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NEW CALDERÓN–ZYGMUND DECOMPOSITION FOR SOBOLEV FUNCTIONS

BY

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Abstract. We give a new Calderón–Zygmund decomposition for Sobolev spaces on a doubling Riemannian manifold. Our hypotheses are weaker than those of the already known decomposition which used classical Poincaré inequalities.

Contents

1. Introduction.

The purpose of this article is to weaken the assumptions of the Calderón–Zygmund decomposition for Sobolev functions. This well-known tool was first stated by P. Auscher in [2]. It exactly corresponds to the Calderón–Zygmund decomposition in the context of Sobolev spaces.

Let us briefly recall the ideas of such a decomposition. In [35], E. Stein stated this decomposition for Lebesgue spaces as follows. Let (X, d, μ) be a space of homogeneous type and $p \geq 1$. Given a function $f \in L^p(X)$, the decomposition gives a precise way of partitioning X into two subsets: one where f is essentially small (bounded in L^∞ norm); the other a countable collection of cubes where f is essentially large, but where some control of

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the function is obtained in L^1 norm. This leads to the associated Calderón–Zygmund decomposition of f, where f is written as the sum of "good" and "bad" functions, using the above subsets.

This decomposition is a basic tool in harmonic analysis and in the study of singular integrals. One of the applications is the following: an L^2-bounded Calderón–Zygmund operator is of weak type $(1,1)$ and so L^p-bounded for every $p \in (1,\infty)$.

In [2], P. Auscher extended these ideas to Sobolev spaces. His decomposition is the following:

THEOREM 1.1. *Let $n \geq 1$, $p \in [1,\infty)$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ be such that $\|\nabla f\|_{L^p} < \infty$. Let $\alpha > 0$. Then one can find a collection $(Q_i)_i$ of cubes and functions g and b_i such that*

$$f = g + \sum_i b_i$$

and the following properties hold:

$$\|\nabla g\|_{L^\infty} \leq C\alpha,$$

$$b_i \in W_0^{1,p}(Q_i), \qquad \int_{Q_i} |\nabla b_i|^p \leq C\alpha^p |Q_i|,$$

$$\sum_i |Q_i| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p, \qquad \sum_i \mathbf{1}_{Q_i} \leq N,$$

where C and N depend only on the dimension n and p.

The important point in this decomposition is that the functions b_i are supported in the corresponding balls, while the original Calderón–Zygmund decomposition applied to ∇f would not give this.

The proof relies on an appropriate use of the Poincaré inequality and was extended to doubling manifolds with the Poincaré inequality by P. Auscher and T. Coulhon in [6].

This decomposition is used in many works and it appears in various forms and extensions, for example in [6] (same proof on manifolds), [8] (on $\mathbb{R}^n$ but with a doubling weight), in B. Ben Ali's PhD thesis [16] and [5], [13] (the Sobolev space is modified to adapt to Schrödinger operators), in N. Badr's PhD thesis [9] and [10, 11] (used toward interpolation of Sobolev spaces on manifolds and measured metric spaces) and in [15] (Sobolev spaces on graphs).

The aim of this article is to extend the proof using another kind of "Poincaré inequality". This work can be integrated with several recent works, where the authors look for replacing mean value operators by others in the definition of Hardy spaces or maximal operators (see [21, 19, 26, 30, 33] etc.). Section 3 is mainly devoted to the proof of Calderón–Zygmund decompositions for Sobolev functions (as in Theorem 1.1) in an abstract framework of doubling Riemannian manifolds under assumptions involving a new kind of Poincaré inequality. Then we give an application to the real interpolation of Sobolev spaces $W^{1,p}$. In Section 4, we focus on a particular case (using the heat semigroup) corresponding to the so-called pseudo-Poincaré inequalities. These are weaker than the classical ones and ensure the Calderón–Zygmund decomposition for Sobolev functions. We give some applications using this improvement.

2. Preliminaries. Throughout this paper we will denote by $\mathbf{1}_E$ the characteristic function of a set E, and by E^c the complement of E. If X is a metric space, Lip will be the set of real Lipschitz functions on X, and Lip_0 the set of compactly supported real Lipschitz functions on X. For a ball Q in a metric space, λQ denotes the ball co-centered with Q and with radius λ times that of Q. Finally, C will be a constant that may change from an inequality to another, and we will write $u \lesssim v$ to mean that there exists a constant C such that $u \le Cv$, and $u \simeq v$ to mean that $u \lesssim v$ and $v \lesssim u$.

In this paper, M denotes a complete Riemannian manifold. We write μ for the Riemannian measure on M, ∇ for the Riemannian gradient, $|\cdot|$ for the length on the tangent space (omitting the subscript x for simplicity) and $\|\cdot\|_{L^p}$ for the norm on $L^p := L^p(M,\mu)$, $1 \le p \le \infty$. We denote by $Q(x,r)$ the open ball of center $x \in M$ and radius $r > 0$.

We will use the positive Laplace–Beltrami operator Δ defined by

$$\forall f,g \in C_0^\infty(M), \quad \langle \Delta f, g\rangle = \langle \nabla f, \nabla g\rangle.$$

We deal with the Sobolev spaces of order 1, $W^{1,p} := W^{1,p}(M)$, where the norm is defined by

$$\|f\|_{W^{1,p}(M)} := \|f\|_p + \big\| \, |\nabla f| \, \big\|_{L^p}.$$

2.1. The doubling property

Definition 2.1 (Doubling property). Let M be a Riemannian manifold. One says that M has the *doubling property* (D) if there exists a constant $C > 0$ such that for all $x \in M$ and $r > 0$ we have

$$(D) \qquad \mu(Q(x,2r)) \le C\mu(Q(x,r)).$$

Lemma 2.2. *Let M be a Riemannian manifold satisfying (D) and let $d = \log_2 C$. Then for all $x, y \in M$ and $\theta \ge 1$,*

$$(1) \qquad \mu(Q(x,\theta R)) \le C\theta^d \mu(Q(x,R)).$$

Observe that if M satisfies (D) then

$$\mathrm{diam}(M) < \infty \;\Leftrightarrow\; \mu(M) < \infty \quad \text{(see [1])}.$$

Therefore if M is a complete Riemannian manifold satisfying (D) then $\mu(M) = \infty$.

THEOREM 2.3 (Maximal theorem, [22]). *Let M be a Riemannian manifold satisfying (D). Denote by $\mathcal{M}$ the uncentered Hardy–Littlewood maximal function over open balls of M defined by*

$$\mathcal{M}f(x) := \sup_{\substack{Q \text{ ball} \\ x \in Q}} |f|_Q$$

where $f_E := \fint_E f\,d\mu := \mu(E)^{-1} \int_E f\,d\mu$. Then for every $p \in (1,\infty]$, $\mathcal{M}$ is L^p-bounded and moreover of weak type $(1,1)$ (¹). Consequently, for $s \in (0,\infty)$, the operator $\mathcal{M}_s$ defined by

$$\mathcal{M}_s f(x) := [\mathcal{M}(|f|^s)(x)]^{1/s}$$

is of weak type (s,s) and L^p-bounded for all $p \in (s,\infty]$.

2.2. Classical Poincaré inequality

DEFINITION 2.4 (Classical Poincaré inequality on M). We say that a complete Riemannian manifold M *admits the Poincaré inequality* (P_q) for some $q \in [1,\infty)$ if there exists a constant $C > 0$ such that, for every function $f \in \mathrm{Lip}_0(M)$ and every ball Q of M of radius $r > 0$, we have

$$(P_q) \qquad \Big(\fint_Q |f - f_Q|^q\,d\mu\Big)^{1/q} \le Cr\Big(\fint_Q |\nabla f|^q\,d\mu\Big)^{1/q}.$$

REMARK 2.5. By density of $C_0^\infty(M)$ in $\mathrm{Lip}_0(M)$, we can replace $\mathrm{Lip}_0(M)$ by $C_0^\infty(M)$.

