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BEYOND LEBESGUE AND BAIRE II: BITOPOLOGY AND MEASURE-CATEGORY DUALITY

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In memoriam Caspar Goffman (1913–2006)

Abstract. We re-examine measure-category duality by a bitopological approach, using both the Euclidean and the density topologies of the line. We give a topological result (on convergence of homeomorphisms to the identity) obtaining as a corollary results on infinitary combinatorics due to Kestelman and to Borwein and Ditor. We hence give a unified proof of the measure and category cases of the Uniform Convergence Theorem for slowly varying functions. We also extend results on very slowly varying functions of Ash, Erdős and Rubel.

1. Introduction. In a topological space one has one space and one topology. One often needs to have one space and two comparable topologies, one stronger and one weaker (as in functional analysis, where one may have the strong and weak topologies in play, or the weak and weak-star topologies). The resulting setting is that of a *bitopological space,* formalized in this language by Kelly [Kel].

Measure-category duality is the theme of the well-known book by Oxtoby [Oxt]. Here one has on the one hand measurable sets or functions, and small sets are null sets (sets of measure zero), and on the other hand sets or functions with the Baire property (briefly, Baire sets or functions), where small sets are meagre sets (sets of the first category).

In some situations, one has a dual theory, which has a measure-theoretic formulation on the one hand and a topological (or Baire) formulation on the other. We present here as a unifying theme the use of two topologies, each of which gives one of the two cases.

Our starting point is the *density topology* (introduced in [HauPau], [GoWa], [Mar] and studied also in [GNN]—see also [CLO], and for textbook treatments [Kech], [LMZ]). Recall that for T measurable, t is a (metric) *density point* of T if $\lim_{\delta\to 0} |T\cap I_\delta(t)|/\delta = 1$, where $I_\delta(t) = (t-\delta/2, t+\delta/2)$.

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By the Lebesgue Density Theorem almost all points of T are density points ([Hal, Section 61], [Oxt, Th. 3.20], or [Goff]). A set U is *d-open* (density-open = open in the density topology d) if (it is measurable and) each of its points is a density point of U. We mention five properties:

(i) The density topology is finer than (contains) the Euclidean topology ([Kech, 17.47(ii)]). See [LMZ] for a textbook treatment of other such fine topologies.
(ii) A set is Baire in the density topology iff it is (Lebesgue) measurable ([Kech, 17.47(iv)]).
(iii) A Baire set is meagre in the density topology iff it is null ([Kech, 17.47(iii)]). So (since a countable union of null sets is null) the conclusion of the Baire theorem holds for the line under d :
(iv) $(\mathbb{R}, d)$ is a Baire space, i.e., the conclusion of the Baire theorem holds ([Eng, Section 3.9]).
(v) A function is d-continuous iff it is approximately continuous in Denjoy's sense ([Den]; [LMZ, p. 1, 149]).

The reader unfamiliar with the density topology may find it helpful to think, in the style of Littlewood's First Principle, of basic opens sets as being intervals less some measurable set. See [Lit, Ch. 4], [Roy, Section 3.6 p. 72].

Both measurability and the Baire property have been used as regularity conditions, to exclude pathological situations. A classic instance is that of *additive functions,* satisfying the *Cauchy functional equation* $f(x+y) = f(x)+f(y)$. Such functions are either very good—continuous, and so linear, $f(x) = cx$ for some c—or very bad (one can construct such functions from Hamel bases, so this is called the *Hamel pathology);* see [BOst12] for details. A further instance is our focus here, the theory of *regular variation* [BGT], where each may be used as a regularity condition to prove the basic result of the theory, the Uniform Convergence Theorem (UCT). The present paper is a sequel to [BOst4] on *generic* regular variation, which gave a common generalization of the measure and Baire cases. The theory is usually developed in parallel, with the measure case regarded as primary and the Baire case as secondary. Here, we develop the two cases together. Our new viewpoint gives the interesting insight that it is in fact the Baire case that is the primary one.

In Section 2 below we give our main result, the Category Embedding Theorem (CET); the natural setting is a *Baire space* (as above). In Section 3 we give our unified treatment of the UCT, and extend to very slowly varying functions in Section 4. We close in Section 5 with some remarks.

2. Category Embedding Theorem (CET). The three results of this section (or four, as Theorem 3 below has two cases) develop a new aspect of measure-category duality. This has powerful applications: see Sections 3 and 4 below for the Uniform Convergence Theorem (UCT) of regular variation, and Remark 1 of Section 5 for numerous other applications.

Theorem 1 below is a topological version of the Kestelman–Borwein–Ditor (KBD) Theorem given at the end of this section (again see Section 5, Remark 1). The latter is a (homeomorphic) *embedding* theorem (see e.g. [Eng, p. 67]); Trautner uses the term covering principle in [Trau]. We need the following definition, in which X is any topological space.

Definition (weak category convergence). A sequence of homeomorphisms $h_n : X \to X$ satisfies the *weak category convergence* condition (wcc) if:

For any non-meagre open set $U \subseteq X$, there is a non-meagre open set $V \subseteq U$ such that, for each $k \in \omega$,

$$\bigcap_{n \geq k} V \setminus h_n^{-1}(V) \text{ is meagre.} \tag{wcc}$$

Equivalently, for each $k \in \omega$, there is a meagre set M such that, for $t \notin M$,

$$t \in V \;\Rightarrow\; (\exists n \geq k)\; h_n(t) \in V.$$

(We use the term 'weak category convergence' to avoid possible confusion with 'category convergence', as in the works of Wilczyński and collaborators; see e.g. [W], [PWW].)

