COLLOQUIUM MATHEMATICUM

VOL. 121 2010 NO. 2

A MEASURE OF AXIAL SYMMETRY OF CENTRALLY SYMMETRIC CONVEX BODIES

BY

MAREK LASSAK and MONIKA NOWICKA (Bydgoszcz)

Abstract. Denote by K_m the mirror image of a planar convex body K in a straight line m. It is easy to show that $K^*_m = \mathrm{conv}(K \cup K_m)$ is the smallest by inclusion convex body whose axis of symmetry is m and which contains K. The ratio $\mathrm{axs}(K)$ of the area of K to the minimum area of K^*_m over all straight lines m is a measure of axial symmetry of K. We prove that $\mathrm{axs}(K) > \frac{1}{2}\sqrt{2}$ for every centrally symmetric convex body and that this estimate cannot be improved in general. We also give a formula for $\mathrm{axs}(P)$ for every parallelogram P.

1. Introduction. Denote by E^2 the Euclidean plane. The mirror image of a convex body K in a straight line m is denoted by K_m. We put $K^*_m = \mathrm{conv}(K \cup K_m)$ and call m the *mirror line*. It is easy to show that K^*_m *is the smallest by inclusion convex body containing* K *whose axis of symmetry is* m. Recall two claims formulated in [11].

Claim 1.1. *Let* $K \subset E^2$ *be a convex body. If the position of a straight line* m *varies continuously, then* $\mathrm{area}(K^*_m)$ *varies continuously.*

Claim 1.2. *Let* $K \subset E^2$ *be a convex body and let* m *and* n *be two parallel straight lines such that only* m *passes through* K. *Then* $\mathrm{area}(K^*_m) < \mathrm{area}(K^*_n)$.

By these claims and by compactness arguments we conclude that the infimum of the area of K^*_m over all straight lines m is attained. So using the term minimum instead of infimum is correct here (the same remark concerns many other places of the paper where we consider compact families of straight lines m).

The number

$$\mathrm{axs}(K) = \frac{\mathrm{area}(K)}{\min\limits_{m} \mathrm{area}(K^*_m)}$$

is the *measure of axial symmetry of* K that we consider in this paper. This measure and miscellaneous other measures of axial symmetry are discussed

2010 *Mathematics Subject Classification*: Primary 52A10; Secondary 52A38.

Key words and phrases: convex body, parallelogram, axis of symmetry, area, measure of axial symmetry.

DOI: 10.4064/cm121-2-12

in [1–12]. In particular, the appendix of [8] gives an overview of measures of symmetry of convex bodies, including axial symmetry measures. Also Section 4.2 of the survey article [9] considers measures of symmetry of convex bodies, and in particular their measures of axial symmetry.

We conjecture that $\operatorname{axs}(K) > \frac{1}{2}\sqrt{2}$ for every convex body $K \subset E^2$.

From [10] we know that $\operatorname{axs}(K) \geq 16/31$ for every convex body $K \subset E^2$. That paper also considers approximation of planar convex bodies by some specific axially symmetric convex bodies like rhombi and isosceles triangles.

In [11] it is proved that $\operatorname{axs}(K) > \frac{1}{2}\sqrt{2}$ for every triangle K and that this estimate cannot be improved. Also final estimates for right-angled and acute triangles are given.

The main aim of the present paper is to prove that $\operatorname{axs}(K) > \frac{1}{2}\sqrt{2}$ for every centrally symmetric convex body K (see Theorem 4.1) and that this estimate cannot be improved. More precisely, it cannot be improved for the family of parallelograms; see Theorem 2.1 which gives the value of $\operatorname{axs}(P)$ for an arbitrary parallelogram P. The proof of the inequality $\operatorname{axs}(K) > \frac{1}{2}\sqrt{2}$ in Theorem 4.1 is based on Theorem 3.1 which says that for any centrally symmetric planar convex body K there exists an axially symmetric convex octagon Q circumscribed about K such that $\operatorname{area}(K) > \frac{1}{2}\sqrt{2} \cdot \operatorname{area}(Q)$.

Our proofs of Theorems 3.1 and 4.1 are based on similar ideas to those in the proof of Theorem 8 of [7]. That theorem and its proof claim the weaker inequalities $\operatorname{axs}(K) \geq \frac{1}{2}\sqrt{2}$ and $\operatorname{area}(K) \geq \frac{1}{2}\sqrt{2} \cdot \operatorname{area}(Q)$ for every centrally symmetric K, but the proof in [7] is partially incorrect (see our comment at the end of Section 3).

We omit an easy proof of the following claim.