Let us recall some known facts about Poincaré inequalities with varying q. It is known that (P_q) implies (P_p) when $p \ge q$ (see [29]). Thus, if the set of q such that (P_q) holds is not empty, then it is an interval unbounded on the right. A recent result of S. Keith and X. Zhong (see [31]) asserts that this interval is open in $[1,\infty)$:

THEOREM 2.6. *Let (X,d,μ) be a complete metric-measure space with μ doubling and admitting a Poincaré inequality (P_q) for some $1 < q < \infty$. Then there exists $\epsilon > 0$ such that (X,d,μ) admits (P_p) for every $p > q - \epsilon$.*

2.3. Estimates for the heat kernel. We recall the following off-diagonal decay of the heat semigroup and the link between this decay and the boundedness of the Riesz transform, the doubling property and the Poincaré inequality. We refer the reader to the work of P. Auscher, T. Coulhon, X. T. Duong and S. Hofmann [7] and [6] for more details about all

(¹) An operator T is of weak type (p,p) if there is $C > 0$ such that for any $\alpha > 0$, $\mu(\{x;\, |Tf(x)| > \alpha\}) \le (C/\alpha^p)\|f\|_p^p$.

these notions and how they are related. Let us consider the following two inequalities:

$$\|\nabla f\|_p \le C(\|\Delta^{1/2} f\|_p + \|f\|_p) \tag{nhR_p}$$

and

$$\|\Delta^{1/2} f\|_p + \|f\|_p \le C\|\nabla f\|_p. \tag{$nhRR_p$}$$

Theorem 2.7. *Let M be a complete doubling Riemannian manifold.*

- *The inequalities (nhR_2) and $(nhRR_2)$ are always satisfied.*
- ([23]) *Assume that the heat kernel p_t of the semigroup $e^{-t\Delta}$ satisfies the following pointwise estimate:*

$$p_t(x,x) \lesssim \frac{1}{\mu(B(x,t^{1/2}))}. \tag{DUE}$$

Then for all $p \in (1,2]$, (nhR_p) and $(nhRR_{p'})$ hold (2).

- ([28, Theorem 1.1]) *Under (D), (DUE) self-improves to the following Gaussian upper bound:*

$$p_t(x,y) \lesssim \frac{1}{\mu(B(y,t^{1/2}))} e^{-cd(x,y)^2/t}. \tag{UE}$$

Note that (UE) implies L^1-L^∞ off-diagonal decay for $(e^{-t\Delta})_{t>0}$.

- *Under (UE), the collection $(\sqrt{t}\nabla e^{-t\Delta})_{t>0}$ has L^2-L^2 off-diagonal decay.*
- *Under (DUE) and by the analyticity of the heat semigroup, the following pointwise upper bound for the kernel $t\frac{\partial}{\partial t}p_t$ of $\Delta e^{-t\Delta}$ holds (see* [25, Theorem 4] *and* [28, Corollary 3.3]*):*

$$t\left|\frac{\partial}{\partial t}p_t(x,y)\right| \lesssim \frac{1}{\mu(B(y,t^{1/2}))} e^{-cd(x,y)^2/t}. \tag{2}$$

Theorem 2.8 ([32, 34]). *The conjunction of (D) and the Poincaré inequality (P_2) on M is equivalent to the following Li–Yau inequality:*

$$\frac{1}{\mu(B(y,t^{1/2}))} e^{-c_1 d(x,y)^2/t} \lesssim p_t(x,y) \lesssim \frac{1}{\mu(B(y,t^{1/2}))} e^{-c_2 d(x,y)^2/t}, \tag{LY}$$

with some constants $c_1, c_2 > 0$.

Theorem 2.9 ([7]). *The L^p-boundedness of the Riesz transform $\nabla\Delta^{-1/2}$ implies*

$$\|\,|\nabla e^{-t\Delta}|\,\|_{L^p\to L^p} \lesssim 1/\sqrt{t}. \tag{G_p}$$

Moreover, under (P_2) and (G_{p_0}) with $p_0 > 2$, the collection $(\sqrt{t}\nabla e^{-t\Delta})_{t>0}$ has L^p-L^p off-diagonal decay for every $p \in [2, p_0)$.

(2) The assumptions in [23] are even weaker.

REMARK 2.10. All these results were proved in their homogeneous version, with homogeneous properties (R_p) and (RR_p). The proofs were essentially based on the well-known Calderón–Zygmund decomposition for Sobolev functions. This tool was then extended to non-homogeneous Sobolev spaces (see [10]). Thus by exactly the same proof, we can obtain non-homogeneous versions of all these results.

2.4. The K-method of real interpolation. We refer the reader to [17], [18] for details on this theory. Here we only recall the essentials to be used in what follows.

Let A_0, A_1 be two normed vector spaces embedded in a topological Hausdorff vector space V. For each $a \in A_0 + A_1$ and $t > 0$, we define the K-functional by

$$K(a, t, A_0, A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

For $0 < \theta < 1$, $1 \leq q \leq \infty$, we set

$$(A_0, A_1)_{\theta,q} = \left\{a \in A_0+A_1 : \|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(a, t, A_0, A_1))^q \frac{dt}{t}\right)^{1/q} < \infty\right\}.$$

This is an exact interpolation space of exponent θ between A_0 and A_1 (see [18, Chapter II]).

DEFINITION 2.11. Let f be a measurable function on a measure space (X, μ). The *decreasing rearrangement* of f is the function f^* defined for every $t \geq 0$ by

$$f^*(t) = \inf\{\lambda;\ \mu(\{x;\ |f(x)| > \lambda\}) \leq t\}.$$

The *maximal decreasing rearrangement* of f is the function f^{**} defined for every $t > 0$ by

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)\, ds.$$

PROPOSITION 2.12.

- $(f + g)^{**} \leq f^{**} + g^{**}$.
- $(\mathcal{M}f)^* \sim f^{**}$.
- $\mu(\{x;\ |f(x)| > f^*(t)\}) \leq t$.
- $\forall p \in (1, \infty]$, $\|f^{**}\|_p \sim \|f\|_p$.

We exactly know the functional K for Lebesgue spaces:

PROPOSITION 2.13. *For* $0 < p_0 < p_1 \leq \infty$ *we have*

$$K(f, t, L^{p_0}, L^{p_1}) \simeq \left(\int_0^{t^\alpha} [f^*(s)]^{p_0}\, ds\right)^{1/p_0} + t\left(\int_{t^\alpha}^\infty [f^*(s)]^{p_1}\, ds\right)^{1/p_1},$$

where $1/\alpha = 1/p_0 - 1/p_1$.

3. New Calderón–Zygmund decompositions for Sobolev functions. In the introduction, we recalled the main use of Calderón–Zygmund decompositions for Sobolev functions. In the previously cited works, this decomposition relies on Poincaré inequalities and some tricks with the mean value operators. Here we present similar arguments with abstract operators, requiring new "Poincaré inequalities". Then we give some applications to real interpolation of Sobolev spaces.

3.1. Decomposition using abstract "oscillation operators". Let $\mathcal{A} := (A_Q)_Q$ be a collection of operators (acting from $W^{1,p}$ to $W^{1,p}_{\mathrm{loc}}$) indexed by the balls of the manifold (A_Q can be thought of to be similar to the mean value operator over the ball Q).

Definition 3.1. We define a new maximal operator associated to this collection: for $1 \le s \le p \le \infty$ and all functions $f \in W^{1,p}$,

$$M_{\mathcal{A},s}(f)(x) := \sup_{Q;\, Q\ni x} \frac{1}{\mu(Q)^{1/s}} \left\| A_Q(f) \right\|_{W^{1,s}(Q)}.$$

Let us now define the assumptions that we need on the collection $\mathcal{A}$.

Definition 3.2. 1) We say that for $q \in [1,\infty]$ (3), the manifold M satisfies the *Poincaré inequality* (P_q) *relative to the collection* $\mathcal{A}$ if there is a constant C such that for every ball Q (of radius r_Q) and for all functions $f \in W^{1,p}$, $p \ge q$,

$$\Big(\fint_Q |f - A_Q(f)|^q \, d\mu \Big)^{1/q} \le C r_Q \sup_{s\ge 1} \Big(\fint_{sQ} (|f| + |\nabla f|)^q \, d\mu \Big)^{1/q}.$$

2) For $1 \le q \le r \le \infty$, we say that the collection $\mathcal{A}$ satisfies L^q-L^r *off-diagonal estimates* if

(a) there are constants $C' > 0$ and $N \in \mathbb{N}^*$ such that for all equivalent balls Q, Q' (i.e. $Q \subset Q' \subset NQ$) and all functions $f \in W^{1,p}$ with $p \ge q$, we have

$$\frac{1}{\mu(Q)^{1/r}} \| A_Q(f) - A_{Q'}(f) \|_{L^r(NQ)} \le C' r_Q \inf_{NQ} \mathcal{M}_q(|f| + |\nabla f|), \tag{3}$$

(b) for every ball Q,

$$\frac{1}{\mu(Q)^{1/r}} \| A_Q(f) \|_{W^{1,r}(Q)} \le C' \inf_{Q} \mathcal{M}_q(|f| + |\nabla f|). \tag{4}$$

Here is our main result:

Theorem 3.3. *Let M be a complete Riemannian manifold satisfying (D) and of infinite measure. Consider a collection $\mathcal{A} = (A_Q)_Q$ of operators defined on M. Assume that M satisfies the Poincaré inequality (P_q) relative*

(3) We take the supremum instead of the L^q average when $q = \infty$.