We will see below in Theorem 2 that this is a weak form of convergence to the identity and indeed Theorems 3E and 3D verify that, for $z_n \to 0$, the homeomorphisms $h_n(x) := x + z_n$ satisfy (wcc) in the Euclidean and in the density topologies. However, it is not true that $h_n(x)$ converges to the identity pointwise in the sense of the density topology; furthermore, whereas addition (a two-argument operation) is not d-continuous (see [HePo]), translation (a one-argument operation) is. In what follows, the words *quasi everywhere (q.e.)*, or *for quasi-all points,* mean *for all points off a meagre set.* We will use *for generically all* to mean for quasi-all in the category case, and for almost all in the measure case.

In Theorem 1 below, the topological space X may be assumed to be non-meagre (of second category) in itself, and the Baire set T to be non-meagre, as otherwise there is nothing to prove. To verify that X is non-meagre, one would typically assume that X is a Baire space (see the Introduction).

Theorem 1 (Category Embedding Theorem, CET). *Let X be a topological space and $h_n : X \to X$ be homeomorphisms satisfying* (wcc). *Then, for any Baire set T, for quasi-all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Take T Baire and non-meagre. We may assume that $T = U \setminus M$ with U non-meagre and open and M meagre. Let $V \subseteq U$ be non-meagre, open and satisfy (wcc). Since the functions h_n are homeomorphisms, the set

$$M' := M \cup \bigcup_n h_n^{-1}(M)$$

is meagre. Writing 'i.o.' for 'infinitely often', put

$$W = \mathbf{h}(V) := \bigcap_{k\in\omega} \bigcup_{n\geq k} V \cap h_n^{-1}(V) = \limsup [h_n^{-1}(V) \cap V]$$
$$= \{x : x \in h_n^{-1}(V) \cap V \text{ i.o.}\} \subseteq V \subseteq U.$$

So for $t \in W$ we have $t \in V$ and

$$v_m := h_m(t) \in V, \tag{1}$$

for infinitely many m—for $m \in \mathbb{M}_t$, say. Now W is co-meagre in V. Indeed

$$V \setminus W = \bigcup_{k\in\omega} \bigcap_{n\geq k} V \setminus h_n^{-1}(V),$$

which by (wcc) is meagre.

Take $t \in W \setminus M' \subseteq U \setminus M = T$, as $V \subseteq U$ and $M \subseteq M'$. Thus $t \in T$. For $m \in \mathbb{M}_t$, we have $t \notin h_m^{-1}(M)$, since $t \notin M'$ and $h_m^{-1}(M) \subseteq M'$; but $v_m = h_m(t)$ so $v_m \notin M$. By (1), $v_m \in V \setminus M \subseteq U \setminus M = T$. Thus $\{h_m(t) : m \in \mathbb{M}_t\} \subseteq T$ for t in a non-meagre set.

To deduce that quasi-all $t \in T$ satisfy the conclusion of the theorem, put $S := T \setminus \mathbf{h}(T)$; then S is Baire and $S \cap \mathbf{h}(T) = \emptyset$. If S is non-meagre, then by the preceding argument there are $s \in S$ and an infinite $\mathbb{M}_s$ such that $\{h_m(s) : m \in \mathbb{M}_s\} \subseteq S$, i.e. $s \in \mathbf{h}(S) \subseteq \mathbf{h}(T)$, a contradiction. (This last step is an implicit appeal to a generic dichotomy—see [BOst8].) ∎

Theorem 1 implies that for $\{h_n\}$ satisfying (wcc) and Baire T the set $\limsup h_n^{-1}(T)$ contains T modulo a meagre set. [In the other direction, suppose, as in the Euclidean case for the shifts considered above, $h_n(x) \to x$ (for all x), $T = U \setminus N$ with U open and N meagre, and without loss of generality $W := X \setminus \mathrm{cl}\, U \neq \emptyset$. Then $S := \limsup h_n^{-1}(T) \subseteq T$ modulo meagre sets. Indeed, if $x \in \limsup_n h_n^{-1}(T)$, then $h_n(x) \in T = U \setminus N$ infinitely often, so $h_n(x) \in U$ infinitely often, and so $x \in \mathrm{cl}\, U$. To see that for quasi-all $x \in S$, we have $x \in T$, it is enough to show that for quasi-all x we have $x \in U$ (as N is meagre). But $x \in \mathrm{cl}\, U$ for $x \in S$, so if $x \in S \setminus U$, then $x \in \mathrm{cl}\, U \setminus U$, which is meagre.]

Clearly the result relativizes to any open subset of T; that is, the embedding property is a *local* one.