CLAIM 1.3. *If $K \subset E^2$ is a centrally symmetric convex body with center o and if two perpendicular straight lines m_1 and m_2 pass through o, then $K^*_{m_1}$ and $K^*_{m_2}$ coincide.*

2. Measure of axial symmetry of parallelograms. In the following theorem we give a formula for the measure $\operatorname{axs}(P)$ of axial symmetry of an arbitrary parallelogram $P = abcd$. For definiteness, we assume that $\angle bad \leq \pi/2$ and $|ad| \leq |ab|$. We use the same symbol to denote an angle and its measure.

THEOREM 2.1. *Let $P = abcd$ be a parallelogram such that $\angle bad \leq \pi/2$ and $|ad| \leq |ab|$. Put $\alpha = \angle bac$ and $\beta = \angle abd$. If $|ac|/|ab| \leq \sqrt{2}$, then*

$$\operatorname{axs}(P) = \frac{\sin(\alpha+\beta)}{2\sin\beta\cos\alpha}. \tag{1}$$

If $|ac|/|ab| \geq \sqrt{2}$, *then*

$$\operatorname{axs}(P) = \frac{\sin\beta}{\sin(\alpha+\beta)\cos\alpha}. \tag{2}$$

Proof. Without loss of generality, we assume that $|ab| = 1$. Denote by o the center of symmetry of P. We put $\gamma = \angle adb$ and $\delta = \angle oad$.

Since $\angle bad \leq \pi/2$ and $|ad| \leq |ab|$, we get $|ob| \leq |ao| \leq |ab|$ and $\angle bao \leq \angle oba$. Thus $\alpha \leq \beta$. Moreover, $\pi/2 \leq \angle aob$, $\angle bad \leq \pi/2$ and $\angle bad = \alpha + \delta$. Hence

$$\alpha + \beta \leq \pi/2 \quad \text{and} \quad \alpha + \delta \leq \pi/2. \tag{3}$$

From $\delta < \alpha + \delta$ and the right inequality in (3) we obtain $\delta < \pi/2$. Of course, $\angle acb = \delta$. Clearly in the triangle abc we have $|bc| \leq |ab| \leq |ac|$, which implies $\alpha \leq \delta$. To summarize,

$$\alpha \leq \delta < \pi/2. \tag{4}$$

By the sine theorem we get

$$|oc| = |ao| = \frac{\sin\beta}{\sin(\alpha+\beta)} \quad \text{and} \quad |od| = |ob| = \frac{\sin\alpha}{\sin(\alpha+\beta)}. \tag{5}$$

Further we limit ourselves to the mirror lines m through o because it is not difficult to see that if n is a straight line parallel to m, then $\operatorname{area}(P_m^*) \leq \operatorname{area}(P_n^*)$.

If p is a point, then we denote by p_m the mirror image of p in the line m.

Let l be the straight line through o parallel to the side ab. Denote by φ the angle between l and m. From Claim 1.3 we see that it is sufficient to minimize $\operatorname{area}(P_m^*)$ over all $\varphi \in [0, \pi/2]$.

In four subintervals of $[0, \pi/2]$ we have different formulas for $\operatorname{area}(P_m^*)$. So we consider four cases. The corresponding formulas $f_1(\varphi), \ldots, f_4(\varphi)$ are discussed in Cases 1–4 below.

Case 1: $\varphi \in [0, \alpha]$. In this case P_m^* is the octagon $ad_mbc_mcb_mda_m$ (see Fig. 1). Obviously,

$$P_m^* = oc_mc \cup ocb_m \cup ob_md \cup oda_m \cup oa_ma \cup oad_m \cup od_mb \cup obc_m.$$

Observe that the angle between m and the straight line through a and c is $\alpha - \varphi$.

From this and since c_m is symmetric to c with respect to m we get $\angle c_moc = 2(\alpha - \varphi)$. Now $|oc| = |oc_m|$ implies $\operatorname{area}(oc_mc) = \frac{1}{2}|oc|^2 \sin(2\alpha - 2\varphi)$. Since the triangles oc_mc and oa_ma are symmetric with respect to o, we see that $\operatorname{area}(oc_mc) = \operatorname{area}(oa_ma)$. Of course, $\angle c_mob_m = \alpha + \beta$. From this, from $\angle c_moc = 2\alpha - 2\varphi$ and $\angle cob_m = \angle c_mob_m - \angle c_moc$ we get $\angle cob_m = \beta - \alpha + 2\varphi$. This and $|ob_m| = |ob|$ lead to $\operatorname{area}(ocb_m) = \frac{1}{2}|oc|\,|ob| \sin(\beta - \alpha + 2\varphi)$. Clearly, $\operatorname{area}(ocb_m) = \operatorname{area}(oda_m) = \operatorname{area}(oad_m) = \operatorname{area}(obc_m)$. Observe that $\angle cod =$

