COLLOQUIUM MATHEMATICUM

VOL. 121 2010 NO. 2

ALMOST PRÜFER v-MULTIPLICATION DOMAINS AND THE RING $D + XD_S[X]$

BY

QING LI (Chengdu)

Abstract. This paper is a continuation of the investigation of almost Prüfer v-multiplication domains (APVMDs) begun by Li [Algebra Colloq., to appear]. We show that an integral domain D is an APVMD if and only if D is a locally APVMD and D is well behaved. We also prove that D is an APVMD if and only if the integral closure $\overline{D}$ of D is a PVMD, $D \subseteq \overline{D}$ is a root extension and D is t-linked under $\overline{D}$. We introduce the notion of an almost t-splitting set. $D^{(S)}$ denotes the ring $D + XD_S[X]$, where S is a multiplicatively closed subset of D. We show that the ring $D^{(S)}$ is an APVMD if and only if $D^{(S)}$ is well behaved, D and $D_S[X]$ are APVMDs, and S is an almost t-splitting set in D.

1. Introduction. Throughout this paper, D will be an integral domain with quotient field K, $\overline{D}$ the integral closure of D, and X an indeterminate over D.

In this paper we shall use the notions of $*$-operations. Let $F(D)$ denote the set of nonzero fractional ideals of D. A function $*\colon F(D) \to F(D)$, written as $A \mapsto A_*$, is called a *$*$-operation* if for all $A, B \in F(D)$ and for all $a \in K - \{0\}$, (i) $(a)_* = (a)$ and $(aA)_* = aA_*$, (ii) $A \subseteq A_*$ and $A \subseteq B$ implies $A_* \subseteq B_*$, and (iii) $(A_*)_* = A_*$.

We review some terminology related to the v-, w- and t-operations. For $I \in F(D)$, set $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, $I_t = \bigcup J_v$, the union being taken over all finitely generated subideals J of I, and $I_w = \{x \in K \mid xJ \subseteq I$ with $J^{-1} = D$ for some finitely generated fractional ideal J of $D\}$. If $I = I_v$ (resp., $I = I_t$, $I = I_w$), then I is said to be a *v-ideal* (resp., a *t-ideal*, a *w-ideal*).

A $*$-ideal I is said to be *of finite type* if $I = J_*$ for some finitely generated ideal $J \in F(D)$. An ideal maximal with respect to being a t-ideal is called a maximal t-ideal. A maximal t-ideal is a prime ideal. For $I \in F(D)$, I is t-invertible if it satisfies $(II^{-1})_t = D$. For details on $*$-operations the readers may consult Sections 32 and 34 of [6].

In [9], M. Zafrullah began to develop a general theory of almost factoriality. One important class of integral domains introduced in [9] was that

2010 *Mathematics Subject Classification*: 13G05, 13A15, 13F20.
Key words and phrases: APVMD, well behaved, AV-domain, almost t-splitting set.

DOI: 10.4064/cm121-2-6

of almost GCD-domains (AGCD-domains). He defined D to be an *AGCD-domain* if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that $a^nD \cap b^nD$ is principal (or equivalently, $(a^n, b^n)_v$ is principal). Recall that a *GCD-domain* D is characterized by the property that for all $a, b \in D \setminus \{0\}$, $(a, b)_v$ is principal. So a GCD-domain is an AGCD-domain. According to [7], an integral domain D is defined to be an *almost Prüfer v-multiplication domain* (APVMD) if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is t-invertible. It is easily seen that an APVMD is a generalization of an AGCD-domain. Recall that D is defined to be a *Prüfer v-multiplication domain* (PVMD) if for each $a, b \in D \setminus \{0\}$, (a, b) is t-invertible. Obviously a PVMD is an APVMD, but an APVMD is not necessarily a PVMD.