The following theorem sheds some light on the significance of the category convergence condition (wcc). The result is capable of improvement, by reference to more general (topological) countability conditions. (Typically these lift category and measure arguments out of the classical context of separable metric spaces; in this connection, for an account of Čech-completeness and metrization theory see e.g. [Eng, §§3.9 and 4.4], and for an account of p-spaces, their common generalization, see [Arh, §7].) Here, for instance, a σ-discrete family could replace the countable family $\mathcal{B}$ of the theorem as the generator of the coarser topology; such a replacement would offer a route to Bing's Metrization Theorem, given sufficient regularity assumptions— see [Eng, Th. 4.4.8].

Theorem 2 (Convergence to the identity). *Assume that the homeomorphisms $h_n : X \to X$ satisfy the weak category convergence condition* (wcc) *and that X is a Baire space. Suppose there is a countable family $\mathcal{B}$ of open subsets of X which generates a (coarser) Hausdorff topology on X. Then, for quasi-all (under the original topology) t, there is an infinite $\mathbb{N}_t$ such that (under the coarser topology)*

$$\lim_{m\in\mathbb{N}_t} h_m(t) = t.$$

Proof. We work in the original topology until further notice, and assume that members of $\mathcal{B}$ are non-empty, so non-meagre (as the original topology is Baire). By (wcc), select for each $U \in \mathcal{B}$ and $k \in \omega$ a non-empty open $V_k(U) \subseteq U$ such that $M_k(U) := \bigcap_{n\geq k} V_k(U) \setminus h_n^{-1}(V_k(U))$ is a meagre subset of $V_k(U)$.

For each $W \in \mathcal{B}$ and $k \in \omega$ choose a maximal family $\mathcal{V}_k^W := \{V_k(U_i^W) : i \in I^W\}$ of disjoint non-empty open subsets of the form $V_k(U)$ for $U \subseteq W$. Let V_k^W denote its union and $F_k(W)$ the closure of V_k^W. Then $W \subseteq F_k(W)$, otherwise $U := W \setminus F_k(W)$ is non-empty, open and disjoint from V_k^W, so that $V_k(U)$ is non-empty and disjoint from the members of $\mathcal{V}_k^W$, contradicting maximality. Now $N_k(W) := F_k(W) \setminus V_k^W$ is closed and nowhere dense as V_k^W is open, so $N := \bigcup_{k\in\omega}\bigcup_{W\in\mathcal{B}} N_k(W)$ is meagre (as $\mathcal{B}$ is countable). Note that $W \setminus N_k(W) \subseteq F_k(W) \setminus N_k(W) \subseteq V_k^W$. Put

$$M_k^W := \bigcup_{i\in I^W} M_k(U_i^W) \quad \text{and} \quad M := \bigcup_{k\in\omega}\bigcup_{W\in\mathcal{B}} M_k^W,$$

which are both meagre—the former by the Banach Category Theorem (for which see [Oxt, Ch. 16], or as the Localization Theorem [Kelley, Th. 6.35]), as $M_k(U_i^W) \subseteq V_k(U_i^W)$ and $\mathcal{V}_k^W$ is disjoint, the latter by countability of $\mathcal{B}$. Consider $t \notin N \cup M$. For $W \in \mathcal{B}$ with $t \in W$ and $k \in \omega$, one has $t \in V_k(U_i^W)$ for some $U_i^W \subseteq W$ with $i \in I^W$, as $t \in W \setminus N_k(W) \subseteq V_k^W$. Also, since $t \notin M$, one has $t \in V_k(U_i^W) \setminus M_k(U_i^W)$, so $t \in h_m^{-1}(V_k(U_i^W))$ for some $m = m(t,k,W,i) \geq k$. So $h_m(t) \in V_k(U_i^W) \subseteq U_i^W \subseteq W$. Passing now to the coarser topology in which $\mathcal{B}_t := \{W \in \mathcal{B} : t \in W\}$ is a basis for the neighbourhoods of t, it follows that there is an infinite set $\mathbb{N}_t$ of integers m for which $h_m(t) \to t$ in the coarser topology. ∎

We now deduce the category and measure cases of the Kestelman–Borwein–Ditor Theorem (Th. KBD, stated below) as two corollaries of the above theorem by applying it first to the usual and then to the density topology on the reals, $\mathbb{R}$.

For our first application we take $X = \mathbb{R}$ with the density topology, a Baire space. Let $z_n \to 0$ be a null sequence. Put

$$h_n(x) := x - z_n, \quad \text{so that} \quad h_n^{-1}(x) = x + z_n.$$

The topology is translation-invariant, and so each h_n is a homeomorphism. To verify the weak category convergence of the sequence h_n, consider U non-empty and d-open; then consider any measurable non-null $V \subseteq U$. To verify (wcc) in relation to V, it now suffices to prove the following result, which is of independent interest (cf. Littlewood's First Principle, as above).

Theorem 3D (Verification Theorem D). *Let V be measurable and non-null. For any null sequence $\{z_n\} \to 0$ and each $k \in \omega$,*

$$H_k := \bigcap_{n \geq k} V \setminus (V + z_n) \text{ is null, so meagre in the d-topology.}$$

Proof. Suppose otherwise. Then for some k, $|H_k| > 0$. Write H for H_k. Since $H \subseteq V$, for $n \geq k$ we have $\emptyset = H \cap h_n^{-1}(V) = H \cap (V + z_n)$ and so a fortiori $\emptyset = H \cap (H + z_n)$.