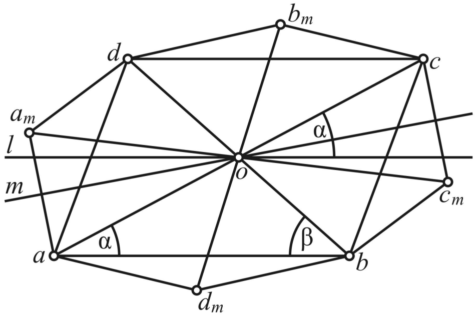

Fig. 1

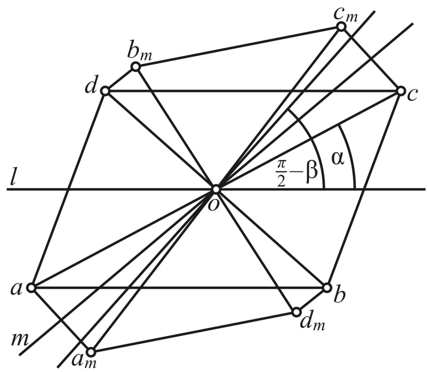

Fig. 2

$\pi-\alpha-\beta$. Hence, by $\angle b_m od = \angle cod - \angle cob_m$ and $\angle cob_m = \beta-\alpha+2\varphi$ we get $\angle b_m od = \pi - 2\beta - 2\varphi$. Now $|ob_m| = |ob|$ and (5) imply that $\mathrm{area}(ob_m d) = \frac{1}{2}|ob|^2 \sin(\pi-2\beta-2\varphi)$. Of course, $\mathrm{area}(ob_m d) = \mathrm{area}(od_m b)$. All this implies that $\mathrm{area}(P_m^*)$ is given by the function

$$f_1(\varphi) = |oc|^2 \sin(2\alpha-2\varphi) + 2|oc||ob|\sin(\beta-\alpha+2\varphi) + |ob|^2 \sin(\pi-2\beta-2\varphi).$$

Since $\varphi \in [0,\alpha]$, we conclude that

$$0 \le 2\alpha - 2\varphi \le 2\alpha,$$
$$\beta-\alpha \le \beta-\alpha+2\varphi \le \alpha+\beta,$$
$$\pi - 2(\alpha+\beta) \le \pi - 2\beta - 2\varphi \le \pi - 2\beta.$$

From the first inequality and $\alpha \le \pi/4$ (which results from $\alpha \le \beta$ and the left inequality of (3)) we obtain $0 \le 2\alpha - 2\varphi \le \pi/2$. From the second displayed inequality, the left inequality of (3) and $\alpha \le \beta$ we get $0 \le \beta-\alpha+2\varphi \le \pi/2$. From the third displayed inequality, the left inequality of (3) and $0 < \beta$ we see that $0 \le \pi - 2\beta - 2\varphi < \pi$. Consequently, $2\alpha - 2\varphi$, $\beta-\alpha+2\varphi$ and $\pi-2\beta-2\varphi$ belong to $[0,\pi]$. Since the sine function is concave in this interval, $f_1(\varphi)$ is concave there.

Case 2: $\varphi \in [\alpha, \pi/2-\beta]$ for $\gamma \le \pi/2$ and $\varphi \in [\alpha, \alpha+\delta]$ for $\gamma \ge \pi/2$. We have

$$P_m^* = aa_m d_m bcc_m b_m d$$
$$= occ_m \cup oc_m b_m \cup ob_m d \cup oda \cup oaa_m \cup oa_m d_m \cup od_m b \cup obc \qquad \text{(see Fig. 2)}.$$

We omit tedious considerations (partially similar to those from Case 1) which lead to the conclusion that $\mathrm{area}(P_m^*)$ equals

$$f_2(\varphi) = |oc|^2 \sin(2\varphi-2\alpha) + 2|oc|\,|ob|\sin(\alpha+\beta) + |ob|^2 \sin(\pi-2\beta-2\varphi).$$

Similarly to Case 1 we show that $f_2(\varphi)$ is concave in $[\alpha, \pi/2-\beta]$ and $[\alpha, \alpha+\delta]$ (when $\gamma \le \pi/2$ apply $0 < \alpha+\beta$, and when $\gamma \ge \pi/2$ apply the right inequality of (4), $\gamma \ge \pi/2$ and $0 < \alpha+\beta$).

