

ABSTRACT PARABOLIC PROBLEMS IN ORDERED
BANACH SPACES

BY

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Abstract. We consider abstract parabolic problems in ordered Banach spaces and give conditions under which they have global attractors. Our approach is via comparison of solutions. Within this approach abstract comparison principles are obtained and bounds on the attractors are given by order intervals in Banach spaces. These results are applied to ordinary differential equations and to parabolic equations for which the main part is given by a sum of fractional powers of sectorial operators having increasing resolvents and integral operators having positive kernels.

1. Introduction. Let X be a Banach space and $A : D(A) \subset X \rightarrow X$ be a *sectorial operator*. Choose $\lambda_0 \in \mathbb{R}$ such that $\operatorname{Re} \sigma(A) > \lambda_0$, that is, $\operatorname{Re} \lambda > \lambda_0$ for all $\lambda \in \sigma(A)$ ($\sigma(A)$ is the spectrum of A). As usual X^α ([13, p. 29]) denote the *fractional power spaces* associated with A . Assume that $f : X^\alpha \rightarrow X$ is Lipschitz continuous in bounded subsets of X^α and consider the following abstract parabolic initial value problem:

$$(1) \quad \dot{u} + Au = f(u), \quad u(0) = u_0 \in X^\alpha.$$

Under these assumptions the problem (1) is locally well posed in X^α (see [13], Theorem 3.3.4), the solution $u(t, u_0)$ of (1) is defined in a maximal interval of existence $[0, \tau_{u_0})$ and either $\tau_{u_0} = \infty$ or $\limsup_{t \rightarrow \tau_{u_0}^-} \|u(t, u_0)\|_{X^\alpha} = \infty$. When X is a Hilbert space, to ensure that the problem (1) is globally well posed, the usual approach is to obtain some energy estimates for the solutions. We are interested in the case when X is not a Hilbert space and in that case different tools to obtain bounds on the solutions are needed. The tools we use are abstract comparison results. For that we need some additional structure to the abstract problem. We start with the following definitions.

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DEFINITION 1. An *ordered Banach space* is a pair (X, \leq) , where X is a Banach space and \leq is an ordering relation in X satisfying

- 1) $x \leq y$ implies $x + z \leq y + z$, $x, y, z \in X$,
- 2) $x \leq y$ implies $\lambda x \leq \lambda y$, $x, y \in X$, and $0 \leq \lambda \in \mathbb{R}$,
- 3) the *positive cone* $C = \{x \in X : x \geq 0\}$ is closed in X (where $x \geq y$ means $y \leq x$).

DEFINITION 2. Let (X, \leq_X) and (Y, \leq_Y) be ordered Banach spaces. A function $T : X \rightarrow Y$ is *increasing* if $x_1 \leq_X x_2$ implies $T(x_1) \leq_Y T(x_2)$, and it is *positive* if $x \geq_X 0$ implies $T(x) \geq_Y 0$.

Note that the two notions in Definition 2 coincide when the map T is linear.

DEFINITION 3. Let X be an ordered Banach space. A vector $\xi \in X$ is said to be an *upper bound* for $B \subset X$ if $b \leq \xi$ for any $b \in B$. In an ordered Banach space, if $a \leq b$, the set $[a, b] = \{x \in X : a \leq x \leq b\}$ is called an *order interval*.

For the rest of this section we assume that X is an ordered Banach space. Assume that $(\lambda + A)^{-1}$ is increasing for all $\mathbb{R} \ni \lambda > \lambda_0$. Assume also that there are numbers $c_1^+, c_1^- \in \mathbb{R}$ and vectors $X \ni c_2^+, -c_2^- \geq 0$ such that the map f satisfies

$$(2) \quad \begin{aligned} f(u) &\leq c_1^+ u + c_2^+, & u &\geq 0, \\ f(u) &\geq c_1^- u + c_2^-, & u &\leq 0. \end{aligned}$$

Under these assumptions and some regularity we prove that the solutions of (1) are globally defined. If in addition we assume that $\operatorname{Re} \sigma(A - c_1^\pm I) > 0$, then (1) has a global attractor.