Let u be a density point of H. Thus for some interval $I_\delta(u) = (u - \delta/2, u + \delta/2)$ we have

$$|H \cap I_\delta(u)| > 3\delta/4.$$

Let $E = H \cap I_\delta(u)$. For any z_n, we have $|(E + z_n) \cap (I_\delta(u) + z_n)| = |E| > 3\delta/4$. For $0 < z_n < \delta/4$, we have $|(E + z_n) \setminus I_\delta(u)| \leq |(u + \delta/2, u + 3\delta/4)| = \delta/4$. Put $F = (E + z_n) \cap I_\delta(u)$; then $|F| > \delta/2$.

But $\delta \geq |E \cup F| = |E| + |F| - |E \cap F| \geq 3\delta/4 + \delta/2 - |E \cap F|$. So

$$|H \cap (H + z_n)| \geq |E \cap F| \geq \delta/4,$$

contradicting $\emptyset = H \cap (H + z_n)$. This completes the proof. ■

A similar but simpler proof establishes the following result which implies (wcc) for the Euclidean topology on $\mathbb{R}$; here for given open U we may take any open interval $V \subseteq U$.

Theorem 3E (Verification Theorem E). *Let V be an open interval in $\mathbb{R}$. For any null sequence $\{z_n\} \to 0$ and each $k \in \omega$,*

$$H_k := \bigcap_{n \geq k} V \setminus (V + z_n) \text{ is empty.}$$

We are now ready to state and prove Th. KBD. As with the CET, the set T here may be assumed to be non-meagre/non-null, since otherwise there is nothing to prove.

THEOREM KBD (Kestelman–Borwein–Ditor Theorem). *Let $\{z_n\} \to 0$ be a null sequence of reals. If T is Baire/Lebesgue measurable, then for generically all $t \in T$ there is an infinite set $\mathbb{M}_t$ such that*

$$\{t + z_m : m \in \mathbb{M}_t\} \subseteq T.$$

Proof. Th. CET may be applied to $h_n(x) := x + z_n$ in view of Th. 3E or 3D respectively in the category/measure cases. ∎

3. Uniform Convergence Theorem (UCT). As an illustration of the power of the results above, we use them to give a short proof of the fundamental theorem of regular variation, the Uniform Convergence Theorem (UCT) below (see [BGT, Section 1.2] for background and references). This has traditionally been proved for the measure and Baire cases separately; an old question, raised in [BGT, p. 11] and answered in [BOst4], is that of finding the minimal common generalization of measurability and the Baire property (the 'No Trumps' property below). Here we handle the two cases together by working bitopologically, reducing the measure case to the Baire case, and greatly simplify the proof.

Recall (see [BGT]) that a function $h : \mathbb{R} \to \mathbb{R}$ is *slowly varying* (in additive notation) if for every sequence $\{x_n\} \to \infty$ and each $u \in \mathbb{R}$

$$\lim_{n\to\infty} (h(u + x_n) - h(x_n)) = 0. \tag{SV}$$

THEOREM 4 (Uniform Convergence Theorem, UCT). *If h is slowly varying, and measurable, or Baire, then* (SV) *holds uniformly in u on compacts.*

Proof. Suppose otherwise. Then for some measurable/Baire slowly varying function h and some $\varepsilon > 0$, there is $\{u_n\} \to u$ and $\{x_n\} \to \infty$ such that

$$|h(u_n + x_n) - h(x_n)| \geq 2\varepsilon. \tag{2}$$

Now, for each point y, $\lim_n |h(y + x_n) - h(x_n)| = 0$ by slow variation, so there is $k = k(y)$ such that, for $n \geq k$,

$$|h(y + x_n) - h(x_n)| < \varepsilon.$$

For $k \in \omega$, define the measurable/Baire set

$$T_k := \bigcap_{n\geq k} \{y : |h(y + u + x_n) - h(x_n)| < \varepsilon\}.$$

Since $\{T_k : k \in \omega\}$ covers $\mathbb{R}$, for some $k \in \omega$ the set T_k is non-null/non-meagre. Since $z_n := u_n - u$ is null, Th. KBD shows that for some $t \in T_k$ and for some infinite $\mathbb{M}_t$, $\{t + z_m : m \in \mathbb{M}_t\} \subseteq T_k$. So, since $u + z_m = u_m$,

$$|h(t + u_m + x_m) - h(x_m)| < \varepsilon.$$

By slow variation of h at t, since $u_m + x_m \to \infty$ it follows that for m large enough,

$$|h(t + u_m + x_m) - h(u_m + x_m)| < \varepsilon.$$

The last two inequalities together imply that for m large enough and in $\mathbb{M}_t$,

$$\begin{aligned}|h(u_m + x_m) - h(x_m)| &\le |h(u_m + x_m) - h(t + u_m + x_m)| \\ &\quad + |h(t + u_m + x_m) - h(x_m)| \\ &< 2\varepsilon,\end{aligned}$$

and this contradicts (2). ■

This strikingly brief proof is inspired by the 'fourth proof' in [BGT], from [BG1], itself based on work of Csiszár and Erdős [CsEr]. It is a much streamlined version of that in [BOst2], the main simplification being enabled by use of CET (Th. 1) to prove Th. KBD (all that the proof above uses explicitly). For another proof, albeit for the measurable case only, see [Trau]. Trautner employs a theorem of Egorov (cf. Littlewood's Third Principle, see [Lit, Ch. 4], [Roy, Section 3.6 and Problem 31], or [Hal, Section 55 p. 243]); see [BGT, p. ixx and p. 10].