According to [3, Theorem 4.17], the domain $\mathbb{Z}+2i\mathbb{Z} = \mathbb{Z}[2i]$ is an AGCD-domain that is not integrally closed. By [7, Theorem 3.1], R is an APVMD with torsion t-class group if and only if R is an AGCD-domain. Hence an APVMD is not integrally closed. Thus an APVMD need not be a PVMD since a PVMD is integrally closed. From [7, Theorem 2.4], we know that D is an integrally closed APVMD if and only if D is a PVMD. In Section 2, we show that a locally APVMD is not necessarily an APVMD. However, we show that D is an APVMD if and only if D is a locally APVMD and D is well behaved. Also, we prove that D is an APVMD if and only if $D \subseteq \overline{D}$ is a root extension, $\overline{D}$ is a PVMD and D is t-linked under $\overline{D}$. Recall that an extension $D \subseteq R$ of integral domains is said to be a *root extension* if for each $x \in R$ there exists a natural number n (depending on x) with $x^n \in D$. According to [11], D is *t-linked* under an integral domain R if for each finitely generated fractional ideal A of D such that $(AR)^{-1} = R$ (or equivalently, $(AR)_v = R$), one has $(AD)^{-1} = D$ (or equivalently, $(AD)_v = D$). Here note that the first "v" is the v-operation on R, but the second "v" is the v-operation on D.

In [7, Theorem 3.10], we have proved that D is an APVMD if and only if $D + XK[X]$ is an APVMD. We know that $K = D_S$ with $S = D \setminus \{0\}$. Our next goal is to study the composite polynomial ring $D + XD_S[X] = \{f(X) \in D_S[X] \mid f(0) \in D\}$ for any multiplicatively closed set S of D when D is an APVMD. For convenience, $D^{(S)}$ will denote the ring $D + XD_S[X]$. In Section 4 we show that if D is an APVMD then $D^{(S)}$ need not be an APVMD. So we investigate the conditions under which $D^{(S)}$ is an APVMD.

Recall that a saturated multiplicatively closed subset S of D is said to be *t-splitting* if for every nonzero $d \in D$ we have $(d) = (AB)_t$, where A and B are integral ideals of D with $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. In [1, Theorem 2.5], it was shown that $D^{(S)}$ is a PVMD if and only if D is a PVMD and S is a t-splitting set.

In Section 3, we introduce the notion of an almost t-splitting set. We say that a saturated multiplicatively closed subset S of D is *almost t-splitting* if for every nonzero $d \in D$, there is a positive integer $n = n(d)$ such that $(d^n) = (AB)_t$, where A and B are integral ideals of D with $A_t \cap sD = sA_t$ (or equivalently, $(A, s)_t = D$) for all $s \in S$ and $B_t \cap S \neq \emptyset$. In Section 4, we prove that the ring $D^{(S)}$ is an APVMD if and only if $D^{(S)}$ is well behaved, D and $D_S[X]$ are APVMDs, and S is an almost t-splitting set in D. At the same time, we show that $D^{(S)}$ is an AP-domain (respectively, AB-domain) if and only if D is an AP-domain (respectively, AB-domain) and $D_S = K$. According to [3], an integral domain D is defined to be an *almost Bézout domain* (*AB-domain*) if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is principal; while D is an *almost Prüfer domain* (*AP-domain*) if for each $a, b \in D \setminus \{0\}$, there is a positive integer $n = n(a, b)$ such that (a^n, b^n) is invertible. Obviously, AB-domains and AP-domains are APVMDs.

2. Basic theory of APVMDs. In [10, Corollary 4.4], it is shown that D is a PVMD if and only if D is a locally PVMD (i.e., if for every maximal ideal P, D_M is a PVMD) and D is well behaved. We shall extend this result to APVMDs. Recall that an integral domain D is *well behaved* (respectively, *conditionally well behaved*) if for every prime (respectively, maximal) t-ideal P, PD_P is also a t-ideal of D_P. Here we say that D is a *locally* APVMD if for every maximal ideal M, D_M is an APVMD (or equivalently, for every prime ideal P of D, D_P is an APVMD). Note that given a prime ideal P of D, there exists a maximal ideal M of D with $P \subseteq M$. Every localization of an APVMD is also an APVMD, by [7, Proposition 3.4]. Therefore, if D_M is an APVMD, then $D_P = (D_M)_{P_M}$ is also an APVMD.

As in [3], D is said to be an *almost valuation domain* (*AV-domain*) if for any nonzero $a, b \in D$, there exists a positive integer $n = n(a, b)$ with $a^n \mid b^n$ or $b^n \mid a^n$. By [7, Theorem 2.3], D is an APVMD if and only if D_P is an AV-domain for each prime t-ideal P of D, and if and only if D_M is an AV-domain for each maximal t-ideal M of D.