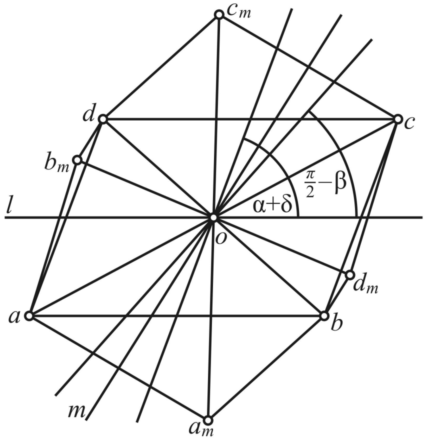

Fig. 3

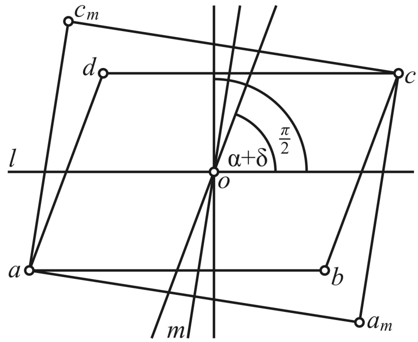

Fig. 4

Case 3: $\varphi \in [\pi/2-\beta, \alpha+\delta]$ for $\gamma \le \pi/2$. We easily conclude that P_m^* is the octagon $aa_mbd_mcc_mdb_m$ (see Fig. 3). Clearly,

$$P_m^* = occ_m \cup oc_md \cup odb_m \cup ob_ma \cup oaa_m \cup oa_mb \cup obd_m \cup od_mc.$$

Observe that area(P_m^*) is given by

$$f_3(\varphi) = |oc|^2 \sin(2\varphi-2\alpha)+2|oc||ob| \sin(\pi+\alpha-\beta-2\varphi)+|ob|^2 \sin(2\beta+2\varphi-\pi).$$

Analogously to Case 1 we show that the function $f_3(\varphi)$ is concave in $[\pi/2-\beta, \alpha+\delta]$ (we apply the left inequality of (3), the right inequality of (4), $\gamma \ge \pi/4$ and $\gamma \ge \delta$).

Case 4: $\varphi \in [\alpha+\delta, \pi/2]$. This time P_m^* is the rectangle aa_mcc_m (see Fig. 4). The area of P_m^* equals

$$f_4(\varphi) = 2|oc|^2 \sin(2\varphi-2\alpha).$$

Similarly to Case 1 we show that the function $f_4(\varphi)$ is concave in the interval $[\alpha+\delta, \pi/2]$ (apply $0<\alpha$ and $0<\delta$). This finishes the considerations of Case 4.

The functions $f_i(\varphi)$ for $i = 1, \ldots, 4$ (when $\gamma \le \pi/2$) and for $i = 1, 2, 4$ (when $\gamma \ge \pi/2$) are concave in the respective intervals considered in our cases. So each attains its smallest value at an end-point (or both) of the corresponding interval. Since the four (if $\gamma \le \pi/2$) and three (if $\gamma \ge \pi/2$) intervals are neighboring, the smallest value of area(P_m^*) is attained at least at one end-point of at least one of the intervals.

First assume $\gamma \le \pi/2$. By the preceding paragraph, to find the smallest value of area(P_m^*) we consider Cases 1–4. We choose the smallest of the numbers

$$\begin{aligned}
f_1(0) &= |oc|^2\sin 2\alpha + 2|oc||ob|\sin(\beta-\alpha) + |ob|^2\sin(\pi-2\beta),\\
f_1(\alpha) &= f_2(\alpha) = 2|oc||ob|\sin(\beta+\alpha) + |ob|^2\sin(\pi-2\beta-2\alpha),\\
f_2(\pi/2-\beta) &= f_3(\pi/2-\beta) = |oc|^2\sin(\pi-2\beta-2\alpha) + 2|oc|\,|ob|\sin(\alpha+\beta),\\
f_3(\alpha+\delta) &= |oc|^2\sin 2\delta + 2|oc||ob|\sin(\pi-\alpha-\beta-2\delta)\\
&\qquad + |ob|^2\sin(2\alpha+2\delta+2\beta-\pi),\\
f_4(\alpha+\delta) &= 2|oc|^2\sin 2\delta,\\
f_4(\pi/2) &= 2|oc|^2\sin(\pi-2\alpha).
\end{aligned}$$

Elementary, but time consuming calculations show that $f_1(0) = f_4(\pi/2) = 2|oc|^2\sin 2\alpha$, $f_1(\alpha) = f_2(\alpha) = \sin 2\alpha$, $f_2(\pi/2-\beta) = f_3(\pi/2-\beta) = \sin 2\beta$ and $f_3(\alpha+\delta) = f_4(\alpha+\delta) = 2|oc|^2\sin 2\delta$. By $\alpha \le \beta$ and the left inequality of (3) we get $\alpha \le \beta \le \pi/2-\alpha$. Since $0 < 2\alpha \le 2\beta \le \pi-2\alpha < \pi$, we have $\sin 2\alpha \le \sin 2\beta$. From the right inequality of (3) and the left inequality of (4) we obtain $\alpha \le \delta \le \pi/2-\alpha$. Hence $0<2\alpha\le 2\delta\le\pi-2\alpha<\pi$. Consequently, $\sin 2\alpha \le \sin 2\delta$ and $2|oc|^2\sin 2\alpha \le 2|oc|^2\sin 2\delta$. Thus the minimum of $\mathrm{area}(P_m^*)$ is equal to $\sin 2\alpha$ or to $2|oc|^2\sin 2\alpha$.