to the collection $\mathcal{A}$ for some $q \in [1,\infty)$, and that $\mathcal{A}$ satisfies L^q-L^r off-diagonal estimates for some $r \in (q,\infty]$. Let $q \le p < r$, $f \in W^{1,p}$ and $\alpha > 0$. Then one can find a collection (Q_i) of balls, functions $g \in W^{1,r}$ and $b_i \in W^{1,q}$ with the following properties:

$$f = g + \sum_i b_i, \tag{5}$$

$$\|g\|_{W^{1,r}} \lesssim \|f\|_{W^{1,p}}^{p/r} \alpha^{1-p/r}, \qquad \int_{\bigcup_i Q_i} (|g|^r + |\nabla g|^r)\, d\mu \lesssim \alpha^r \mu\Big(\bigcup_i Q_i\Big), \tag{6}$$

$$\operatorname{supp}(b_i) \subset Q_i, \qquad \|b_i\|_{W^{1,q}} \lesssim \alpha \mu(Q_i)^{1/q}, \tag{7}$$

$$\sum_i \mu(Q_i) \le C\alpha^{-p} \int (|f| + |\nabla f|)^p\, d\mu, \tag{8}$$

$$\sum_i \mathbf{1}_{Q_i} \le N. \tag{9}$$

Remark 3.4. From the assumed L^q-L^r off-diagonal estimates for $\mathcal{A}$ and Theorem 2.3, we deduce that the maximal operator $M_{\mathcal{A},q}$ is continuous from $W^{1,q}$ to $L^{q,\infty}$ and from $W^{1,p}$ to L^p for $p \in (q,r]$.

Proof. We follow the ideas of [10] where the result is proved for the particular case

$$A_Q(f) := \fint_Q f\, d\mu.$$

Let $f \in W^{1,p}$ and $\alpha > 0$. Consider the set

$$\Omega := \{x \in M;\ \mathcal{M}_q(|f| + |\nabla f|)(x) + M_{\mathcal{A},q}(f)(x) > \alpha\}.$$

We can assume that it is non-empty (otherwise the result is obvious with $g = f$). With this assumption, the different maximal operators are of weak type (p,p) so

$$\mu(\Omega) \le C\alpha^{-p}\Big(\int |f|^p\, d\mu + \int |\nabla f|^p\, d\mu\Big) < \infty. \tag{10}$$

In particular $\Omega \ne M$ as $\mu(M) = \infty$. Let F be the complement of Ω. Since Ω is an open set distinct from M, we can take a Whitney decomposition $(\underline{Q_i})$ of Ω. That is, the balls $\underline{Q_i}$ are pairwise disjoint and there exist two constants $C_2 > C_1 > 1$, depending only on the metric, such that

1. $\Omega = \bigcup_i Q_i$ with $Q_i = C_1\underline{Q_i}$ and the balls Q_i have the bounded overlap property;
2. $r_i = r(Q_i) = \frac{1}{2} d(x_i, F)$ and x_i is the center of Q_i;
3. each ball $C_2\underline{Q_i}$ intersects F ($C_2 = 4C_1$ works) and we define $\overline{Q_i} = 2C_2\underline{Q_i}$.

For $x \in \Omega$, denote $I_x = \{i;\ x \in Q_i\}$. By the bounded overlap property of the balls Q_i, we have $\sharp I_x \le N$ with an integer N. Fixing $j \in I_x$ and using the properties of the Q_i's, we easily see that $\frac{1}{3}r_i \le r_j \le 3r_i$ for all $i \in I_x$. In particular, $Q_i \subset 7Q_j$ for all $i \in I_x$.

Condition (9) is nothing but the bounded overlap property of the Q_i's and (8) follows from (9) and (10).

Observe that the doubling property and the fact that $\overline{Q_i} \cap F \neq \emptyset$ yield

$$\begin{aligned}(11)\quad \int_{Q_i} (|f|^q + |\nabla f|^q + |A_{\overline{Q_i}}(f)|^q + |\nabla A_{\overline{Q_i}}(f)|^q)\, d\mu \\ &\le \int_{\overline{Q_i}} (|f|^q + |\nabla f|^q + |A_{\overline{Q_i}}(f)|^q + |\nabla A_{\overline{Q_i}}(f)|^q)\, d\mu \\ &\le \inf_{\overline{Q_i}} [\mathcal{M}_q(|f| + |\nabla f|) + M_{\mathcal{A},q}(f)]^q \mu(\overline{Q_i}) \\ &\le \alpha^q \mu(\overline{Q_i}) \lesssim \alpha^q \mu(Q_i).\end{aligned}$$

We now define the functions b_i. Let $(\chi_i)_i$ be a partition of unity of Ω associated to the covering $(\underline{Q_i})$, such that for all i, χ_i is a Lipschitz function supported in Q_i with $\| \, |\nabla \chi_i| \, \|_\infty \lesssim r_i^{-1}$. Set

$$b_i := (f - A_{\overline{Q_i}}(f))\chi_i.$$

It is clear that $\operatorname{supp}(b_i) \subset Q_i$. Let us estimate $\|b_i\|_{W^{1,q}(Q_i)}$. We have

$$\int_{Q_i} |b_i|^q\, d\mu = \int_{Q_i} |f - A_{\overline{Q_i}}(f)|^q\, d\mu \lesssim \int_{Q_i} |f|^q\, d\mu + \int_{Q_i} |A_{\overline{Q_i}}(f)|^q\, d\mu \lesssim \alpha^q \mu(Q_i).$$

We applied (11) in the last inequality. Since

$$\nabla((f - A_{\overline{Q_i}}(f))\chi_i) = \chi_i(\nabla f - \nabla A_{\overline{Q_i}}(f)) + (f - A_{\overline{Q_i}}(f))\nabla \chi_i,$$

we have

$$\int_{Q_i} |\nabla b_i|^q\, d\mu \lesssim \int_{Q_i} |\nabla f - \nabla A_{\overline{Q_i}}(f)|^q\, d\mu + \frac{1}{r_i^q} \int_{Q_i} |f - A_{\overline{Q_i}} f|^q\, d\mu.$$

The first term is estimated as above for b_i. Thus

$$\int_{Q_i} |\nabla f - \nabla A_{\overline{Q_i}}(f)|^q\, d\mu \lesssim \alpha^q \mu(Q_i).$$

For the second term, the Poincaré inequality (P_q) (relative to the collection $\mathcal{A}$) shows that

$$\frac{1}{r_i^q} \int_{Q_i} |f - A_{\overline{Q_i}}(f)|^q\, d\mu \lesssim \sup_{s \ge 1} \frac{\mu(Q_i)}{\mu(sQ_i)} \int_{sQ_i} (|f|^q + |\nabla f|^q)\, d\mu \lesssim \alpha^q \mu(Q_i).$$

We used that for all $s \ge 1$, $s\overline{Q_i}$ meets F and (11) for sQ_i in place of Q_i. Thus (7) is proved.

Set $g = f - \sum_i b_i$. It remains to prove (6). Since the sum is locally finite on Ω, g is defined almost everywhere on M and $g = f$ on F. Observe that g is a locally integrable function on M. This follows from the fact that $b = f - g \in L^q$ here (for the homogeneous case, one can easily prove that $b \in L^1_{\mathrm{loc}}$). Note that $\sum_i \chi_i = \mathbf{1}_\Omega$ and $\sum_i \nabla\chi_i = \nabla \mathbf{1}_\Omega$. We then have

$$\begin{aligned} \nabla g &= \nabla f - \sum_i \nabla b_i \\ &= \nabla f - \Big(\sum_i \chi_i [\nabla f - \nabla A_{\overline{Q_i}}(f)]\Big) - \sum_i (f - A_{\overline{Q_i}}(f))\nabla\chi_i \\ &= \mathbf{1}_F(\nabla f) + \sum_i \chi_i \nabla A_{\overline{Q_i}}(f) - \sum_i A_{\overline{Q_i}}(f)\nabla\chi_i - f\nabla \mathbf{1}_\Omega. \end{aligned} \tag{12}$$

The definition of F and the Lebesgue differentiation theorem yield $\mathbf{1}_F(|f| + |\nabla f|) \leq \alpha$ μ-a.e. We deduce (with an interpolation inequality) that for $1/r = \theta/p$,

$$\begin{aligned} \|\mathbf{1}_F(|f| + |\nabla f|)\|_{L^r} &\lesssim \|\mathbf{1}_F(|f| + |\nabla f|)\|^\theta_{L^p} \|\mathbf{1}_F(|f| + |\nabla f|)\|^{1-\theta}_{L^\infty} \\ &\lesssim \|f\|^{p/r}_{W^{1,p}} \alpha^{1-p/r}. \end{aligned}$$