4. UCT for very slowly varying functions. We recall from [AER] that h is *very slowly varying* if for some non-decreasing positive φ

(VSV) $$\{h(x + u) - h(x)\}\varphi(x) \to 0 \quad (x \to \infty) \quad \forall u \in \mathbb{R};$$

h is *φ-slowly varying* if this holds for a specific φ (see also [BGT, 1.2.5 p. 11], where φ is $1/g$, and pp. 134–135 there). If φ is bounded (in which case we may take $\varphi \equiv 1$), this is just slow variation; if $\varphi \uparrow \infty$, h is more than slowly varying, whence the terminology. Note that as φ is monotone, φ itself is both measurable and Baire.

It is a remarkable fact, due to Ash, Erdős and Rubel [AER], that if φ grows fast enough we can obtain a uniform convergence theorem with no regularity condition on h whatever. We summarize their results as follows.

THEOREM AER.

(i) *If h is φ-slowly varying and measurable, then h is φ-slowly varying uniformly on compact sets in* (VSV).

(ii) *If h is φ-slowly varying, and φ satisfies*

(AER) $$\varphi(x) \sum_{n=0}^{\infty} 1/\varphi(x + n) \le B < \infty \quad \forall x \ge 0$$

then h is uniformly φ-slowly varying on compact sets.

(iii) *If* (AER) *does not hold* (*e.g. for $\varphi \equiv 1$*), *there is a function $h = h(\varphi)$ which is φ-slowly varying but not uniformly so.*

To formulate our generalization of Theorem AER(i), we recall some combinatorial terminology from [BOst4], concerning 'No Trumps' or **NT** (see Section 5 for an explanation of this term), applied to sequences of subsets of the line and to functions, which will provide our desired common generalization of the measurable and Baire cases: see Th. 5 and 6 below.

DEFINITION. For $\{T_k : k \in \omega\}$ a countable family of subsets of $\mathbb{R}$, write $\mathbf{NT}(\{T_k : k \in \omega\})$ to mean that, for every bounded/convergent sequence $\{u_n\}$ in $\mathbb{R}$, some T_k contains a translate of a subsequence of $\{u_n\}$, i.e. there are $k \in \omega$, infinite $\mathbb{M} \subseteq \omega$, and $t \in \mathbb{R}$ such that

$$\{t + u_n : n \in \mathbb{M}\} \subseteq T_k.$$

The term appears in [BOst1] on subadditive functions. When $T_k = S$ for all k, we write this as $\mathbf{NT}(S)$. This allows a formulation of when a function may be regarded as having 'nice' level sets:

$$H^k(h) := \{t : |h(t)| < k\} \quad (k \in \omega),$$

as in [BOst4]. Thus, since $\mathbb{R}$ is the union of the level sets of a function, we have as an immediate corollary of the Kestelman–Borwein–Ditor Theorem:

THEOREM 5 (No Trumps Theorem, cf. [BOst4]). *For $\mathbb{R}$ under either the density or the Euclidean topology and $h : \mathbb{R} \to \mathbb{R}$ measurable or Baire respectively, $\mathbf{NT}(\{H^k(h) : k \in \omega\})$ holds.*

Proof. Since $\mathbb{R} = \bigcup_{k\in\omega} H^k(h)$ and $\mathbb{R}$ is a Baire space under either topology, $H^k(h)$ is non-meagre/non-null for some k. For this k, any convergent sequence $u_n \to u$, and $z_n := u_n - u$, Th. KBD implies that the set $H^k(h)$ contains t-translates of a subsequence of z_m (for quasi-all/almost all members $t \in H^k(h)$) and hence the $(t - u)$-translates of u_m. (Here we do not assert that $t - u$ is in $H^k(h)$, merely that it is in $\mathbb{R}$.) ∎

Thus the **NT** property is a common generalization of both measurability and the Baire property. We now extend this to the setting of [AER].

DEFINITIONS. 1. For given increasing φ, $\{x_n\} \to \infty$ and $\varepsilon > 0$, we define the $\{x_n\}$*-stabilized* ε*-level sets* of h by

$$T_k^\varepsilon(h) := \bigcap_{n\ge k} \{z \in \mathbb{R}_+ : |h(z + x_n) - h(x_n)|\varphi(x_n) < \varepsilon\}.$$

2. Say that h is a φ-**NT** *function* if for each $\varepsilon > 0$ and each $\{x_n\} \to \infty$, $\mathbf{NT}\{T_k^\varepsilon(h) : k \in \omega\}$ holds.

If h is both φ-**NT** and φ-slowly varying, say that h is φ-**NT**-*slowly varying.*

THEOREM 6 (φ-No Trumps Theorem). *For $\mathbb{R}$ under either the density or the Euclidean topology and $h : \mathbb{R} \to \mathbb{R}$ measurable/Baire and φ-slowly varying,* $\mathbf{NT}(\{T_k^\varepsilon(h) : k \in \omega\})$ *holds for every* $\{x_n\} \to \infty$, *i.e. h is* φ-**NT**-*slowly varying.*

Proof. For h Baire/measurable the sets $T_k^\varepsilon(h)$ are Baire/measurable (since φ is Baire/measurable). If h is φ-slowly varying then $\mathbb{R}_+ = \bigcup_{k\in\omega} T_k^\varepsilon(h)$. Continuing as in Th. 5, by Th. KBD, $\mathbf{NT}(\{T_k^\varepsilon(h) : k \in \omega\})$ holds. ∎

The following result contains the UCT as the case $\varphi \equiv 1$, and its proof is similar to that of Th. 4.