Now assume $\gamma \ge \pi/2$. From Cases 1, 2, 4 and from the concavity of $f_1(\varphi), f_2(\varphi), f_4(\varphi)$, we see that in order to find the smallest value of $\mathrm{area}(P_m^*)$ we have to choose the smallest of the numbers $f_1(0) = f_4(\pi/2) = 2|oc|^2\sin 2\alpha$, $f_1(\alpha) = f_2(\alpha) = \sin 2\alpha$, $f_2(\alpha+\delta) = f_4(\alpha+\delta) = 2|oc|^2\sin 2\delta$. As in the preceding paragraph, $2|oc|^2\sin 2\alpha \le 2|oc|^2\sin 2\delta$. Again we have the same two candidates $\sin 2\alpha$ and $2|oc|^2\sin 2\alpha$ to be the minimum of $\mathrm{area}(P_m^*)$.

It remains to compare the two candidates obtained in each of the two preceding paragraphs. Clearly, $|oc| = \frac{1}{2}|ac|$. Hence the inequality $2|oc|^2\sin 2\alpha \le \sin 2\alpha$ is equivalent to $\frac{1}{2}|ac|^2\sin 2\alpha \le \sin 2\alpha$. Consequently, the last inequality holds true if and only if $|ac| \le \sqrt{2}$. The first conclusion is that if $|ac| \le \sqrt{2}$, then the area of P_m^* is the smallest for $\varphi = 0$ and for $\varphi = \pi/2$ and equals

$$\frac{1}{2}|ac|^2\sin 2\alpha = 2\cdot\frac{\sin^2\beta\sin 2\alpha}{\sin^2(\alpha+\beta)}.$$

The second conclusion is that if $|ac| \ge \sqrt{2}$, then the area of P_m^* is the smallest for $\varphi = \alpha$ and it equals $\sin 2\alpha$. From the above two conclusions and from $\mathrm{area}(P) = 2\cdot\frac{\sin\beta\sin\alpha}{\sin(\alpha+\beta)}$, we obtain the assertion of Theorem 2.1. ∎

Corollary 2.2. *We have*

$$\mathrm{axs}(P) = \frac{1}{\cos\alpha}\cdot\max\left\{\frac{\sin(\alpha+\beta)}{2\sin\beta}, \frac{\sin\beta}{\sin(\alpha+\beta)}\right\} = \frac{1}{\cos\alpha}\cdot\max\left\{\frac{|ab|}{|ac|}, \frac{|ac|}{2|ab|}\right\}.$$

COROLLARY 2.3. *For $|ac|/|ab| \le \sqrt{2}$, the value on the right side of* (1) *is attained if and only if the mirror line m passes through the center of P, and is parallel or perpendicular to ab. For $|ac|/|ab| \ge \sqrt{2}$, the value on the right side of* (2) *is attained if and only if m passes through the center of P, and contains ac or is perpendicular to it. For $|ac|/|ab| = \sqrt{2}$ formulas* (1) *and* (2) *give the same value, and* $\mathrm{axs}(P)$ *is attained for all the four positions of m described above, and only for them.*

This corollary follows from the last paragraph of the proof of Theorem 2.1. Just observe that the minimum of $\mathrm{area}(P_m^*)$ is attained only for the values of φ given there.

COROLLARY 2.4. *For every parallelogram P we have* $\mathrm{axs}(P) > \frac{1}{2}\sqrt{2}$. *This estimate cannot be improved for the family of parallelograms.*

Proof. From Corollary 2.2 and from the well known inequality $\max\{s,t\} \ge \sqrt{st}$ we find that $\mathrm{axs}(P) \ge \frac{1}{\sqrt{2}\cos\alpha} > \frac{1}{2}\sqrt{2}$.

To see that the estimate $\sqrt{2}/2$ cannot be improved, we take parallelograms for which $|ac|/|ab| = \sqrt{2}$, that is, for which $|ao|/|ab| = \sqrt{2}/2$. Applying the sine theorem to the triangle abo we see that $\sin(\alpha+\beta)/\sin\beta = \sqrt{2}$. By (1) we get

$$\mathrm{axs}(P) = \frac{\sin(\alpha+\beta)}{2\sin\beta\cos\alpha} = \frac{1}{2}\sqrt{2}\cdot\frac{1}{\cos\alpha}.$$

Letting α tend to 0, we conclude that $\mathrm{axs}(P)$ can be arbitrarily close to $\frac{1}{2}\sqrt{2}$. ∎

3. Circumscribed axially symmetric convex octagons. Here is an alternative proof of a property formulated in [7] as Theorem 8 (see the comment after our proof).