We control the second term on the right hand side of (12) using the off-diagonal decay of $\mathcal{A}$, (4). We recall that $\overline{Q_i} = 2C_2\underline{Q_i}$. We deduce that

$$\big\| |\nabla A_{\overline{Q_i}}(f)| \big\|_{L^r(Q_i)} \lesssim \mu(Q_i)^{1/r} \inf_{\overline{Q_i}} \mathcal{M}_q(|f| + |\nabla f|) \lesssim \alpha\mu(Q_i)^{1/r}. \tag{13}$$

The last inequality is due to the fact that $\overline{Q_i} \cap F \neq \emptyset$. Then the bounded overlap property of the covering $(Q_i)_i$ gives us

$$\begin{aligned} \Big\| \sum_i \chi_i |\nabla A_{\overline{Q_i}}(f)| \Big\|_{L^r} &\lesssim \Big(\sum_i \big\| |\nabla A_{\overline{Q_i}}(f)| \big\|^r_{L^r(Q_i)} \Big)^{1/r} \\ &\lesssim \Big(\alpha^r \sum_i \mu(Q_i) \Big)^{1/r} \lesssim \alpha\mu(\Omega)^{1/r}. \end{aligned}$$

We claim that a similar estimate holds for $h = \sum_i [A_{\overline{Q_i}}(f) - f]\nabla\chi_i$, that is, $\|h\|_{L^r} \lesssim \alpha(\mu(\Omega))^{1/r}$.

To prove this, we fix a point $x \in \Omega$ and let Q_j be a Whitney ball containing x. For all $i \in I_x$, as $r_{Q_i} \simeq r_{Q_j}$, we have

$$\|A_{\overline{Q_i}}(f) - A_{\overline{Q_j}}(f)\|_{L^r(Q_i)} \lesssim r_j \mu(Q_j)^{1/r} \alpha.$$

Indeed, since $Q_i \subset 7Q_j$, this is a direct consequence of the assumed off-diagonal decay and the fact that $10\overline{Q_i} \cap F \neq \emptyset$. Using $\sum_i \nabla\chi_i(x) = 0$, we deduce that

$$\begin{aligned}(14)\qquad \|h\|_{L^r(Q_j)} &\lesssim \sum_{i\in I_x} \|A_{\overline{Q_i}}(f) - A_{\overline{Q_j}}(f)\|_{L^r(Q_j)} r_j^{-1} \\ &\lesssim N\alpha\mu(Q_j)^{1/r} \lesssim \alpha\mu(Q_j)^{1/r}.\end{aligned}$$

Using again the bounded overlap property of the $(Q_i)_i$'s, it follows that

$$\|h\|_{L^r} \lesssim \alpha\mu(\Omega)^{1/r}.$$

Hence

$$\big\| \, |\nabla g| \, \big\|_{L^r(\Omega)} \lesssim \alpha\mu(\Omega)^{1/r}.$$

Then (8) and the L^r estimate of $|\nabla g|$ on F yield $\|\nabla g\|_{L^r} \lesssim \|f\|_{W^{1,p}}^{p/r} \alpha^{1-p/r}$.

Let us now estimate $\|g\|_{L^r}$. We have $g = f\mathbf{1}_F + \sum_i A_{\overline{Q_i}}(f)\chi_i$. Since $|f|\mathbf{1}_F \le \alpha$, we still need to estimate $\|\sum_i A_{\overline{Q_i}}(f)\chi_i\|_{L^r}$. Note that as in (13), we similarly have, for every i,

$$(15)\qquad \|A_{\overline{Q_i}}(f)\|_{L^r(Q_i)} \lesssim \alpha\mu(Q_i)^{1/r}.$$

As above, this last inequality yields (thanks to the bounded overlap property of the $(Q_i)_i$)

$$\|g\|_{L^r(\Omega)} \lesssim \alpha(\mu(\Omega))^{1/r}.$$

Finally, (8) and the L^r estimate of g on F yield $\|g\|_{L^r} \lesssim \|f\|_{W^{1,p}}^{p/r} \alpha^{1-p/r}$. Thus we proved that g belongs to $W^{1,r}$ with the desired boundedness. ∎

REMARK 3.5. Note that in this decomposition, $\nabla\mathbf{1}_\Omega$ corresponds to a singular distribution, supported in $\partial\Omega$. In the previous proof, we assumed that the distribution $\nabla\mathbf{1}_\Omega$ corresponds to a function, vanishing almost everywhere. The estimate (14) shows that h (considered as an L^1_{loc}-function) satisfies the right property. We also have to check that h can be considered as an L^1_{loc}-function. This is due to the fact that

$$\sum_{i,j} [A_{\overline{Q_j}}(f)\chi_j - f]\nabla\chi_i = 0$$

in the distributional sense. This equality shows that when we are close to $\mathrm{supp}(\sum \nabla\chi_i) = \partial\Omega$, the corresponding operator $A_{\overline{Q_j}}$ tends to the identity operator, due to the Poincaré inequality. We do not detail this technical problem and refer to [4].

REMARK 3.6. In the case where the operator A_Q is the mean value operator over the ball Q, the assumption "$M_{\mathcal{A},q} = \mathcal{M}_q$ is continuous from $W^{1,p}$ to $L^{p,\infty}$" is always satisfied. The Poincaré inequality (P_q) corresponds to the "classical one" (in fact it is weaker since the classical one only involves the $L^q(Q)$ norm of the gradient of the function). Moreover L^q-L^∞ off-diagonal estimates hold obviously. Thus, we recover the well-known Calderón–Zygmund decomposition in Sobolev spaces.

3.2. Application to real interpolation of Sobolev spaces. As described in [11], such a Calderón–Zygmund decomposition in Sobolev spaces is sufficient to obtain a real interpolation result for Sobolev spaces.

Theorem 3.7. *Let M be a complete Riemannian manifold of infinite measure satisfying (D) and admitting the Poincaré inequality (P_q) for some $q \in [1,\infty)$ relative to the collection $\mathcal{A}$. Assume that $\mathcal{A}$ satisfies L^q-L^r off-diagonal estimates for an $r \in (q,\infty]$. Then for $1 \le s \le p < r \le \infty$ with $p > q$, the space $W^{1,p}$ is a real interpolation space between $W^{1,s}$ and $W^{1,r}$. More precisely*

$$W^{1,p} = (W^{1,s}, W^{1,r})_{\theta,p}$$

where $\theta \in (0,1)$ is such that

$$\frac{1}{p} = \frac{1-\theta}{s} + \frac{\theta}{r} < \frac{1}{q}.$$

We do not detail the proof and refer the reader to [11] for the link between such a Calderón–Zygmund decomposition and interpolation results. We just briefly explain the main steps of the proof.

Proof. It is sufficient to prove that there exists $C > 0$ such that for every $f \in W^{1,p}$ and $t > 0$,

$$K(f,t,W^{1,s},W^{1,r}) \lesssim t^{\frac{r}{r-s}}[|f|^{q**} + |\nabla f|^{q**}]^{1/q}(t^{\frac{rs}{r-s}}) + t\Big[\int_{t^{\frac{rs}{r-s}}}^{\infty} (\mathcal{M}(|f|+|\nabla f|)^q)^{*r/q}(u)\,du\Big]^{1/r}. \tag{16}$$

We consider the previous Calderón–Zygmund decomposition for f with

$$\alpha = \alpha(t) = [\mathcal{M}_q(|f|+|\nabla f|) + M_{A,q}(f)]^{q*\frac{1}{q}}(t^{\frac{rs}{r-s}}).$$

We write $f = \sum_i b_i + g = b + g$ where $(b_i)_i$ and g satisfy the properties of Theorem 3.3. From the bounded overlap property of the B_i's, it follows that

$$\|b\|_{W^{1,s}}^s \le N\sum_i \|b_i\|_{W^{1,s}}^s \lesssim \alpha(t)^s \sum_i \mu(B_i) \lesssim \alpha(t)^s \mu(\Omega_t),$$

with $\Omega_t = \bigcup_i B_i$. For g, we have as in [11, proof of Theorem 4.2, p. 15]

$$\begin{aligned}\int_{F_t} (|g|^r + |\nabla g|^r)\,d\mu &= \int_{F_t} (|f|^r + |\nabla f|^r)\,d\mu \\ &\lesssim \int_{t^{\frac{rs}{r-s}}}^{\infty} (\mathcal{M}(|f|+|\nabla f|)^q)^{*\frac{r}{q}}(u)\,du \\ &\quad + t^{\frac{rs}{r-s}}(|f|^{q**} + |\nabla f|^{q**})^{\frac{r}{q}}(t^{\frac{rs}{r-s}})\end{aligned}$$

where F_t is the complement of Ω_t. For the Sobolev norm of g in Ω, we use the estimate of the Calderón–Zygmund decomposition. Moreover, since $(\mathcal{M}f)^* \sim f^{**}$ and $(f+g)^{**} \le f^{**} + g^{**}$ (cf. [17], [18]) and thanks to the L^q-L^r off-diagonal assumption on $\mathcal{A}$, we have

$$\alpha(t) \lesssim |f|^{q**\frac{1}{q}}(t^{\frac{rs}{r-s}}) + |\nabla f|^{q**\frac{1}{q}}(t^{\frac{rs}{r-s}}).$$

The choice of $\alpha(t)$ implies $\mu(\Omega_t) \le t^{\frac{rs}{r-s}}$ (cf. [17], [18]). Finally (16) follows from the fact that

$$K(f,t,W^{1,s},W^{1,r}) \le \|b\|_{W^{1,s}} + t\|g\|_{W^{1,r}}$$

and the good estimates of $\|b\|_{W^{1,s}}$ and $\|g\|_{W^{1,r}}$. ∎

Remark 3.8. As explained in [10, 11], to interpolate the non-homogeneous Sobolev spaces, it is sufficient to assume local doubling (D_{loc}) and local Poincaré inequality ($P_{q\mathrm{loc}}$) relative to $\mathcal{A}$. Under these assumptions, we restrict to balls Q of sufficiently small radius.