THEOREM 7 (φ-**UCT**, cf. [AER]). *Suppose that h is* φ-**NT**-*slowly varying. Then h is uniformly* φ-*slowly varying on compact sets.*

Proof. As usual suppose for some $x_n \to \infty$ and some bounded u_n we have $|h(u_n + x_n) - h(x_n)|\varphi(x_n) > 2\varepsilon$. Note that, as φ is increasing and $u_n > 0$, we have $\varphi(x_n)/\varphi(x_n + u_n) \le 1$. As h is φ-slowly varying, for each $z \in \mathbb{R}_+$, $|h(z + x_n) - h(x_n)|\varphi(x_n)$ tends to 0, so for $\varepsilon > 0$ is less than ε for large n. So for

$$T_k^\varepsilon = T_k^\varepsilon(h) := \bigcap_{n\ge k} \{z \in \mathbb{R}_+ : |h(z + x_n) - h(x_n)|\varphi(x_n) < \varepsilon\},$$

$\mathbb{R}_+ = \bigcup_{k\in\omega} T_k^\varepsilon$. By $\mathbf{NT}\{T_k^\varepsilon(h) : k \in \omega\}$ there are $k \in \omega$, $t \in \mathbb{R}$ and an infinite $\mathbb{M}_t$ such that $\{t + u_m : m \in \mathbb{M}_t\} \subset T_k^\varepsilon$. Now, as $u_n + x_n \to \infty$, for some $N \ge k$, and all $n \ge N$, $|h(t + u_n + x_n) - h(u_n + x_n)|\varphi(x_n) < \varepsilon$ (since h is φ-slowly varying at t). So for $m > N$ with m in $\mathbb{M}_t$ we also have $|h(t + u_m + x_m) - h(x_m)|\varphi(x_m) < \varepsilon$, since φ is increasing. Combining,

$$\begin{aligned}
&|h(u_m + x_m) - h(x_m)|\varphi(x_m)\\
&\quad\le |h(t+u_m+x_m) - h(x_m)|\varphi(x_m) + |h(t+u_m+x_m) - h(u_m+x_m)|\varphi(x_m)\\
&\quad\le |h(t+u_m+x_m) - h(x_m)|\varphi(x_m)\\
&\qquad + |h(t+u_m+x_m) - h(u_m+x_m)|\varphi(x_m+u_m)\cdot(\varphi(x_m)/\varphi(x_m+u_m))\\
&\quad\le |h(t+u_m+x_m) - h(x_n)|\varphi(x_m)\\
&\qquad + |h(t+u_m+x_m) - h(u_m+x_m)|\varphi(x_m+u_m)\\
&\quad\le 2\varepsilon,
\end{aligned}$$

a contradiction. ∎

By Th. 6 we have two immediate corollaries, the first of which is new.

THEOREM 7B (Baire φ-**UCT**). *Suppose that h is* φ-*slowly varying and Baire. Then h is uniformly* φ-*slowly varying on compact sets.*

THEOREM 7M (Measurable φ-**UCT**, [AER]). *Suppose that h is φ-slowly varying and measurable. Then h is uniformly φ-slowly varying on compact sets.*

For other results related to [AER], see our recent sequel to it, [BOst3].

5. Remarks

1. The Category Embedding Theorem and infinite combinatorics. Results of van der Waerden type for the reals are derived from the CET in [BOst8] and an Interior Points Theorem of Steinhaus type (see [BGT, Th. 1.1.1] for background) in [BOst12]. For applications beyond the real line including the theory of *topological regular variation* see [BOst10], [BOst11] and [Ost2]. Applications of Th. KBD are wide ranging: in addition to the UCT of Sections 3 and 4 they include automatic continuity ([BOst5], [BOst7], [BOst12]), the theory of subadditive functions [BOst1], combinatorics in function spaces [BOst6] and more generally in topological groups and normed groups [BOst10]. For an extension see [BOst9].

The KBD Theorem in the measure case is due to Borwein and Ditor [BoDi], but was already known much earlier albeit in somewhat weaker form by Kestelman ([Kes, Th. 3]), and rediscovered by Trautner [Trau] (see the end of Section 3).

We thank the referee for drawing our attention to the paper of Wilczyński and Kharazishvili [WKh]. Their setting is general (topological groups), as above, except in Theorems 3D, 3E, where we restrict for simplicity to the line. Their results may be compared with much earlier work of Kemperman [Kem] in Euclidean space; see [BOst8] for implications of results of this kind, and Halmos [Hal, Ch. XII, p. 266].