THEOREM 3.1. *For every centrally symmetric planar convex body K there exists an axially symmetric convex octagon Q which is circumscribed about K and satisfies* $\mathrm{area}(K) > \frac{1}{2}\sqrt{2}\cdot\mathrm{area}(Q)$.

Proof. Let k be a straight line through the center o of symmetry of K. We circumscribe about K the rectangle $R_k = a_kb_kc_kd_k$ whose side a_kb_k is parallel to k. Put $\lambda_k = |a_kb_k|/|b_kc_k|$ and $\alpha_k = \arctan\lambda_k$. Clearly, $0 < \alpha_k < \pi/2$.

Next, we circumscribe about K the parallelogram P_k whose sides are parallel to the diagonals of R_k (see Fig. 5). Clearly, o is the center of symmetry of the circumscribed octagon $Q_k = R_k \cap P_k = q_kr_ks_kt_ku_kv_kw_kz_k$ (the notation is chosen such that $q_kr_k \subset a_kb_k$ and $s_kt_k \subset b_kc_k$, as in Fig. 5). Notice that for non-smooth K and for some specific k it may also happen that the octagon Q_k reduces to a hexagon or to a parallelogram.

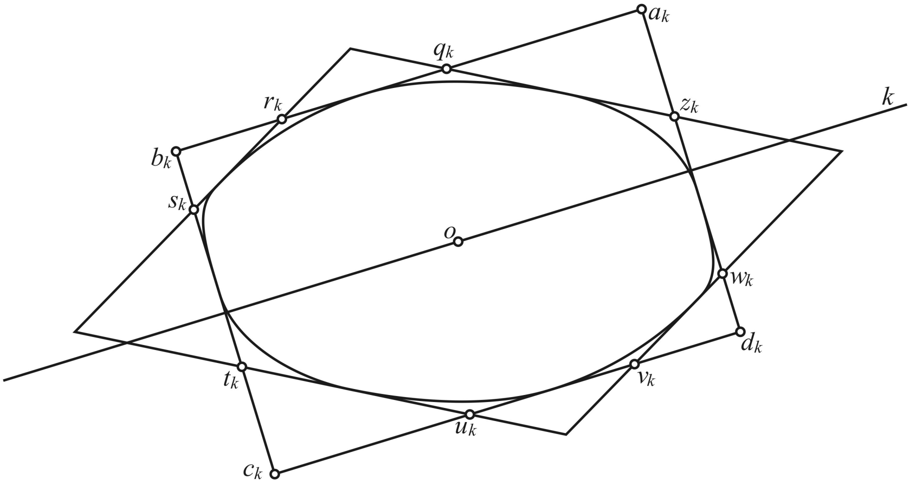

Fig. 5

It is well known that the position of a supporting line of a convex body in a direction is a continuous function of the direction. Hence when k rotates, R_k varies continuously. Moreover, since the function arctan is continuous, also P_k varies continuously. As a consequence, Q_k varies continuously when k rotates. We intend to show that there exists a direction k_0 such that

$$|a_{k_0} q_{k_0}| = |r_{k_0} b_{k_0}|. \tag{6}$$

Assume that for instance $|a_{k_1} q_{k_1}| < |r_{k_1} b_{k_1}|$ for a direction k_1 (if we have the opposite inequality, we proceed analogously, and if we have equality, we take k_1 in place of k_0). Denote by k_2 the direction obtained from k_1 by its rotation by $\pi/2$. Clearly, $R_{k_2} = R_{k_1}$. Hence the definition of P_k implies that $P_{k_2} = P_{k_1}$. From both these equalities we get $Q_{k_2} = Q_{k_1}$ with, in particular, $a_{k_2} = b_{k_1}$ and $b_{k_2} = c_{k_1}$. Moreover, $q_{k_2} = s_{k_1}$ and $r_{k_2} = t_{k_1}$. Of course, the triangle $t_{k_1} c_{k_1} u_{k_1}$ is symmetric (with respect to o) to $z_{k_1} a_{k_1} q_{k_1}$, and the triangle $r_{k_1} b_{k_1} s_{k_1}$ is an enlarged image of $z_{k_1} a_{k_1} q_{k_1}$ by a similarity. Consequently, $|b_{k_1} s_{k_1}| > |t_{k_1} c_{k_1}|$. Since $Q_{k_2} = Q_{k_1}$, this inequality can be written as $|a_{k_2} q_{k_2}| > |r_{k_2} b_{k_2}|$. So by the construction of Q_k and by the continuity of the shape of Q_k we find that between k_1 and k_2 (during the process of rotating of k by $\pi/2$) there exists k_0 for which (6) holds true.

From (6) and from the construction of Q_{k_0} (in particular, from its central symmetry) it follows that Q_{k_0} is axially symmetric. The straight line through the midpoints of $s_{k_0} t_{k_0}$ and $w_{k_0} z_{k_0}$, and the straight line through the midpoints of $q_{k_0} r_{k_0}$ and $u_{k_0} v_{k_0}$, are its axes of symmetry.