LEMMA 2.1. *Let D be an AV-domain. Then* $\mathrm{Spec}(D)$ *is totally ordered.*

Proof. Assume that $P_1, P_2 \in \mathrm{Spec}(D)$ and P_1 is not included in P_2. Then there exists $a \in P_1 \setminus P_2$. For each nonzero $b \in P_2$, we have $(b^n) \subseteq (a^n)$ for some positive integer n. Indeed, if (b^n) is not included in (a^n), then $(a^n) \subseteq (b^n)$ since D is an AV-domain. So $(a^n) \subseteq (b^n) \subseteq P_2$, and hence $a \in P_2$, a contradiction. Therefore, $(b^n) \subseteq (a^n) \subseteq P_1$, so $b \in P_1$. Thus $P_2 \subseteq P_1$. So $\mathrm{Spec}(D)$ is totally ordered. ∎

LEMMA 2.2. *Let D be an integral domain with* $\mathrm{Spec}(D)$ *totally ordered. Then every nonzero prime ideal of D is a t-ideal.*

Proof. This follows from [5, Theorem 9.1.2]. ∎

Example 2.3. According to [11, Example 4.7], let E be the ring of entire functions and S be the multiplicatively closed set generated by principal nonzero primes of E. Then $E^{(S)} = E + XE_S[X]$ is a locally GCD-domain that is not a PVMD. By [12, Proposition 4.3], $E^{(S)}$ is not well behaved. So $E^{(S)}$ is not an APVMD by Theorem 2.4. Note that a locally GCD-domain is a locally AGCD-domain, and thus a locally APVMD. So $E^{(S)}$ is a locally APVMD. Hence we conclude that a locally APVMD need not be an APVMD. However we have

Theorem 2.4. *The following are equivalent:*

(1) *D is an APVMD.*
(2) *D is a locally APVMD and D is well behaved.*
(3) *D is a locally APVMD and D is conditionally well behaved.*

Proof. (1)$\Rightarrow$(2): If D is an APVMD, then D is a locally APVMD by [7, Proposition 3.4]. For each nonzero prime t-ideal P of D, D_P is an AV-domain. Then by Lemmas 2.1 and 2.2, PD_P is a t-ideal of D_P. Therefore, D is well behaved.

(2)$\Rightarrow$(3): This is clear.

(3)$\Rightarrow$(1): For each nonzero maximal t-ideal M of D, D_M is an APVMD and MD_M is a t-ideal of D_M. So $D_M = (D_M)_{MD_M}$ is an AV-domain. Thus D is an APVMD. ∎

Analogously, D is a PVMD if and only if D is a locally PVMD and D is well behaved, by [10, Corollary 4.4]. Now recall that a nonzero prime ideal U of the polynomial ring $D[X]$ (in one indeterminate X) with $U \cap D = 0$ is called an *upper to zero*. The domain D is said to be a *UMT-domain* if every upper to zero in $D[X]$ is a maximal t-ideal.

Proposition 2.5. *Let D be an APVMD. Then D is t-linked under $\overline{D}$.*

Proof. Assume that $(I\overline{D})^{-1} = \overline{D}$ for each finitely generated fractional ideal I of D. We need to show that $(ID)^{-1} = D$, or equivalently, $(ID)_t = D$. If $(ID)_t \neq D$, then $(ID)_t \subseteq M$ for some maximal t-ideal of D. Since $D \subseteq \overline{D}$ is an integral extension, there exists a prime ideal P of $\overline{D}$ such that $M = P \cap D$. Because an APVMD is a UMT-domain by [7, Theorem 3.8], P is a t-ideal by [5, Proposition 1.4]. As $I\overline{D} = (ID)\overline{D} \subseteq M\overline{D} = (P \cap D)\overline{D} \subseteq P$, we have $\overline{D} = (I\overline{D})_t \subseteq P_t = P$, a contradiction. Therefore, $(ID)_t = D$. So D is t-linked under $\overline{D}$. ∎

Corollary 2.6. *D is an APVMD if and only if $D \subseteq \overline{D}$ is a root extension, $\overline{D}$ is a PVMD and D is t-linked under $\overline{D}$.*

Proof. ($\Leftarrow$) This follows from [7, Theorem 3.7].