This paper is organized as follows. In Section 2 we give introductory results concerning positivity and comparison, which will be of help when proving the desired result on the asymptotics of (1). In Section 3 we prove existence of global attractors. In Section 4 we give several examples of equations to which the abstract results developed in Section 3 apply, including ordinary differential equations, parabolic equations with pseudodifferential operators and integral-pseudodifferential equations. Finally, in the Appendix we state several results which enable us to state that a given operator has increasing resolvent as well as to produce new operators with increasing resolvent starting from operators for which this property is known.

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2. Abstract monotonicity and comparison. In this section we follow [6] to establish abstract comparison results for parabolic problems of the form (1), assuming that the base space X is an ordered Banach space and that the resolvent of A is *increasing*, i.e. $(\lambda + A)^{-1}$ is increasing for all $\lambda > \lambda_0$.

2.1. Nonlinear perturbations of increasing resolvent operators. We start from positivity and comparison results for nonlinear equations

$$(3) \quad \dot{u} + Au = f(t, u), \quad u(t_0) = u_0 \in X^\alpha,$$

where $f : [t_0, t_1] \times X^\alpha \rightarrow X$ is locally Hölder in t , locally Lipschitz in u and $0 \leq \alpha < 1$. The unique solution of (3) (see [13], [18]) is denoted by $u_f(t, u_0)$. All the initial value problems studied below will be such that the nonlinearity satisfies the above condition.

Concerning positivity, we have

THEOREM 1. *Let (X, \leq) be an ordered Banach space and A a sectorial operator in X with increasing resolvent. Assume that for every $r > 0$ there exists a constant $\beta = \beta(r) > 0$ such that $f(t, \cdot) + \beta I$ is positive for each $t \in [t_0, t_1]$ in the ball of radius r in X^α . If $u_0 \geq 0$, then $u_f(t, u_0)$ is positive as long as it exists.*

Concerning comparison, we have

THEOREM 2. *Let (X, \leq) be an ordered Banach space, and A be a sectorial operator with increasing resolvent.*

(i) *If for every $r > 0$ there exists $\beta = \beta(r) > 0$ such that for $t \in [t_0, t_1]$, $f(t, \cdot) + \beta I$ is increasing in a ball of radius r in X^α , then $u_0 \geq u_1$ implies $u_f(t, u_0) \geq u_f(t, u_1)$ as long as both solutions exist.*

(ii) *If $f(t, \cdot) \geq g(t, \cdot)$ for every t and if for each $r > 0$ there is $\beta(r) > 0$ such that $f(t, \cdot) + \beta I$ is increasing in a ball of radius r in X^α (or g has the latter property), then $u_f(t, u_0) \geq u_g(t, u_0)$ as long as both solutions exist.*

(iii) *If for every $r > 0$ there exists $\beta(r) > 0$ and an increasing function $h(t, \cdot)$ such that $f(t, \cdot) + \beta I \geq h(t, \cdot) \geq g(t, \cdot) + \beta I$ in a ball of radius r , then $u_0 \geq u_1$ implies $u_f(t, u_0) \geq u_g(t, u_1)$ as long as both solutions exist.*

The proofs of the above results are based on the study of successive approximations (see [6]).

2.2. Quasi-monotone maps and increasing resolvent matrices. In what follows we mention how Theorem 2 relates to the classical results concerning ordinary differential equations in \mathbb{R}^n having the form

$$(4) \quad \frac{d}{dt} \vec{z} = F(\vec{z}), \quad \vec{z}(0) = \vec{z}_0.$$

We aim to give conditions on the vector field F implying that the solution operator for (4) is increasing with respect to initial conditions. The approach is the one taken by J. Szarski in [19] (see also [17, 22]).