We now give a homogeneous version of all these results and then give applications.

3.3. Homogeneous version. We begin by recalling the definition of homogeneous Sobolev spaces on a manifold.

Let M be a C^∞ Riemannian manifold of dimension n. For $1 \le p \le \infty$, we define $\dot{E}^{1,p}$ to be the vector space of distributions φ with $|\nabla\varphi| \in L^p$, where $\nabla\varphi$ is the distributional gradient of φ. We equip $\dot{E}^{1,p}$ with the seminorm

$$\|\varphi\|_{\dot{E}^{1,p}} = \big\| |\nabla\varphi| \big\|_{L^p}.$$

The homogeneous Sobolev space $\dot{W}^{1,p}$ is then the quotient space $\dot{E}^{1,p}/\mathbb{R}$.

Remark 3.9. 1. For all $\varphi \in \dot{E}^{1,p}$, $\|\overline{\varphi}\|_{\dot{W}^{1,p}} = \big\| |\nabla\varphi| \big\|_{L^p}$, where $\overline{\varphi}$ denotes the class of φ.

2. The space $\dot{W}^{1,p}$ is a Banach space (see [27]).

We then have the homogeneous versions of all our results. We only state them, their proofs being the same as in the non-homogeneous case with a few modifications due to the homogeneous norm.

Let $\mathcal{A} := (A_Q)_Q$ be a collection of operators (acting from $\dot{W}^{1,p}$ to $\dot{W}^{1,p}_{\mathrm{loc}}$) indexed by the balls of the manifold. We define analogously a new homogeneous maximal operator associated to this collection: for $1 \le s \le p \le \infty$ and all functions $f \in \dot{W}^{1,p}$,

$$\dot{M}_{\mathcal{A},s}(f)(x) := \sup_{Q;\, Q \ni x} \frac{1}{\mu(Q)^{1/s}} \big\| |\nabla A_Q(f)| \big\|_{L^s(Q)}.$$

The assumptions that we need on the collection $\mathcal{A}$ are then the following:

Definition 3.10. 1) We say that for $q \in [1,\infty)$, the manifold M satisfies the *homogeneous Poincaré inequality* $(\dot{P}_q)$ *relative to the collection* $\mathcal{A}$ if there is a constant C such that for every ball Q (of radius r_Q) and for all functions $f \in \dot{W}^{1,p}$ with $p \geq q$,

$$(\dot{P}_q) \qquad \Big(\fint_Q |f - A_Q(f)|^q \, d\mu\Big)^{1/q} \leq C r_Q \sup_{s\geq 1}\Big(\fint_{sQ} |\nabla f|^q \, d\mu\Big)^{1/q}.$$

2) We say that the collection $\mathcal{A}$ satisfies L^q-L^r *homogeneous off-diagonal estimates* if

(a) there are constants $C' > 0$ and $N \in \mathbb{N}^*$ such that for all equivalent balls Q, Q' (i.e. $Q \subset Q' \subset NQ$, $N \in \mathbb{N}^*$) and all functions $f \in \dot{W}^{1,p}$ with $p \geq q$, we have

$$\frac{1}{\mu(Q)^{1/r}} \|A_Q(f) - A_{Q'}(f)\|_{L^r(NQ)} \leq C' r_Q \inf_{NQ} \mathcal{M}_q(|\nabla f|);$$

(b) for every ball Q,

$$\frac{1}{\mu(Q)^{1/r}} \big\| \, |\nabla A_Q(f)| \, \big\|_{L^r(Q)} \leq C' \inf_Q \mathcal{M}_q(|\nabla f|). \tag{17}$$

Then we get the homogeneous version of the Calderón–Zygmund decomposition:

Theorem 3.11. *Let M be a complete Riemannian manifold satisfying (D) and of infinite measure. Consider a collection $\mathcal{A} = (A_Q)_Q$ of operators defined on M. Assume that M satisfies the Poincaré inequality $(\dot{P}_q)$ relative to the collection $\mathcal{A}$ for some $q \in [1,\infty)$ and that $\mathcal{A}$ satisfies L^q-L^r homogeneous off-diagonal estimates for an $r \in (q,\infty]$. Let $f \in \dot{W}^{1,p}$ and $\alpha > 0$. Then one can find a collection (Q_i) of balls and functions $g \in \dot{W}^{1,r}$ and $b_i \in \dot{W}^{1,q}$ with the following properties:*

$$f = g + \sum_i b_i, \tag{18}$$

$$\|g\|_{\dot{W}^{1,r}} \lesssim \|f\|_{\dot{W}^{1,p}}^{p/r} \alpha^{1-p/r}, \qquad \int_{\bigcup_i Q_i} |\nabla g|^r \, d\mu \lesssim \alpha^r \mu\Big(\bigcup_i Q_i\Big), \tag{19}$$

$$\operatorname{supp}(b_i) \subset Q_i, \qquad \|b_i\|_{\dot{W}^{1,q}} \lesssim \alpha \mu(Q_i)^{1/q}, \tag{20}$$

$$\sum_i \mu(Q_i) \leq C\alpha^{-p} \int |\nabla f|^p \, d\mu, \tag{21}$$

$$\sum_i \mathbf{1}_{Q_i} \leq N. \tag{22}$$

This decomposition will give us the following homogeneous interpolation result:

THEOREM 3.12. *Let M be a complete Riemannian manifold of infinite measure satisfying (D) and admitting the Poincaré inequality $(\dot{P}_q)$ for some $q \in [1,\infty)$ relative to the collection $\mathcal{A}$. Assume that $\mathcal{A}$ satisfies L^q-L^r homogeneous off-diagonal estimates for an $r \in (q,\infty]$. Then for $1 \leq s \leq p < r \leq \infty$ with $p > q$, the space $\dot{W}^{1,p}$ is a real interpolation space between $\dot{W}^{1,s}$ and $\dot{W}^{1,r}$. More precisely*

$$\dot{W}^{1,p} = (\dot{W}^{1,s}, \dot{W}^{1,r})_{\theta,p}$$

where $\theta \in (0,1)$ is such that

$$\frac{1}{p} = \frac{1-\theta}{s} + \frac{\theta}{r} < \frac{1}{q}.$$

4. Pseudo-Poincaré inequalities and applications

4.1. The particular case of "pseudo-Poincaré inequalities". From [2, 3], we know that under (D), the Poincaré inequality (P_q) guarantees the assumptions of Theorem 3.3 when A_Q is the mean value operator over the ball Q. This permits proving a Calderón–Zygmund decomposition for Sobolev functions.

The aim of this subsection is to show, using a particular choice of operators A_Q, that our assumptions are weaker than the classical Poincaré inequality used in the already known decomposition.

Let Δ be the positive Laplace–Beltrami operator and set $A_Q := e^{-r_Q^2\Delta}$ for each ball Q of radius r_Q. In all this section, we work with these operators. In order to obtain a Calderón–Zygmund decomposition as in Theorem 3.3, we need to put some assumptions on $(A_Q)_Q$ as those in Section 3.

According to this choice of operators, we define "pseudo-Poincaré inequalities".

DEFINITION 4.1 (Pseudo-Poincaré inequality on M). We say that a complete Riemannian manifold M admits *the pseudo-Poincaré inequality* $(\widetilde{P}_q)$ for some $q \in [1,\infty)$ if there exists a constant $C > 0$ such that, for every function $f \in C_0^\infty$ and every ball Q of M of radius $r > 0$, we have

$$(\tilde{P}_q) \qquad \Big(\fint_Q |f - e^{-r^2\Delta} f|^q \, d\mu\Big)^{1/q} \leq Cr \sup_{s\geq 1} \Big(\fint_{sQ} |\nabla f|^q \, d\mu\Big)^{1/q}.$$

Pseudo-Poincaré inequalities correspond to what we called the Poincaré inequality relative to this collection $\mathcal{A}$ (the homogeneous version; we can also consider the non-homogeneous one).