($\Rightarrow$) This follows from Proposition 2.5 and [7, Theorem 3.6]. ∎

3. Almost t-splitting sets. We say $d \in D^* = D \setminus \{0\}$ is an *almost t-split by S* if there exists an $n = n(d)$ with $(d^n) = (AB)_t$ for some integral ideals A and B of D, where $A_t \cap sD = sA_t$ (or equivalently, $(A, s)_t = D$) for all $s \in S$ and $B_t \cap S \neq \emptyset$. Note that A, B are both t-invertible. We say that S is an *almost t-splitting set* in D if for each $d \in D^*$ is an almost t-split by S.

LEMMA 3.1. *Suppose that D is an integral domain, S is a multiplicatively closed subset of D and $d \in D^*$ is an almost t-split by S. Then there exists an $n = n(d)$ with $(d^n) = (AB)_t$ for some integral ideals A and B of D, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. Thus $A_t = d^n D_S \cap D$, hence $d^n D_S \cap D$ is a t-invertible t-ideal. Also, $B_t = d^n A^{-1}$.*

Proof. We only need to show that $A_t = d^n D_S \cap D$. Since $A_t B_t \subseteq (AB)_t \subseteq (d^n)$, we have $A_t \subseteq A_t D_S \cap D = A_t B_t D_S \cap D \subseteq (AB)_t D_S \cap D \subseteq d^n D_S \cap D$. Note that $B_t D_S = D_S$ since $B_t \cap S \neq \emptyset$. On the other hand, let $x \in d^n D_S \cap D$, so that $sx \in (d^n)$ for some $s \in S$. Then $sx \in (AB)_t \subseteq A_t$. So $sx \in A_t \cap sD = sA_t$, and hence $x \in A_t$. Therefore $A_t = d^n D_S \cap D$. ∎

LEMMA 3.2. *Suppose that D is an integral domain and S is a multiplicatively closed subset of D. Let $d \in D^*$ be such that $d^n D_S \cap D$ is t-invertible for some $n = n(d)$. Then d is an almost t-split by S.*

Proof. Let $A = d^n D_S \cap D$. Hence A is a t-ideal. Clearly $(d^n) \subseteq A$. Set $B = d^n A^{-1}$. Then B is an integral t-invertible t-ideal of D and $(d^n) = (AB)_t$. Now $B_S = (d^n A^{-1})_S = d^n D_S (A^{-1})_S = d^n D_S (A_S)^{-1} = d^n D_S (d^n D_S)^{-1} = D_S$. Hence $B \cap S \neq \emptyset$. Next we show that $A \cap sD = sA$. Clearly it suffices to show that $A \cap sD \subseteq sA$. Let $x \in A \cap sD$. Then $x = sb$ for some $b \in D$. Hence $b = x/s \in A_S \cap D = d^n D_S \cap D = A$. So $x = sb \in sA$. ∎

The following is a straightforward consequence of Lemmas 3.1 and 3.2.

COROLLARY 3.3. *Suppose that D is an integral domain and S is a multiplicatively closed subset of D. Then $d \in D^*$ is an almost t-split by S if and only if $d^n D_S \cap D$ is t-invertible for some $n = n(d)$. Hence S is an almost t-splitting set in D if and only if for each $d \in D^*$, $d^n D_S \cap D$ is t-invertible for some $n = n(d)$.*

Let S be a multiplicatively closed subset of D. Recall that a prime ideal Q of D with $Q \cap S \neq \emptyset$ is said to *intersect S in detail* if $P \cap S \neq \emptyset$ for each prime ideal $P \subseteq Q$.

LEMMA 3.4. *Suppose that D is an integral domain, S is an almost t-splitting set in D and Q is a prime t-ideal of D with $Q \cap S \neq \emptyset$. Then Q intersects S in detail.*

Proof. Let $0 \neq P \subseteq Q$ be a prime ideal of D. Let $0 \neq x \in P$. Then we can shrink P to a prime ideal minimal over (x) which is a t-ideal. Thus we can assume that P is a t-ideal. Suppose that $P \cap S = \emptyset$. As S is an almost t-splitting set, $(x^n) = (AB)_t$, where $B_t \cap S \neq \emptyset$ and $(A, s)_t = D$ for each $s \in S$. Then $A_t B_t \subseteq (AB)_t = (x^n) \in P$ and B_t is not included in P since $B_t \cap S \neq \emptyset$. Thus $A_t \subseteq P \subseteq Q$. Let $s \in Q \cap S$. Then $D = (A, s)_t \subseteq Q$, a contradiction. So $P \cap S \neq \emptyset$. Therefore Q intersects S in detail. ∎