Let $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$ be vectors in \mathbb{R}^n . We write

$$\vec{x} \leq \vec{y} \quad \text{if} \quad x_j \leq y_j, \quad j = 1, \dots, n,$$

and

$$\vec{x} \stackrel{i}{\leq} \vec{y} \quad \text{if} \quad x_j \leq y_j, \quad j = 1, \dots, n, \quad \text{and} \quad x_i = y_i.$$

We say that $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *quasi-monotone increasing* if

$$\vec{x} \stackrel{i}{\leq} \vec{y} \Rightarrow F_i(\vec{x}) \leq F_i(\vec{y}), \quad i = 1, \dots, n.$$

THEOREM 3. *If F is a locally Lipschitz function which is quasi-monotone increasing, $\vec{x}_0 \leq \vec{y}_0 \in \mathbb{R}^n$, $\vec{x}(t, \vec{x}_0)$ and $\vec{y}(t, \vec{y}_0)$ are the solutions of (4) starting at \vec{x}_0 and \vec{y}_0 respectively, then*

$$\vec{x}(t, \vec{x}_0) \leq \vec{y}(t, \vec{y}_0)$$

for as long as both solutions exist.

Proof. Since for any bounded set B there is a constant $L_B > 0$ such that $F + L_B I$ is increasing in B , the result follows from Theorem 2. ■

If $F(\vec{x}) = A\vec{x}$ with A being an $n \times n$ matrix, then requiring that F is quasi-monotone increasing is equivalent to requiring that all off-diagonal terms of A are nonnegative. Hence we have

COROLLARY 1. *Let A be an $n \times n$ matrix with nonnegative off-diagonal entries. Then $e^{At} \geq 0$, $t \geq 0$.*

This enables us to construct many examples of finite-dimensional operators with increasing resolvent.

3. Order and attractors. With the aid of the considerations of Section 2 we now prove the results stated in the introduction.

3.1. Basic estimates. We start with simple lemmas from the theory of semigroups.

LEMMA 1. *Let A be a sectorial operator in a Banach space X and, for $c_1 \in \mathbb{R}$, consider the linear problem*

$$\dot{u} + Au - c_1 u = 0, \quad u(0) = u_0.$$

For every $T > 0$ there exists $M = M(T, c_1)$ such that if $u_0 \in X^\beta$ and $\alpha \geq \beta$ then

$$\|u(t)\|_{X^\alpha} \leq M t^{-(\alpha-\beta)} \|u_0\|_{X^\beta}$$

for every $t \in (0, T]$. Moreover, if $\operatorname{Re} \sigma(A - c_1 I) > \mu_1 > 0$, then we can take $T = \infty$ and replace M by $M_0 e^{-\mu_1 t}$.

LEMMA 2. Suppose that A and X are as in Lemma 1, $\operatorname{Re} \sigma(A - c_1 I) > \mu_1 > 0$ and consider the equation

$$(5) \quad \dot{w} + Aw = c_1 w + c_2, \quad w(0) = w_0.$$

Let $\phi = (A - c_1)^{-1}c_2$ and assume that Y is a Banach space,

- $\phi \in Y$,
- there is $\alpha_0 > 0$ (which may be greater than 1) such that

$$\|x\|_Y \leq c_Y \|x\|_{X^{\alpha_0}}, \quad x \in X^{\alpha_0}.$$

If $w(t, w_0)$ denotes the solution of (5), then there are positive constants M and K such that

$$(6) \quad \|w(t, w_0)\|_Y \leq Mt^{-\alpha_0} \|w_0 - \phi\|_X + K, \quad t > 0.$$

Proof. Consider the change of variables $w = v + \phi$ where ϕ is the solution of

$$A\phi - c_1\phi - c_2 = 0.$$

Then v satisfies

$$\dot{v} + Av = c_1 v, \quad v(0) = w_0 - \phi.$$

From Lemma 1 and the embedding of X^{α_0} into Y we have

$$(7) \quad \|w(t, w_0)\|_Y \leq c_Y M_0 t^{-\alpha_0} \|w_0 - \phi\|_X + \|\phi\|_Y, \quad t > 0. \blacksquare$$

3.2. B -monotone maps. In Subsection 2.1 we mentioned the maps f with the property that given a bounded set B in the domain of f , there is a constant $\beta_B > 0$ such that $f + \beta_B I$ is increasing in B . These maps will be called B -monotone maps.

We start by studying the behavior of solutions of parabolic problems of the form

$$(8) \quad \dot{u} + Au = f(u), \quad u(0) = u_0,$$

where (X, \geq) is an ordered Banach space, A is a sectorial operator with increasing resolvent and $f : X^\alpha \rightarrow X$ is Lipschitz continuous on bounded sets for some $\alpha \in [0, 1)$.