We begin by showing that the pseudo-Poincaré inequalities are implied by the classical Poincaré inequalities. We denote

$$(q_0) \qquad q_0 := \inf\{q \in [1,\infty);\ (P_q) \text{ holds}\}.$$

PROPOSITION 4.2. *Let M be a complete manifold satisfying (D) and admitting the Poincaré inequality (P_q) for some $1 \leq q < \infty$.*

1. *If $q_0 < 2$ then the pseudo-Poincaré inequality $(\widetilde{P}_q)$ holds.*
2. *If $q_0 \geq 2$, we moreover assume (DUE). Then $(\widetilde{P}_q)$ also holds.*

Before proving this proposition, we give the following covering lemma.

LEMMA 4.3. *Let M be a complete manifold satisfying (D). Let Q a ball of radius r_Q. Then there exists a bounded covering $(Q_j)_j$ of Q with balls of radius $t^{1/2}$ for $0 < t \leq r_Q^2$. Moreover, for $s \geq 1$, the collection $(sQ_j)_j$ is an s-covering of sQ, that is,*

$$\sup_{x \in sQ} \sharp\{j;\, x \in sQ_j\} \lesssim s^d,$$

where d is the homogeneous dimension of the manifold.

Proof. We choose a maximal collection $(Q(x_j, t^{1/2}/3))_j$ of disjoint balls in Q. Then we set $Q_j = Q(x_j, t^{1/2})$, which is a covering of Q.

Fix $x \in sQ$ and denote $J_x := \{j;\, x \in sQ_j\}$. Pick $j_0 \in J_x$ (if $J_x \neq \emptyset$, there is nothing to prove). By (D), we have

$$\begin{aligned}(\sharp J_x)\mu(sQ_{j_0}) &\lesssim (\sharp J_x) s^d \mu\Big(\frac{1}{3}Q_{j_0}\Big) \lesssim s^d \sum_{j \in J_x} \mu\Big(\frac{1}{3}Q_j\Big) \\ &\lesssim s^d \mu\Big(\bigcup_{j \in J_x} \frac{1}{3}Q_j\Big) \lesssim s^d \mu(Q(x, 2st^{1/2})) \lesssim s^d \mu(sQ_{j_0}),\end{aligned}$$

where we used the fact that the balls $\frac{1}{3}Q_j$ are disjoint and all have equivalent measure when $j \in J_x$. ∎

Proof of Proposition 4.2. Consider a ball Q of radius $r > 0$. We deal with the semigroup and write the oscillation as follows:

$$f - e^{-r^2 \Delta} f = -\int_0^{r^2} \frac{d}{dt} e^{-t\Delta} f \, dt = \int_0^{r^2} \Delta e^{-t\Delta} f \, dt.$$

Now we apply arguments used in [7, Lemma 3.2]. Using the completeness of the manifold, we have

$$\begin{aligned}\Big(\frac{1}{\mu(Q)} \int_Q \Big| \int_0^{r^2} \Delta e^{-t\Delta} f \, dt \Big|^q d\mu\Big)^{1/q} &\lesssim \int_0^{r^2} \Big(\frac{1}{\mu(Q)} \int_Q |\Delta e^{-t\Delta} f|^q \, d\mu\Big)^{1/q} dt \\ &\lesssim \int_0^{r^2} \Big(\frac{1}{\mu(Q)} \sum_j \int_{Q_j} |\Delta e^{-t\Delta}(f - f_{Q_j})|^q \, d\mu\Big)^{1/q} dt,\end{aligned}$$

where $(Q_j)_j$ is a bounded covering of Q with balls of radius $t^{1/2}$ as in Lemma 4.3. Fix $t \in (0, r^2)$ and set $C_k(Q_j) := 2^{k+1}Q_j \setminus 2^k Q_j$ for $k \geq 1$ and

$C_0(Q_j) = 2Q_j$. Then, arguing as in Lemma 3.2 of [7],

$$
\begin{aligned}
S &:= \sum_j \int_{Q_j} |\Delta e^{-t\Delta}(f - f_{Q_j})|^q \, d\mu \\
&\lesssim \sum_j \int_{Q_j} t^{-q} \bigg| \int_M \frac{e^{-cd(x,y)^2/t}}{\mu(Q(y,\sqrt{t}))} (f(y) - f_{Q_j}) \, d\mu(y) \bigg|^q \, d\mu(x) \\
&\lesssim \sum_{j,k;k\ge 0} \int_{Q_j} t^{-q} \mu(2^{k+1}Q_j)^{q-1} \int_{C_k(Q_j)} \frac{e^{-cqd(x,y)^2/t}}{\mu(Q(y,\sqrt{t}))^q} |f(y) - f_{Q_j}|^q \, d\mu(y) \, d\mu(x).
\end{aligned}
$$

Hence

$$
\begin{aligned}
S &\lesssim \sum_{j,k;k\ge 1} t^{-q} \mu(2^{k+1}Q_j)^{q-1} \\
&\quad \times \int_{C_k(Q_j)} \Big(\int_{\{x;\, d(x,y) \ge 2^{k-1}\sqrt{t}\}} e^{-cqd(x,y)^2/t} \, d\mu(x) \Big) \frac{|f(y) - f_{Q_j}|^q}{\mu(Q(y,\sqrt{t}))^q} \, d\mu(y) \\
&\quad + \sum_j t^{-q} \frac{1}{\mu(Q_j)^q} \mu(2Q_j)^{q-1} \int_{2Q_j} \Big(\int_{Q_j} d\mu(x) \Big) |f(y) - f_{Q_j}|^q \, d\mu(y) \\
&\lesssim \sum_j t^{-q} \sum_{k\ge 1} e^{-cq4^k} 2^{kdq} \int_{C_k(Q_j)} |f(y) - f_{Q_j}|^q \, d\mu(y) \\
&\quad + \sum_j t^{-q} \int_{2Q_j} |f(y) - f_{Q_j}|^q \, d\mu(y) \\
&\lesssim \sum_j t^{-q} \sum_{k\ge 1} e^{-cq4^k} 2^{kdq} \int_{2^{k+1}Q_j} |f(y) - f_{2^{k+1}Q_j}|^q \, d\mu(y) \\
&\quad + \sum_{l=1}^{k+1} \frac{\mu(2^{k+1}Q_j)}{\mu(2^l Q_j)} |f_{2^l Q_j} - f_{2^{l-1}Q_j}| + \sum_j t^{-q} \int_{2Q_j} |f(y) - f_{Q_j}|^q \, d\mu(y) \\
&\lesssim \sum_j t^{-q} \sum_{k\ge 1} e^{-cq4^k} 2^{Mk} t^{q/2} \sum_{l=1}^{k+1} \int_{2^l Q_j} |\nabla f|^q \, d\mu + \sum_j t^{-q} t^{q/2} \int_{2Q_j} |\nabla f|^q \, d\mu.
\end{aligned}
$$

We used (2), (P_q), and the fact that $\mu(Q(y,\sqrt{t})) \sim \mu(Q_j)$ for $y \in 2Q_j$, and

$$
\frac{1}{\mu(Q(y,\sqrt{t}))} \le C \frac{2^{kd}}{\mu(2^{k+1}Q_j)} \qquad \text{for } y \in C_k(Q_j),\ k \ge 1.
$$

We also used the fact that for $s, t > 0$,

$$
\int_{\{x;\, d(x,y) \ge \sqrt{t}\}} e^{-cd(x,y)^2/s} \, d\mu(x) \le C e^{-ct/s} \mu(Q(y,\sqrt{s}))
$$

thanks to (D) (see Lemma 2.1 in [24]).