PROPOSITION 3.5. *Let D be an APVMD and S a saturated almost t-splitting set in D. Then S is also a saturated almost t-splitting set in $\overline{D}$.*

Proof. For each $d \in \overline{D} \setminus \{0\}$, $d^n \in D$ for some $n \geq 1$ since $D \subseteq \overline{D}$ is a root extension. As S is an almost t-splitting set in D, there exists a positive integer m such that $(d^n)^m = (d^{nm}) = (AB)_t$ for some finitely generated ideals A and B of D, where $A_t \cap sD = sA_t$ (or equivalently, $(A, s)_t = D$) and $B_t \cap S \neq \emptyset$. Because $\overline{D}$ is t-linked over D by the proof of [7, Theorem 3.6], $((A, s)\overline{D})_t = \overline{D}$. ∎

4. The ring $D + XD_S[X]$. Recall that an overring T of a domain D is said to be a *w-domain* over D if T, as a D-module, is a w-module. Clearly, if T is a flat D-module, then T is a w-domain over D. From [8, Theorem 8.8.2], it follows that T is a *w-domain* over D if and only if for every w-ideal I of T, $I \cap D$ is a w-ideal of D.

According to [1, Theorem 2.5], $D + XD_S[X]$ is a PVMD if and only if D is a PVMD and S is a t-splitting set. Now we shall consider the $D + XD_S[X]$ construction from an APVMD. By Example 2.3, we know that if E is the ring of entire functions and S the multiplicatively closed set generated by the principal primes of E, then $E^{(S)} = E + XE_S[X]$ is not an APVMD. But we note that E is an APVMD. Thus we conclude that $D^{(S)} = D + XD_S[X]$ is not necessarily an APVMD when D is an APVMD. However we have

THEOREM 4.1. *Let D be an integral domain and S a saturated multiplicatively closed subset of D. Then $D^{(S)} = D + XD_S[X]$ is an APVMD if and only if*

(1) *$D^{(S)}$ is well behaved,*
(2) *D and $D_S[X]$ are APVMDs,*
(3) *S is an almost t-splitting set in D.*

Proof. ($\Rightarrow$) Suppose that $D^{(S)} = D + XD_S[X]$ is an APVMD. Then $D^{(S)}$ is well behaved by Theorem 2.4. Since $D_S[X] = (D^{(S)})_S$, $D_S[X]$ is an APVMD by [7, Proposition 3.4].

We next show that D is an APVMD. Let $x, y \in D$. Then there exists an integer $n \geq 1$ such that $(x^n, y^n)D^{(S)}$ is t-invertible, and hence $(x^n, y^n)^{-1}D^{(S)} = ((x^n, y^n)D^{(S)})^{-1}$ is a t-invertible t-ideal of $D^{(S)}$. So by [4, Proposition 3.9], $(x^n, y^n)^{-1}$ is a t-invertible t-ideal of D. Hence D is an APVMD.