2. No Trumps. The term No Trumps in Theorem 5, a combinatorial principle, is used in close analogy with earlier combinatorial principles, in particular Jensen's Diamond $\diamondsuit$ [Je] and Ostaszewski's Club $\clubsuit$ [Ost1]. It also plays a key role in the analysis of the UCT, as is shown in [BOst2]. Our proof of Th. 5 makes explicit an argument implicit in [BG1, p. 482] (and repeated in [BGT, p. 9]), itself inspired by [CsEr] (see also [BOst2], [BOst4]). The intuition behind our formulation may be gleaned from forcing arguments in [Mil1]–[Mil4].

3. Measure-category duality. The duality between measure and category emerged in the 1920s, largely in the work of Sierpiński. See the commentary by Hartman [Hart] in Sierpiński's selected works ([Sie1], [Sie2]). The theme is explored at textbook length in [Oxt]; see Ch. 19 for duality (including the Sierpiński–Erdős Duality Principle under the Continuum Hypothesis), Ch. 17 (in ergodic theory, duality extends to some but not all forms of the Poincaré recurrence theorem) and Ch. 21 (in probability theory, duality extends as far as the zero-one law but not to the strong law of large numbers). Duality also fails to extend to the theory of random series [Kah]. For further limitations of duality, see [DoF], [Bart], [BGJS]. For the theory of a.e.-convergence associated with σ-ideals studied by Wilczyński's school, see [PWW]. For a set-theoretic explanation of the duality in regular variation in terms of forcing see [BOst2, Section 5], [Mil1, Section 6].

Added in proof. 1. The forthcoming paper [Mil-Ost] is relevant to this one. For reasons which are clarified there the name CET is there replaced by 'Bitopological Shift-compactness Theorem', which we now prefer.

2. For the lack of d-continuity of addition, for which we cite [HePo] in the text, see also [Sch, Prop. 1.9].

REFERENCES

[Arh] A. V. Arhangel'skii, *Paracompactness and metrization. The method of covers in the classification of spaces*, in: General Topology III, A. V. Arhangel'skii (ed.), Encyclopaedia Math. Sci. 51, Springer, 1995, 1–70.

[AER] J. M. Ash, P. Erdős and L. Rubel, *Very slowly varying functions*, Aequationes Math. 10 (1974), 1–9.

[Bart] T. Bartoszyński, *A note on duality between measure and category*, Proc. Amer. Math. Soc. 128 (2000), 2745–2748.

[BGJS] T. Bartoszyński, M. Goldstern, H. Judah and S. Shelah, *All meager filters may be null*, Proc. Amer. Math. Soc. 117 (1993), 515–521.

[BG1] N. H. Bingham and C. M. Goldie, *Extensions of regular variation, I: Uniformity and quantifiers*, Proc. London Math. Soc. (3) 44 (1982), 473–496.

[BGT] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular Variation*, 2nd ed., Encyclopedia Math. Appl. 27, Cambridge Univ. Press, Cambridge, 1989 (1st ed. 1987).

[BOst1] N. H. Bingham and A. J. Ostaszewski, *Generic subadditive functions*, Proc. Amer Math. Soc. 136 (2008), 4257–4266.

[BOst2] —, —, *Infinite combinatorics and the foundations of regular variation*, J. Math. Anal. Appl. 360 (2009), 518–529.

[BOst3] —, —, *Very slowly varying functions – II*, Colloq. Math. 116 (2009), 105–117.

[BOst4] —, —, *Beyond Lebesgue and Baire: generic regular variation*, ibid. 116 (2009), 119–138.

[BOst5] —, —, *New automatic properties: subadditivity, convexity, uniformity*, Aequationes Math. 78 (2009), 257–270.

[BOst6] —, —, *Infinite combinatorics in function spaces: category methods*, Publ. Inst. Math. Beograd 86 (100) (2009), 55–73.

[BOst7] —, —, *Automatic continuity via analytic thinning*, Proc. Amer Math. Soc. 138 (2010), 907–919.

[BOst8] —, —, *Kingman, category and combinatorics*, in: Probability and Mathematical Genetics, J. F. C. Kingman Festschrift (N. H. Bingham and C. M. Goldie, eds.), London Math. Soc. Lecture Note Ser. 378, Cambridge Univ. Press, 2010, 135–168.

[BOst9] N. H. Bingham and A. J. Ostaszewski, *Homotopy and the Kestelman Borwein Ditor Theorem*, Canad. Math. Bull., to appear.

[BOst10] —, —, *Normed versus topological groups: dichotomy and duality*, Dissertationes Math. 472 (2010).

[BOst11] —, —, *Topological regular variation: I. Slow variation*, Topology Appl. 157 (2010), 1999–2013.

[BOst12] —, —, *Dichotomy and infinite combinatorics: the theorems of Steinhaus and Ostrowski*, Math. Proc. Cambridge Philos. Soc., to appear.

[BoDi] D. Borwein and S. Z. Ditor, *Translates of sequences in sets of positive measure*, Canad. Math. Bull. 21 (1978), 497–498.

[CLO] K. Ciesielski, L. Larson and K. Ostaszewski, *$\mathcal{I}$-density continuous functions*, Mem. Amer. Math. Soc. 107 (1994), no. 515.

[CsEr] I. Csiszár and P. Erdős, *On the function $g(t) = \limsup_{x\to\infty}(f(x+t) - f(x))$*, Magyar Tud. Akad. Kut. Int. Közl. A 9 (1964), 603–606.