From now on we omit the subscripts k_0 and thus we write $Q = qrstuvwz$. Since the octagon Q is circumscribed about K, every side of Q contains a point of K. Namely, $q' \in qr,\ r' \in rs,\ \dots\ , z' \in zq$. Put $S = q'r's't'u'v'w'z'$.

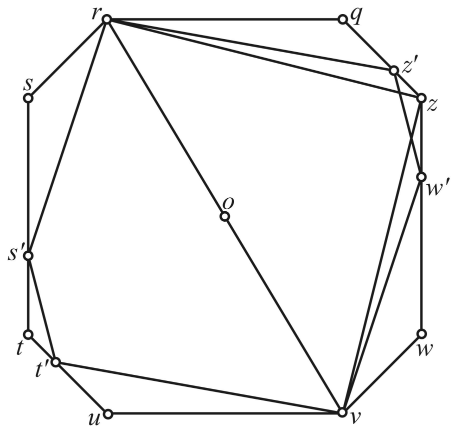

Fig. 6

By the central symmetry of K, we may assume that these points are chosen so that o is the center of symmetry of S.

Assume first that S is not a parallelogram. We then show that

$$\text{area}(S) > \frac{1}{2}\sqrt{2} \cdot \text{area}(Q). \tag{7}$$

Observe that it is sufficient to verify (7) in the special case when o is the center of Cartesian coordinates and $q = (h, 1)$, $r = (-h, 1)$, $s = (-1, h)$, $t = (-1, -h)$, $u = -q$, $v = -r$, $w = -s$ and $z = -t$, where $0 \le h \le 1$; the reason is that this special case is obtained from the general case by an affine transformation and from the fact that the ratio of areas of figures does not change under affine transformations. Assume that $|rr'| \le |z'q|$ (the opposite case is analogous). Since $\text{area}(r'z'r) \le \text{area}(r'z'q')$, the area of S does not increase when q' moves to r and u' moves to v, while the remaining six vertices are unchanged (see Fig. 6). So imagine that $q' = r$ and $u' = v$. In this particular situation the area of S again does not increase when r' moves to r and v' moves to v, while the remaining four vertices are unchanged. So we get the hexagon $H = rs't'vw'z'$ which is a special position of S and we conclude that $\text{area}(H) \le \text{area}(S)$.

From the preceding we see that in order to prove (7) it is sufficient to show that for every $h \in [0, 1]$ we have

$$\text{area}(H) \ge \frac{1}{2}\sqrt{2} \cdot \text{area}(Q). \tag{8}$$

For this purpose it is sufficient to show that

$$\frac{1}{2}\sqrt{2} \cdot \text{area}(Q) \le \text{area}(rtvz), \tag{9}$$

and that

$$\text{area}(rtvz) \le \text{area}(H). \tag{10}$$

To verify (9) we first easily establish that $\text{area}(rtvz) = 2h^2 + 2$ and $\text{area}(Q) = -2h^2 + 4h + 2$. Consequently, (9) is equivalent to the inequality $\frac{1}{2}\sqrt{2}(-h^2+2h+1) \le h^2+1$, that is, $(h-\sqrt{2}+1)^2 \ge 0$. Thus (9) is confirmed.

Having in mind the central symmetry of Q and H, to prove (10) it is sufficient to show that for every fixed $h \in [0,1]$,

$$\text{area}(rvz) \le \text{area}(rvw'z'). \tag{11}$$

Observe that $w' = (1, \lambda)$ for a $\lambda \in [-h, h]$ and $z' = (\mu, 1+h-\mu)$ for a $\mu \in [h, 1]$ (see Fig. 6). We omit an elementary proof that

$$\text{area}(rvw'z') = \frac{1}{2}h^2 + h + 1 + \frac{1}{2}h\lambda - \frac{1}{2}h\mu - \frac{1}{2}\lambda\mu. \tag{12}$$

Applying the partial derivative test for absolute extrema of functions of two variables we easily show that this function of $(\lambda, \mu) \in [-h, h] \times [h, 1]$ has the global minimum only at the critical point $(h, 1)$, that is, for $\lambda = h$ and $\mu = 1$. Consequently, (11) and thus also (10) are true.

From (9) and (10) we see that (8) holds true.

As a consequence of (8), we obtain (7) with the weak inequality. Since S is not a parallelogram, we exclude the case $\lambda = h$ and $\mu = 1$. Since the global minimum of our function (12) is attained only for $\lambda = h$ and $\mu = 1$, we obtain the strict inequality in (7).