Next we claim that S is an almost t-splitting set. By [7, Theorem 3.6], the integral closure $\overline{D^{(S)}}$ of $D^{(S)}$is a PVMD. The integral closure of $D^{(S)}$ is $\overline{D^{(S)}} = \overline{D} + X\overline{D}_S[X]$ by [2, Theorem 2.7], where $\overline{D}^{(S)} = \overline{D} + X\overline{D}_S[X]$. Then by [1, Theorem 2.5], $\overline{D}$ is a PVMD and S is a t-splitting set in $\overline{D}$. Let $d \in D^* \subseteq \overline{D}$. By [1, Lemma 2.4], $(d, X)\overline{D}^{(S)}$ is t-invertible in $\overline{D}^{(S)}$. Because $\overline{D}^{(S)} = \overline{D^{(S)}}$, $(d, X)\overline{D^{(S)}}$ is t-invertible in $\overline{D^{(S)}}$. Thus there exists a finitely generated ideal J of $\overline{D^{(S)}}$ such that $((d, X)J\overline{D^{(S)}})_t = \overline{D^{(S)}}$. Set $J = (f_1, \ldots, f_n) \subseteq \overline{D^{(S)}}$. Since $D^{(S)}$ is an APVMD, $D^{(S)} \subseteq \overline{D^{(S)}}$ is a root extension. So there exists a positive integer m with $(f_i)^m \in D^{(S)}$ for $i = 1, \ldots, n$. By [3, Lemma 3.3], $(((f_1)^m, \ldots, (f_n)^m)\overline{D^{(S)}})_t = ((f_1, \ldots, f_n)^m\overline{D^{(S)}})_t = (J^m\overline{D^{(S)}})_t$. We have $((d, X)^m J^m\overline{D^{(S)}})_t = \overline{D^{(S)}}$ because $((d, X)J\overline{D^{(S)}})_t = \overline{D^{(S)}}$. Hence $((d, X)^m((f_1)^m, \ldots, (f_n)^m)\overline{D^{(S)}})_t = \overline{D^{(S)}}$. As we know, $D^{(S)}$ is t-linked under $\overline{D^{(S)}}$, so $((d, X)^m((f_1)^m, \ldots, (f_n)^m)D^{(S)})_t = D^{(S)}$. Therefore $(d, X)D^{(S)}$ is t-invertible in $D^{(S)}$. Then by [1, Lemma 2.4], S is a t-splitting set in D. Thus S is an almost t-splitting set in D.

($\Leftarrow$) Let P be a prime t-ideal of $D^{(S)}$. To show that $D^{(S)}$ is an APVMD, it suffices to show that $(D^{(S)})_P$ is an AV-domain. If $P \cap D = 0$, then $(D^{(S)})_P$ is a DVR and thus an AV-domain. Assume that $P \cap D \neq 0$. We claim that $p = P \cap D$ is a prime t-ideal of D. Since $D \subseteq D^{(S)}$ is a flat extension, $D^{(S)}$ is a flat D-module. Therefore $D^{(S)}$ is a w-domain over D, hence $p = P \cap D$ is a prime w-ideal of D. Because D is an APVMD, D is a UMT-domain by [7, Theorem 3.8]. Hence a prime w-ideal is a t-ideal. So $p = P \cap D$ is a prime t-ideal of D.

Case 1: Suppose that $P \cap S \neq \emptyset$. Then $(D^{(S)})_{D-p} = D_p + XD_{S(D-p)}[X] = D_p + XK[X]$ is a AB-domain by [3, Theorem 4.9]. Thus $(D^{(S)})_P$ is a localization of the AB-domain $D_p + XK[X]$. Here the equality $D_{S(D-p)} = K$ follows from Lemma 3.4 and hence $(D^{(S)})_P$ is a quasi-local AB-domain by [3, Theorem 4.6]. So $(D^{(S)})_P$ is an AV-domain.

Case 2: Suppose that $P \cap S = \emptyset$. Since $D^{(S)}$ is well behaved, it follows that $P(D^{(S)})_S = PD_S[X]$ is a prime t-ideal by [12, Corollary 1.3]. Then $(D^{(S)})_P = ((D^{(S)})_S)_{PD_S[X]} = (D_S[X])_{PD_S[X]}$ is an AV-domain because $D_S[X]$ are APVMDs. Therefore $D^{(S)}$ is an APVMD . ∎

From [3, Theorem 4.9], we know that D is an AB-domain (respectively, AP-domain) if and only if $D + XK[X]$ is an AB-domain (respectively, AP-domain). To generalize these results, we naturally consider the conditions under which $D^{(S)} = D + XD_S[X]$ is an AB-domain (respectively, AP-domain) for any saturated multiplicatively closed subset S of D.

Theorem 4.2. *Let D be an integral domain and S a saturated multiplicatively closed subset of D. Then $D^{(S)} = D + XD_S[X]$ is an AP-domain if and only if D is an AP-domain and $D_S = K$.*

Proof. ($\Leftarrow$) This follows from [2, Theorem 4.9].

($\Rightarrow$) If $D^{(S)} = D + XD_S[X]$ is an AP-domain, then the integral closure $\overline{D^{(S)}}$ of $D^{(S)}$ is a Prüfer domain and $D^{(S)} \subseteq \overline{D^{(S)}}$ is a root extension by [3, Corollary 4.8]. Note that $\overline{D^{(S)}} = \overline{D} + X\overline{D}_S[X]$ by [2, Theorem 2.7]. Thus by [1, Theorem 3.6], $\overline{D}$ is a Prüfer domain and $\overline{D}_S = K$.