[Den] A. Denjoy, *Sur les fonctions dérivées sommables*, Bull. Soc. Math. France 43 (1915), 161–248.

[DoF] R. Dougherty and M. Foreman, *Banach–Tarski decompositions using sets with the Baire property*, J. Amer. Math. Soc. 7 (1994), 75–124.

[Eng] R. Engelking, *General Topology*, Heldermann, Berlin 1989.

[Goff] C. Goffman, *On Lebesgue's density theorem*, Proc. Amer. Math. Soc. 1 (1950), 384–388.

[GNN] C. Goffman, C. J. Neugebauer and T. Nishiura, *Density topology and approximate continuity*, Duke Math. J. 28 (1961), 497–505.

[GoWa] C. Goffman and D. Waterman, *Approximately continuous transformations*, Proc. Amer. Math. Soc. 12 (1961), 116–121.

[Hal] P. R. Halmos, *Measure Theory*, Grad. Texts in Math. 18, Springer, 1974 (1st ed. Van Nostrand, 1969).

[Hart] S. Hartman, *Travaux de W. Sierpiński sur la théorie des ensembles et ses applications IV. Mesure et catégorie. Congruence des ensembles*, in [Sie1], 20–25.

[HauPau] O. Haupt et C. Pauc, *La topologie approximative de Denjoy envisagée comme vraie topologie*, C. R. Acad. Sci. Paris 234 (1952), 390–392.

[HePo] R. W. Heath and T. Poerio, *Topological groups and semi-groups on the reals with the density topology*, Conference papers, Oxford 2006 (Conference in honour of Peter Collins and Mike Reed).

[Je] R. B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic 4 (1972), 229–308.

[Kah] J.-P. Kahane, *Probabilities and Baire's theory in harmonic analysis*, in: Twentieth Century Harmonic Analysis, A Celebration, J. S. Byrnes (ed.), Kluwer, 2001, 57–72.

[Kech] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Texts in Math. 156, Springer, 1995.

[Kelley] J. L. Kelley, *General Topology*, Van Nostrand, 1955.

[Kel] J. C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. 13 (1963), 71–89.

[Kem] J. H. B. Kemperman, *A general functional equation*, Trans. Amer. Math. Soc. 86 (1957), 28–56.

[Kes] H. Kestelman, *The convergent sequences belonging to a set*, J. London Math. Soc. 22 (1947), 130–136.

[Lit] J. E. Littlewood, *Lectures on the Theory of Functions*, Oxford Univ. Press, 1944.

[LMZ] J. Lukeš, J. Malý and L. Zajíček, *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Math. 1189, Springer, 1986.

[Mar] N. F. G. Martin, *A topology for certain measure spaces*, Trans. Amer. Math. Soc. 112 (1964), 1–18.

[Mil1] A. W. Miller, *Infinite combinatorics and definability*, Ann. Pure Appl. Math. Logic 41 (1989), 179–203 (updated web version: http://www.math.wisc.edu /˜miller/res/).

[Mil2] —, *Special sets of reals*, 415–432, in: H. Judah (ed.), Set Theory of the Reals (Ramat Gan, 1993), Israel Math. Conf. Proc. 6 (web version: http://www. math.wisc.edu/˜miller/res/).

[Mil3] —, *Descriptive Set Theory and Forcing*, Springer, 1995.

[Mil4] —, *Souslin's hypothesis and convergence in measure*, Proc. Amer. Math. Soc. 124 (1996), 1529–1532.

[Mil-Ost] H. I. Miller and A. J. Ostaszewski, *Group action and shift-compactness*, preprint, to appear.

[Ost1] A. J. Ostaszewski, *On countably compact perfectly normal spaces*, J. London Math. Soc. 14 (1976), 505–516.

[Ost2] —, *Regular variation, topological dynamics, and the uniform boundedness theorem*, Topology Proc. 36 (2010), 1–32.

[Oxt] J. C. Oxtoby, *Measure and Category,* 2nd ed., Grad. Texts in Math. 2, Springer, New York, 1980.

[PWW] W. Poreda, E. Wagner-Bojakowska and W. Wilczyński, *A category analogue of the density topology*, Fund. Math. 125 (1985), 167–173.

[Roy] H. L. Royden, *Real Analysis*, 3rd ed., Prentice-Hall, 1988.

[Sch] S. Scheinberg, *Topologies which generate a complete measure algebra*, Adv. Math. 7 (1971), 231–239.

[Sie1] W. Sierpiński, *Oeuvres choisies, II: Théorie des ensembles et ses applications, travaux des années 1908-1929*, PWN, Warszawa, 1975.

[Sie2] —, *Oeuvres choisies III: Théorie des ensembles et ses aplications, travaux des années 1930-1966*, PWN, Warszawa, 1976.

[Trau] R. Trautner, *A covering principle in real analysis*, Quart. J. Math. Oxford (2) 38 (1987), 127–130.

[W] W. Wilczyński, *Density topologies*, in: Handbook of Measure Theory, Vols. I, II, North-Holland, 2002, 675–702.

[WKh] W. Wilczyński and A. B. Kharazishvili, *Translations of measurable sets and sets having the Baire property*, Soobshch. Akad. Nauk Gruzii 145 (1992), 43–46.

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