Consider the following auxiliary problem:

$$(9) \quad \dot{u} + Au = f^+(u),$$

where f^+ is Lipschitz continuous in bounded subsets of X^α satisfying

$$(10) \quad f(u) \leq f^+(u), \quad u \in X^\alpha.$$

Under these assumptions, Theorem 2 immediately yields

LEMMA 3. If either f or f^+ is B -monotone and $u_0 \leq u_1$, then

$$(11) \quad u_f(t, u_0) \leq u_{f^+}(t, u_1)$$

for as long as $u_f(t, u_0)$ and $u_{f^+}(t, u_1)$ exist.

3.3. Attractors and bounds. We are now prepared to prove existence of attractors for (1) and obtain bounds for them.

Assume that A is a sectorial operator in an ordered Banach space X and, for some $\alpha \in [0, 1)$, $f : X^\alpha \rightarrow X$ is Lipschitz continuous in bounded sets and B -monotone. Assume also that (2) holds and for any $u_0 \in X^\alpha$ there are $u_0^+, u_0^- \in X^\alpha$ satisfying $u_0^- \leq u_0 \leq u_0^+$ and $u_0^+ \geq 0, -u_0^- \geq 0$.

Consider the problem (8) and a pair of auxiliary problems

$$(12) \quad \dot{u}^+ + Au^+ = c_1^+ u^+ + c_2^+, \quad u^+(0) = u_0^+,$$

$$(13) \quad \dot{u}^- + Au^- = c_1^- u^- + c_2^-, \quad u^-(0) = u_0^-.$$

Let $u(t, u_0)$, $u^+(t, u_0^+)$ and $u^-(t, u_0^-)$ denote the solutions of (8), (12) and (13) respectively. Since $u^+(t, u_0^+) \geq 0$ from Lemma 3, we have

$$u(t, u_0) \leq u^+(t, u_0^+)$$

for as long as $u(t, u_0)$ exists. Proceeding similarly in the case of u^- , we get

$$(14) \quad u^-(t, u_0^-) \leq u(t, u_0) \leq u^+(t, u_0^+)$$

for as long as $u(t, u_0)$ exist. Since $u^+(t, u_0^+)$ and $u^-(t, u_0^-)$ exist for all $t \geq 0$, we would like to establish that the same happens for $u(t, u_0)$. This is going to be the case in a number of applications. Meanwhile, remaining abstract, we state the following result.

THEOREM 4. *Assume that A has compact resolvent and $\operatorname{Re} \sigma(A - c_1^\pm) > 0$.*

(i) *If $u(t, u_0)$ exists for all $t \geq 0$ and remains bounded in X^α , then $\omega(u_0) \neq \emptyset$ and*

$$(A - c_1^+)^{-1} c_2^+ = \Phi^+ \geq \phi \geq \Phi^- = (A - c_1^-)^{-1} c_2^-, \quad \phi \in \omega(u_0).$$

(ii) *If (8) has a global attractor \mathcal{A} , then*

$$\Phi^+ \geq \phi \geq \Phi^-, \quad \phi \in \mathcal{A}.$$

In both cases the above results say that the order interval $\{\phi \in X^\alpha : \Phi^+ \geq \phi \geq \Phi^-\}$ contains the asymptotic dynamics of (8).

The proof of Theorem 4 follows immediately from the fact that Φ^+ (or Φ^-) is a global attractor for (12) (or (13)), from the definition of attractor and from (14).

Next we work towards obtaining the existence of global attractors for the problem (1). For that we need to introduce some additional structure. We have already checked, in Lemma 2 and in (14), that if $X^{\alpha_0} \subset Y$ for some $\alpha_0 > 0$ and (2) holds, then (8) has the following property:

(H) Given a solution $u(t, u_0)$ of (8) in X^α on its maximal interval of existence $[0, \tau_{u_0})$, there are continuous functions $\eta^+, \eta^- : (0, \infty) \rightarrow Y$ with $\limsup_{t \rightarrow \infty} \|\eta^\pm(t)\|_Y \leq K$ such that $\eta^+(t) \geq u(t, u_0) \geq \eta^-(t)$, $t \in (0, \tau_{u_0})$.