Since $(2^lQ_j)_j$ is a 2^l-bounded covering of 2^lQ, we deduce that

$$\sum_j \int_{2^lQ_j} |\nabla f|^q \, d\mu \lesssim 2^{ld} \int_{2^lQ} |\nabla f|^q \, d\mu \leq 4^{ld}\mu(Q) \sup_{s\geq 1} \fint_{sQ} |\nabla f|^q \, d\mu,$$

where d is the homogeneous dimension of the doubling manifold. Thus, it follows that

$$\biggl(\frac{1}{\mu(Q)} \int_Q \biggl|\int_0^{r^2} \Delta e^{-t\Delta}(f)\, dt\biggr|^q d\mu\biggr)^{1/q} \lesssim \biggl[\int_0^{r^2} t^{-1/2}\, dt\biggr] \sup_{s\geq 1}\biggl(\fint_{sQ} |\nabla f|^q \, d\mu\biggr)^{1/q},$$

which ends the proof. ∎

Before we prove off-diagonal estimates under the classical Poincaré inequality, let us recall the following result:

PROPOSITION 4.4 ([6]). *Let M be a complete Riemannian manifold satisfying (D) and (P_2). Then there exists $p_0 > 2$ such that the Riesz transform $\mathcal{R} := \nabla\Delta^{-1/2}$ is L^p-bounded for $1 < p < p_0$.*

We now let

$$(p_0) \qquad p_0 := \sup\{p \in (2,\infty);\ \nabla\Delta^{-1/2} \text{ is } L^p\text{-bounded}\},$$

$$(s_0) \qquad s_0 := \sup\{s \in (1,\infty];\ (G_s) \text{ holds}\}.$$

REMARK 4.5. Note that the doubling property (D) and (DUE) imply, for $p \in (1,2]$, the L^p-boundedness of $\nabla\Delta^{-1/2}$, which implies (G_p) (see Subsection 2.3) and that $s_0 \geq p_0 > 2$.

For the second off-diagonal condition (4), we obtain:

PROPOSITION 4.6. *Let M be a complete manifold. Assume that M satisfies (D) and admits the classical Poincaré inequality (P_q) for some $q \in [1,\infty)$ as in Definition 2.4. Consider the estimate*

$$M_{\mathcal{A},r}(f) \lesssim \mathcal{M}_q(|f| + |\nabla f|). \tag{23}$$

1. *If $q_0 < 2$, then (23) holds for all $r \in (q, s_0)$.*
2. *If $q_0 \geq 2$, assume moreover (DUE) and that $s_0 > q$. Then (23) holds for all $r \in (q, s_0)$.*

Consequently, (4) holds for all $r \in (q, s_0)$.

Proof. It is sufficient to prove the following inequalities:

$$\biggl(\fint_Q |e^{-r^2\Delta} f|^r \, d\mu\biggr)^{1/r} \leq C\mathcal{M}_q(|f|)(x) \tag{24}$$

and

$$\biggl(\fint_Q |\nabla e^{-r^2\Delta} f|^r \, d\mu\biggr)^{1/r} \leq C\mathcal{M}_q(|\nabla f|)(x) \tag{25}$$

for every $x \in M$ and every ball Q containing x. We do not detail the proof as it uses an argument analogous to [7, Subsection 3.1, Lemma 3.2 and the end of that subsection]. For example, (25) is essentially inequality (3.12) in Section 3 of [7] where $q_0 = 2$. We just mention that for (24), we use the L^r contractivity of the heat semigroup, (D) and (DUE). For (25), we moreover need the following L^r Gaffney estimates for $\nabla e^{-t\Delta}$ with $r \in (q_0, s_0)$. We say that $(\nabla e^{-t\Delta})_{t>0}$ satisfies the L^p *Gaffney estimate* if there exist $C, \alpha > 0$ such that for all $t > 0$, E, F closed subsets of M and f supported in E,

$$(Ga_p) \qquad \big\| \sqrt{t} |\nabla e^{-t\Delta} f| \big\|_{L^p(F)} \leq C e^{-\alpha d(E,F)^2/t} \|f\|_{L^p(E)}.$$

In the case where $q_0 \geq 2$, interpolating the already known (Ga_2) with (G_s) for every $2 < s < s_0$, we get (Ga_p) for $2 < p < s_0$. When $q_0 < 2$, since in this case (G_s) holds for all $1 < s < 2$ and $2 < s < s_0$, interpolating again (G_s) and (Ga_2), we obtain (Ga_p) for all $1 < p < s_0$. ∎

It remains to check (3).

Proposition 4.7. *Let M be a complete manifold satisfying (D) and admitting the classical Poincaré inequality (P_q) for some $1 \leq q < \infty$. Then*

1. *If $q_0 < 2$, for $r > q$, the collection $\mathcal{A}$ satisfies the L^q-L^r off-diagonal estimates* (3).
2. *If $q_0 \geq 2$, the same result holds under the additional assumption (DUE).*

Proof. Take two equivalent balls Q_0, Q_1, say $Q_0 \subset Q_1 \subset 10Q_0$ with radius r_0 (resp. r_1). We have chosen a numerical factor 10 just for convenience. We have to prove that

$$(26) \qquad \bigg(\frac{1}{\mu(Q_0)} \int_{10Q_0} |e^{-r_0^2\Delta} f - e^{-r_1^2\Delta} f|^r \, d\mu \bigg)^{1/r} \lesssim r_0 \inf_{10Q_0} \mathcal{M}_q(|f| + |\nabla f|).$$

This is a consequence of

$$(27) \qquad \bigg(\frac{1}{\mu(Q_0)} \int_{10Q_0} |e^{-r_0^2\Delta} f - e^{-400 r_0^2\Delta} f|^r \, d\mu \bigg)^{1/r} \lesssim r_0 \inf_{10Q_0} \mathcal{M}_q(|f| + |\nabla f|)$$

and

$$(28) \qquad \bigg(\frac{1}{\mu(Q_0)} \int_{10Q_0} |e^{-400 r_0^2\Delta} f - e^{-r_1^2\Delta} f|^r \, d\mu \bigg)^{1/r} \lesssim r_0 \inf_{10Q_0} \mathcal{M}_q(|f| + |\nabla f|).$$

We use the fact that

$$e^{-r_0^2\Delta} f - e^{-400 r_0^2\Delta} f = e^{-r_0^2\Delta}[1 - e^{-399 r_0^2\Delta}](f)$$

and

$$e^{-400 r_0^2\Delta} f - e^{-r_1^2\Delta} f = -e^{-r_1^2\Delta}[1 - e^{-((20r_0)^2 - r_1^2)\Delta}](f).$$

We only deal with (27); (28) is handled similarly. From (D) and (DUE), we know that (UE) holds and so we have very fast L^1-L^∞ decay for the semigroup, which permits us to gain integrability from L^q to L^r. It follows that

$$\left(\frac{1}{\mu(Q_0)}\int_{10Q_0}|e^{-r_0^2\Delta}f-e^{-400r_0^2\Delta}f|^r\,d\mu\right)^{1/r} \lesssim \sum_{j\geq 0}e^{-\gamma 4^j}\left(\frac{1}{\mu(Q_0)}\int_{C_j(Q_0)}|f-e^{-399r_0^2\Delta}f|^q\,d\mu\right)^{1/q},$$

where the dyadic coronas $C_j(Q_0)$ appear (see again [7, Lemma 3.2 and the end of Subsection 3.1]). Then we use (D) and (P_q). For each j, we choose a bounded covering $(Q_i^j)_i$ of $2^{j+1}Q_0$ with balls of radius $\sqrt{399}r_0$ and obtain

$$\begin{aligned}\frac{1}{\mu(Q_0)}\int_{C_j(Q_0)}|f-e^{-399r_0^2\Delta}f|^q\,d\mu &\lesssim \frac{1}{\mu(Q_0)}\sum_i\int_{Q_i^j}|f-e^{-399r_0^2\Delta}f|^q\,d\mu\\
&\lesssim \frac{1}{\mu(Q_0)}\sum_i\int_{Q_i^j}|f-e^{-399r_0^2\Delta}f|^q\,d\mu\\
&\lesssim \frac{1}{\mu(Q_0)}\sum_i r_0^q\mu(Q_i^j)\sup_{s\geq 1}\fint_{sQ_i^j}|\nabla f|^q\,d\mu\\
&\lesssim \frac{1}{\mu(Q_0)}\sum_i r_0^q\mu(Q_i^j)\sup_{s\geq 1}2^{dj}\Big(\fint_{s2^{j+1}Q_0}|\nabla f|^q\,d\mu\Big)\\
&\lesssim \frac{1}{\mu(Q_0)}\sum_i r_0^q\mu(Q_i^j)2^{dj}\inf_{Q_0}\mathcal{M}(|\nabla f|^q)\\
&\lesssim r_0^q2^{dj}\frac{\mu(2^{j+1}Q_0)}{\mu(Q_0)}(\inf_{Q_0}\mathcal{M}_q(|\nabla f|))^q\\
&\lesssim r_0^q2^{2dj}(\inf_{Q_0}\mathcal{M}_q(|\nabla f|))^q.\end{aligned}$$

We applied (P_q) in the third inequality. In the fourth inequality, we used that $sQ_i^j\subset 2^{j+1}sQ_0$ and thanks to (D), $\mu(2^{j+1}sQ_0)\lesssim\mu(sQ_i^j)2^{jd}$. Then we applied the bounded overlap property in the sixth inequality.

