 6.1 shows that uniqueness arguments may be a strategy for proving existence. A related question is to what extent uniqueness is probable, at least for finite metric spaces embedded in normed spaces. For instance, Example 2.3 shows that if X is embedded in the interval $[-1, 1]$, then uniqueness holds with the exception of one case ($a = 0$), so it holds with probability 1. The cases where it does not hold may be analogous in some sense to eigenvalues of a linear operator, and it would be of interest to investigate them.

Finally, it would be interesting to find some applications of zone diagrams to other parts of mathematics and elsewhere. We think that zone diagrams do have this potential, say in optimization theory and computer science, and even in the natural sciences (see [2, p. 1183] or [1, p. 341]), including the general case of m-spaces. A reader familiar with Voronoi diagrams might have noticed that given a tuple $P = (P_k)_{k\in K}$, $\mathrm{Dom}(P)$ is none other than the Voronoi diagram induced by P. Hence Voronoi diagrams are related to zone diagrams, and since they have many applications, including the case of generalized distances (see [5] and the references therein), this may also be true for zone diagrams. The half-spaces $\mathrm{Eq}^+(x,y)$ and $\mathrm{Eq}^-(x,y)$ in the Hilbert ball with the hyperbolic metric, which appear in [3, pp. 112–115], provide another example of a dominance region, and thus may be related to zone diagrams. We also note that the combinatorial interpretation and the examples given at the beginning of the paper may also point to some applications.

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