ASSUMPTION 1. Assume (H) implies that $u(t, u_0)$ is globally defined, $u(t, u_0) \in Y$ for $t > 0$ and

$$\limsup_{t \rightarrow \infty} \|u(t, u_0)\|_Y \leq K.$$

Assume also that $\|f(u)\|_X \leq N_\eta$ whenever $\|u\|_Y \leq K + \eta$, $\eta > 0$.

Assumption 1 asserts that we are able to strengthen a global-in-time a priori estimate of $u(t, u_0)$ in X (expressed as $\eta^+(t) \geq u(t, u_0) \geq \eta^-(t)$) to its global estimate in the X^α norm. Such a property of solutions is usually a consequence of the structure of the nonlinear term and the smoothing action of the solutions to (8).

COROLLARY 2. Assume that $f : X^\alpha \rightarrow X$ is Lipschitz continuous in bounded sets and B -monotone, (2) holds, $\operatorname{Re} \sigma(A - c_1^\pm) > \mu_1 > 0$, A has compact resolvent and Assumption 1 is satisfied. Then the semigroup corresponding to (8) in X^α has a global attractor.

Proof. To prove this we proceed as follows. For $t \geq t_0 \geq 0$ we write

$$u(t, u_0) = e^{-A(t-t_0)}u(t_0, u_0) + \int_{t_0}^t e^{-A(t-s)}f(u(s, u_0)) ds.$$

We now take t_0 large enough such that $\|u(s, u_0)\|_Y \leq K + \eta$ for $s \geq t_0$, which gives

$$\begin{aligned} \|u(t, u_0)\|_{X^\alpha} &\leq M_0 e^{-\mu_1(t-t_0)}(t-t_0)^{-\alpha} \|u(t_0, u_0)\|_{X^\alpha} \\ &\quad + M_0 \int_{t_0}^t e^{-\mu_1(t-s)}(t-s)^{-\alpha} N_\eta ds \end{aligned}$$

with $N_\eta = \sup_{\|s\|_Y \leq K+\eta} \|f(s)\|_X$. Since

$$\limsup_{t \rightarrow \infty} \|u(t, u_0)\|_{X^\alpha} \leq M_0 \int_0^\infty e^{-\mu_1 z} z^{-\alpha} N_\eta dz = M_0 N_\eta \frac{\Gamma(1-\alpha)}{\mu_1^{1-\alpha}}$$

and $\eta > 0$ is arbitrary, we conclude that

$$\limsup_{t \rightarrow \infty} \|u(t, u_0)\|_{X^\alpha} \leq M_0 N_0 \frac{\Gamma(1-\alpha)}{\mu_1^{1-\alpha}}.$$

This shows point dissipativeness and, since the semigroup e^{-At} is compact, we conclude that (8) has a global attractor (see [12], Theorem 4.2.4). ■

4. Applications

4.1. Ordinary differential equations. In this section we consider comparison results and existence of global attractors for systems of ordinary differential equations.

Let $F^+, F^- : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz functions. Consider the initial value problems

$$(15) \quad \frac{d}{dt} \vec{x} = F^+(\vec{x}), \quad \vec{x}(0) = \vec{x}_0,$$

$$(16) \quad \frac{d}{dt} \vec{y} = F^-(\vec{y}), \quad \vec{y}(0) = \vec{y}_0.$$

Denote by $\vec{x}(t, \vec{x}_0)$ and $\vec{y}(t, \vec{y}_0)$ the solutions of (15) and (16), respectively.

THEOREM 5. *Assume that $F^+ \geq F^-$ and either F^+ or F^- is quasi-monotone increasing (as in Subsection 2.2). If $\vec{x}_0 \geq \vec{y}_0$, then $\vec{x}(t, \vec{x}_0) \geq \vec{y}(t, \vec{y}_0)$ as long as both solutions exist.*

Proof. Note that, for a fixed bounded set B , either $F^+ + \beta_B I$ or $F^- + \beta_B I$ is increasing in B for some $\beta_B \in \mathbb{R}$; therefore either F^+ or F^- is B -monotone. Since $F^+ \geq F^-$, the theorem follows from the results in Subsection 2.1. ■

This comparison result has the following consequence concerning the problem (4).