We now show that $D_S = K$. Since $D \subseteq \overline{D}$ is an integral extension, for each prime ideal P of D there exists a prime ideal M of $\overline{D}$ such that $M \cap D = P$. Note that $\overline{D}_S = K$ if and only if each nonzero prime ideal of $\overline{D}$ meets S. So $M \cap S \neq \emptyset$. Hence $P \cap S = (M \cap D) \cap S = M \cap S \neq \emptyset$. Therefore $D_S = K$.

We claim that $D \subseteq \overline{D}$ is a root extension. For each $x \in \overline{D} \subseteq \overline{D^{(S)}}$, we have $x^n \in D^{(S)}$ for some integer $n \geq 1$. Also $x^n \in K$. So $x^n \in D^{(S)} \cap K = D$. Therefore, D is an AP-domain by [3, Corollary 4.8]. ∎

Recall that the *t-class group* $Cl_t(D)$ is defined to be the group of t-invertible t-ideals of D modulo the subgroup of principal ideals of D.

COROLLARY 4.3. *Let D be an integral domain and S a saturated multiplicatively closed subset of D. Then $D^{(S)} = D + XD_S[X]$ is an AB-domain if and only if D is an AB-domain and $D_S = K$.*

Proof. ($\Leftarrow$) This follows from [3, Theorem 4.9].

($\Rightarrow$) If $D^{(S)} = D + XD_S[X]$ is an AB-domain, then $D^{(S)}$ is an AP-domain. Hence by Theorem 4.2, D is an AP-domain and $D_S = K$. Thus $\overline{D}$ is a Prüfer domain and $D \subseteq \overline{D}$ is a root extension by [3, Corollary 4.8]. Since $\overline{D}$ is integrally closed, the groups $Cl_t(\overline{D})$ and $Cl_t(\overline{D^{(S)}})$ are isomorphic by [4, Corollary 4.5]. Since $Cl_t(\overline{D^{(S)}})$ is torsion by [3, Corollary 4.8], so is $Cl_t(\overline{D})$. Thus by [3, Corollary 4.8], D is an AB-domain. ∎

Acknowledgments. The author would like to thank the referee for valuable comments and suggestions.

REFERENCES

[1] D. D. Anderson, D. F. Anderson and M. Zafrullah, *The ring $D + XD_S[X]$ and t-splitting sets*, Arab. J. Sci. Engrg. Sect. C Theme Issues 26 (2001), 3–16.

[2] —, —, —, *Rings between $D[X]$ and $K[X]$*, Houston J. Math. 17 (1991), 109–129.

[3] D. D. Anderson and M. Zafrullah, *Almost Bézout domains*, J. Algebra 142 (1991), 285–309.

[4] D. F. Anderson, S. El Baghdadi and S. E. Kabbaj, *On the class group of $A+XB[X]$ domains*, in: Advances in Commutative Ring Theory, Lecture Notes in Pure Appl. Math. 205, Dekker, New York, 1999, 73–85.

[5] M. Fontana, S. Gabelli and E. Houston, *UMT-domains and domains with Prüfer integral closure*, Comm. Algebra 26 (1998), 1017–1039.

[6] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.

[7] Q. Li, *On almost Prüfer v-multiplication domains*, Algebra Colloq., to appear.
[8] F. G. Wang, *Commutative Rings and the Theory of Star Operations*, Science Press, Beijing, 2006 (in Chinese).
[9] M. Zafrullah, *A general theory of almost factoriality*, Manuscripta Math. 51 (1985), 29–62.
[10] —, *The* $D + XD_S[X]$ *construction from GCD-domains*, J. Pure Appl. Algebra 50 (1988), 93–107.
[11] —, *Various facets of rings between* $D[X]$ *and* $K[X]$, Comm. Algebra 31 (2003), 2497–2540.
[12] —, *Well behaved prime t-ideals*, J. Pure Appl. Algebra 65 (1990), 199–207.

Qing Li
College of Computer Science and Technology
Southwest University for Nationalities
Chengdu 610041, P.R. China
E-mail: lqop80@163.com

Received 22 October 2009;
revised 15 March 2010 (5291)