Summing over j, we obtain the desired inequality (27). Similarly we prove (28), which completes the proof of (26). ∎

We get the following corollary:

Corollary 4.8. *Assume that M is complete, satisfies (D) and admits the classical Poincaré inequality (P_q) for some $q\in[1,\infty)$. If $q_0\geq 2$, we moreover assume (DUE) and $s_0>q$. Then the assumptions of Theorem* 3.3 *and* 3.7 *hold. We have the pseudo-Poincaré inequality $(\widetilde{P}_q)$ and $\mathcal{A}$ satisfies*

L^q-L^r off-diagonal estimates for $r \in (q, s_0)$.

Conclusion. When $q < 2$, the assumptions of Theorem 3.3 (for this particular choice of $\mathcal{A}$) are weaker than the Poincaré inequality and are sufficient to get the Calderón–Zygmund decomposition.

We also have the homogeneous version:

COROLLARY 4.9. *Assume that M is complete, satisfies (D) and admits the classical Poincaré inequality (P_q) for some $1 \le q < \infty$. If $q_0 \ge 2$, we moreover assume (DUE). Let $\mathcal{A} := (A_Q)_Q$ with $A_Q := e^{-r_Q^2 \Delta}$. Then the assumptions of Theorems 3.11 and 3.12 hold. We have the pseudo-Poincaré inequality $(\widetilde{P}_q)$ and $\mathcal{A}$ satisfies homogeneous L^q-L^r off-diagonal estimates for $r \in (q, s_0)$.*

4.2. Application to reverse Riesz transform inequalities. We refer the reader to [6, 7] for the study of the so-called (RR_p) inequalities:

$$(RR_p) \qquad \|\Delta^{1/2} f\|_{L^p} \lesssim \big\||\nabla f|\big\|_{L^p}.$$

We know that (RR_2) is always satisfied and that (D) and (DUE) implies (RR_p) for all $p \in (2, \infty)$. For the exponents lower than 2, P. Auscher and T. Coulhon obtained the following result ([6]):

THEOREM 4.10. *Let M be a complete non-compact doubling Riemannian manifold. Moreover assume that the classical Poincaré inequality (P_q) holds for some $q \in (1, 2)$. Then for all $p \in (q, 2)$, (RR_p) is satisfied.*

This result is based on a Calderón–Zygmund decomposition for Sobolev functions. Using our new assumptions, we also obtain the following improvement:

THEOREM 4.11. *Assume that M is complete, satisfies (D) and admits the pseudo-Poincaré inequality $(\widetilde{P}_q)$ for some $q \in (1, 2)$. If in addition, the collection $\mathcal{A}$ satisfies L^q-L^2 off-diagonal estimates, then (RR_p) holds for all $p \in (q, 2)$.*

REMARK 4.12. Corollary 4.8 shows that these new assumptions are weaker than the Poincaré inequality (P_q).

We do not prove this result and refer the reader to [6]. The proof is exactly the same as it relies on the Calderón–Zygmund decomposition.

REMARK 4.13. We mention two other works of the authors [20, 14]. In [20], the assumption (RR_p) plays an important role in proving some maximal inequalities in dual Sobolev spaces $W^{-1,p}$, which do not require Poincaré inequalities. So it might be important to know how to prove (RR_p) without the Poincaré inequality.

4.3. Application to Gagliardo–Nirenberg inequalities. We devote this subsection to the study of Gagliardo–Nirenberg inequalities. We refer the reader to [12] for a recent work on this subject.

DEFINITION 4.14. For $\alpha < 0$, we define the Besov space $B^{\alpha}_{\infty,\infty}$ to be the set of all measurable functions f such that

$$\|f\|_{B^{\alpha}_{\infty,\infty}} := \sup_{t>0} t^{-\alpha/2} \|e^{-t\Delta} f\|_{L^\infty} < \infty.$$

We have the following equivalence (Lemma 2.1 in [12]):

$$\|f\|_{B^{\alpha}_{\infty,\infty}} \sim \sup_{t>0} t^{-\alpha/2} \|e^{-t\Delta}(f - e^{-t\Delta} f)\|_{L^\infty}.$$

Then the so-called *Gagliardo–Nirenberg inequalities* are

$$\|f\|_l \lesssim \big\| |\nabla f| \big\|_p^{\theta} \|f\|^{1-\theta}_{B^{\theta/(\theta-1)}_{\infty,\infty}} \tag{29}$$

where $\theta = p/l$ for some $p, l \in [1, \infty)$.

We first recall one of the main results of [12]:

THEOREM 4.15. *Let M be a complete non-compact Riemannian manifold satisfying (D) and (P_q) for some $1 \le q < \infty$. Moreover, assume that M satisfies the global pseudo-Poincaré inequalities (P'_q) and (P'_∞). Then (29) holds for all $q \le p < l < \infty$.*

Here, the *global* pseudo-Poincaré inequality (P'_q) for some $q \in [1, \infty]$ corresponds to

$$(P'_q) \qquad \|f - e^{-t\Delta} f\|_{L^q} \le C t^{1/2} \big\| |\nabla f| \big\|_{L^q}.$$

Theorem 4.15 requires global pseudo-Poincaré inequalities and some Poincaré inequalities with respect to balls. These two kinds of inequalities are quite different as they deal with oscillations with respect to the semigroup (for the pseudo-Poincaré inequalities) and to the mean value operators (for the Poincaré inequalities). We saw in the previous subsection that Poincaré inequalities imply pseudo-Poincaré inequalities. That is why we are looking for assumptions requiring only the Poincaré inequality, getting around the assumed global pseudo-Poincaré inequalities.

We begin by showing that pseudo-Poincaré inequalities related to balls yield global pseudo-Poincaré inequalities.

PROPOSITION 4.16. *Let M be a complete Riemannian manifold satisfying (D) and admitting the pseudo-Poincaré inequality $(\widetilde{P}_q)$ for some $1 \le q < \infty$. Then the global pseudo-Poincaré inequality (P'_q) holds.*

Proof. Let $t > 0$. Pick a countable set $\{x_j\}_{j \in J} \subset M$ such that

$$M = \bigcup_{j \in J} Q(x_j, \sqrt{t}) =: \bigcup_{j \in J} Q_j$$

and for all $x \in M$, x belongs to no more than N_1 balls Q_j. Then

$$\begin{aligned}\|f - e^{-t\Delta} f\|_q^q &\le \sum_j \int_{Q_j} |f - e^{-t\Delta} f|^q \, d\mu \lesssim \sum_j t^{q/2} \int_{Q_j} |\nabla f|^q \, d\mu \\ &\lesssim N_1 t^{q/2} \int_M |\nabla f|^q \, d\mu. \ \blacksquare\end{aligned}$$

REMARK 4.17. It is easy to see that the global pseudo-Poincaré inequality (P'_∞) is satisfied under (D) and (DUE) (see for instance [12, p. 499]).

Using Propositions 4.16, 4.2 and Theorem 4.15, we get the following improvement of Theorem 1.2 in [12]:

THEOREM 4.18. *Let M be a complete Riemannian manifold satisfying (D) and admitting the Poincaré inequality (P_q) for some $1 \le q < \infty$. If $q_0 \ge 2$, we moreover assume (DUE). Then* (29) *holds for all $q \le p < l < \infty$.*

Using our new assumptions, we also get the following Gagliardo–Nirenberg theorem:

THEOREM 4.19. *Assume that M satisfies the hypotheses of Theorem* 3.12 *with $A_Q = e^{-r_Q^2 \Delta}$ and that $r = \infty$. Moreover, assume (DUE). Then* (29) *holds for all $q \le p < l < \infty$.*

Proof. The proof is analogous to that of Theorems 1.1 and 1.2 in [12]. We use our homogeneous interpolation result of Theorem 3.12. Also we need our non-homogeneous interpolation result of Theorem 3.7. It holds thanks to (24) which is true under (D) and (DUE). Moreover, (P'_q) is satisfied and (P'_∞) holds thanks to (D) and (DUE). $\blacksquare$

As a corollary, we obtain

THEOREM 4.20. *Consider a complete Riemannian manifold M satisfying (D) and (P_q) for some $1 \le q < \infty$, and assume that there exists $C > 0$ such that for every $x, y \in M$ and $t > 0$,*

$$(G) \qquad |\nabla_x p_t(x,y)| \le \frac{C}{\sqrt{t}\,\mu(B(y,\sqrt{t}))}.$$

((G) is equivalent to the assumption (G_∞).) In the case where $q_0 > 2$, we moreover assume (DUE). Then inequality (29) *holds for all $q \le p < l < \infty$.*

Proof. In the case where $q \le 2$, this result is already in [12]. For $q_0 \ge 2$, we are under the hypotheses of Theorem 4.19 thanks to Subsection 4.1 and since (G) implies that $r = \infty$. $\blacksquare$

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