THEOREM 6. *Let $F^+, F^-, F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be locally Lipschitz functions. Assume that either F^+ and F^- are quasi-monotone or F is quasi-monotone and $F^+ \geq F \geq F^-$. Under these assumptions:*

(i) *If $x_0 \geq z_0 \geq y_0$ and the solutions $\vec{x}(t, \vec{x}_0)$ of (15) and $\vec{y}(t, \vec{y}_0)$ of (16) are globally defined, then the solution $\vec{z}(t, \vec{z}_0)$ of (4) is globally defined.*

(ii) *If $\vec{x}_0 \geq \vec{z}_0 \geq \vec{y}_0$ and $\vec{x}(t, \vec{x}_0)$ and $\vec{y}(t, \vec{y}_0)$ are globally bounded, then so is $\vec{z}(t, \vec{z}_0)$, and $\omega(\vec{y}_0) \geq \omega(\vec{z}_0) \geq \omega(\vec{x}_0)$, that is, for any $a \in \omega(\vec{z}_0)$ there are $a^+ \in \omega(\vec{x}_0)$ and $a^- \in \omega(\vec{y}_0)$ such that $a^+ \geq a \geq a^-$.*

(iii) *If the problems (15) and (16) have global attractors \mathcal{A}^+ and \mathcal{A}^- , then (4) has a global attractor \mathcal{A} satisfying $\mathcal{A}^+ \geq \mathcal{A} \geq \mathcal{A}^-$, that is, for any $a \in \mathcal{A}$ there are $a^+ \in \mathcal{A}^+$ and $a^- \in \mathcal{A}^-$ such that $a^+ \geq a \geq a^-$.*

This result follows from the results in Subsections 2.2, 2.1 and properties of global attractors.

4.2. Systems of pseudodifferential equations. As a second application we study global solvability and asymptotics of the problem

$$(17) \quad \begin{aligned} u_t + (-\Delta_N)^\alpha u &= f(u), & t > 0, x \in \Omega, \\ u(0, x) &= u_0(x), & x \in \Omega, \end{aligned}$$

where $0 < \alpha \leq 1$ and Δ_N denotes the Laplace operator with Neumann boundary conditions in a bounded smooth domain $\Omega \subset \mathbb{R}^n$. This kind of problems have been studied recently in [7] and [8] in the case of $\Omega = \mathbb{R}^n$ and in [10] for bounded Ω with Dirichlet boundary condition. Here we obtain the existence of a global attractor as a consequence of the comparison technique

developed in the previous sections. The results in this subsection remain true for the case $\alpha = 1$, treated in [6]. Thus, we concentrate our attention on the case $\alpha \in (0, 1)$.

In what follows, X_p^θ , $0 \leq \theta \leq 1$, denote the fractional power spaces associated with $-\Delta_N$ in $L^p(\Omega, \mathbb{R}^m)$. In particular $X_p^1 = \{u \in W^{2,p}(\Omega, \mathbb{R}^m) : \partial u / \partial n = 0, x \in \partial \Omega\}$, and $X_p^0 = X_p = L^p(\Omega, \mathbb{R}^m)$. For local well posedness of (17) in X_p^β with $\beta < \alpha$, we assume that f is locally Lipschitz and either $2\beta > n/p$ or

$$(18) \quad |f(u) - f(v)| \leq c|u - v|(|u|^{\varrho-1} + |v|^{\varrho-1} + 1),$$

$$u, v \in \mathbb{R}^m, 1 \leq \varrho \leq \frac{n}{n - 2\beta p}.$$

The order relation in X is induced by the usual order in \mathbb{R}^m . From Proposition 4 (see Appendix) the resolvent of $-\Delta_N$ is increasing in $L^2(\Omega, \mathbb{R}^m)$. This property extends immediately to all $L^p(\Omega, \mathbb{R}^m)$ with $p \in (1, \infty)$ by the usual density argument (see also [11], Theorem 1.3.9). As a consequence of formula (27) the resolvent of $(-\Delta_N)^\alpha$ remains increasing for all $\alpha \in (0, 1)$.