Finally, consider the case when S is a parallelogram. Then the existence of the octagon Q promised in the conclusion of Theorem 3.1 follows from the proof of Theorem 2.1 and from Corollary 2.4. ∎

As mentioned in the introduction, the inequality $\text{area}(K) \ge \frac{1}{2}\sqrt{2}\cdot\text{area}(Q)$, slightly weaker than ours, is formulated in Theorem 8 of [7, pp. 128–130]. Its proof is, however, incorrect. In the rest of this paragraph we use the notation of [7]. In the proof in question the angles between AB and EF and also GH are not defined. The author only says that they should be equal, but this does not define them and the octagon Q'' uniquely. Thus for a given K, the author obtains a class of circumscribed axially symmetric octagons Q'' (instead of one, as in our proof). For them the inequality $\text{area}(P)/\text{area}(Q'') \ge \frac{1}{2}\sqrt{2}$ is not true in general (for instance for $x = y = 1$ and $k = 2$).

4. Measure of axial symmetry of centrally symmetric bodies. From Theorem 3.1 and Corollary 2.4 we immediately obtain the following theorem.

THEOREM 4.1. *For every centrally symmetric planar convex body K we have* $\text{axs}(K) > \frac{1}{2}\sqrt{2}$ *and in general this inequality cannot be improved.*

Important centrally symmetric convex bodies are *affine-regular hexagons*, i.e., hexagons which are affine images of regular hexagons. Here is an estimate for the measure of axial symmetry of affine-regular hexagons.

CLAIM 4.2. *For every affine-regular hexagon H we have* $\operatorname{axs}(H) > 3/4$.

To see this, consider a longest chord, say ad, of $H = abcdef$. We construct the rectangle J with two opposite sides containing bc and ef, and the other two passing through a and d. From $|be| \le |ad|$ and $|cf| \le |ad|$ we see that H is a proper subset of J. Moreover, since H is an affine-regular hexagon, we have $|bc| = |fe| = \frac{1}{2}|ad|$. Hence $\operatorname{area}(H) < \frac{3}{4} \cdot \operatorname{area}(J)$. Since J is axially symmetric, we obtain the assertion.

We conjecture that the estimate $\frac{3}{4}$ for affine-regular hexagons cannot be enlarged.

PROBLEM 4.3. *Provide extensions of Corollary* 2.4 *and of Theorem* 4.1 *to d-dimensional space, i.e., give estimates for the measure of mirror symmetry of d-dimensional parallelotopes and centrally symmetric convex bodies (and also of arbitrary convex bodies).*

REFERENCES

[1] H. K. Ahn, P. Brass, O. Cheong, H. S. Na, C. S. Shin and A. Vigneron, *Approximation algorithms for inscribing or circumscribing an axially symmetric polygon to a convex polygon*, in: Lecture Notes in Comput. Sci. 3106, Springer, 2004, 259–267.
[2] —, —, —, —, —, —, *Inscribing an axially symmetric polygon and other approximation algorithms for planar convex sets*, Comput. Geom. 33 (2006), 152–164.
[3] A. B. Buda and K. Mislow, *On a measure of axiality for triangular domains*, Elem. Math. 46 (1991), 65–73.
[4] G. D. Chakerian and S. K. Stein, *Measures of symmetry of convex bodies*, Canad. J. Math. 17 (1965), 497–504.
[5] B. A. deValcourt, *Measures of axial symmetry for ovals*, Bull. Amer. Math. Soc. 72 (1966), 289–290.
[6] —, *Measures of axial symmetry for ovals*, Israel J. Math. 4 (1966), 65–82.
[7] —, *Axially symmetric polygons inscribed in and circumscribed about convex sets*, Elem. Math. 22 (1967), 121–133.
[8] B. Grünbaum, *Measures of symmetry of convex bodies*, in: Proc. Sympos. Pure Math. 7, Amer. Math. Soc., 1963, 233–270.
[9] E. Heil and H. Martini, *Special convex bodies*, in: Handbook of Convex Geometry, Vol. A, P. M. Gruber and J. M. Wills (eds.), North-Holland, Amsterdam, 1993, 347–385.
[10] M. Lassak, *Approximation of convex bodies by axially symmetric bodies*, Proc. Amer. Math. Soc. 130 (2002), 3075–3084; erratum, 131 (2003), 2301.
[11] M. Lassak and M. Nowicka, *Minimum-area axially symmetric convex bodies containing a triangle and its measure of axial symmetry*, Beiträge Algebra Geom. 50 (2009), 541–554.

[12] W. Nohl, *Die innere axiale Symmetrie zentrischer Eibereiche der Euklidischen Ebene*, Elem. Math. 17 (1962), 59–63.

Marek Lassak, Monika Nowicka
Institute of Mathematics and Physics
University of Technology
85-796 Bydgoszcz, Poland
E-mail: lassak@utp.edu.pl
mnowicka@utp.edu.pl

Received 31 July 2009;
revised 25 June 2010 (5252)