Next we summarize the comparison results for (17). Assume that $f^+, f^- : \mathbb{R}^m \rightarrow \mathbb{R}^m$ are locally Lipschitz continuous and consider the problems

$$(19) \quad \begin{aligned} u_t^+ + (-\Delta_N)^\alpha u^+ &= f^+(u^+), \quad t > 0, x \in \Omega, \\ u^+(0) &= u_0^+ \in X_p^\beta, \end{aligned}$$

and

$$(20) \quad \begin{aligned} u_t^- + (-\Delta_N)^\alpha u^- &= f^-(u^-), \quad t > 0, x \in \Omega, \\ u^-(0) &= u_0^- \in X_p^\beta. \end{aligned}$$

COROLLARY 3. *Assume that $2\beta p > n$, $f^+ \geq f^-$, and either f^+ or f^- is quasi-monotone increasing. If $u^+(t, u_0^+)$, $u^-(t, u_0^-)$ denote the solutions of (19) and (20) respectively and $u_0^+ \geq u_0^-$, then $u^+(t, u_0^+) \geq u^-(t, u_0^-)$ for as long as both solutions exist.*

Proof. Since $X_p^\beta \hookrightarrow L^\infty(\Omega, \mathbb{R}^m)$ and either f^+ or f^- is quasi-monotone increasing, we find that the Nemytskiĭ operator associated with one of these functions is B -monotone in X_p^β . Therefore, the comparison for initial data in X_p^β follows from Lemma 3. ■

The remaining results in this section have much simpler proofs in the case $2\beta p > n$. Hence, we concentrate on the case $2\beta p \leq n$. Assume that f satisfies (18) and is quasi-monotone increasing.

REMARK 1. A density argument may allow an extension of the above corollary to X_p^β , $2\beta p \leq n$, but we will not use this extension in what follows. Instead, we use regularity properties of the solution.

For existence of attractors we assume that the following dissipativeness condition is satisfied:

$$(21) \quad \limsup_{|u_j| \rightarrow \infty} f_j(u)/u_j < 0, \quad j = 1, \dots, m.$$

For the long time behavior of the solutions it is important to obtain some a priori bounds. This is often obtained with the aid of an energy functional. This idea may not be easily applicable for systems of equations where usually more delicate estimates are needed. This is why we are going to use comparison techniques to get the required bounds.

From (21) the condition (2) is satisfied with c_1^\pm negative.

Observe that if $u_0 \in X_p^\beta$ then $u(t, u_0) \in X_p^\alpha$ for $t > 0$. Since from Remark 2,

$$X_p^\alpha \hookrightarrow X_q^\beta, \quad p \leq q < p_1 = \frac{np}{n - 2(\alpha - \beta)p},$$

we have $u(t, u_0) \in X_q^\beta$, $p \leq q < p_1$. Repeating this argument with an initial data in X_q^β , $q < p_1$, we deduce that the solution enters X_q^β , $p \leq q < p_2 = np_1/(n - 2(\alpha - \beta)p_1)$. Inductively, $u(t, u_0)$ enters X_q^β , $p \leq q < p_j = np_{j-1}/(n - 2(\alpha - \beta)p_{j-1})$ for any $j \in \mathbb{N}$. The sequence p_j is increasing and may not be bounded. If it were, its limit r would satisfy $r = nr/(n - 2(\alpha - \beta)r)$. This however would lead to the equality $\alpha = \beta$ contradicting our hypothesis. Hence $u(t, u_0) \in X_q^\beta$ for any $q \geq p$. In particular, given $u_0 \in X_p^\beta$, $u(t, u_0) \in X_q^\beta$ for $t > 0$ and $2\beta q > n$.

Using any upper (resp. lower) bound $w_0^+ \in C \cap X_q^\beta$ (resp. $-w_0^- \in C \cap X_q^\beta$) for $u(t_0, u_0)$, $t_0 \in [0, \tau_{u_0})$ fixed, we find as in Lemma 2 that

$$(22) \quad \|u(t, u(t_0, u_0))\|_{L^\infty(\Omega, \mathbb{R}^m)} \leq M(u_0)t^{-\beta} + K,$$

where

$$K = \max\{\|((- \Delta_N)^\alpha - c_1^+)^{-1}c_2^+\|_{L^\infty(\Omega, \mathbb{R}^m)}, \|((- \Delta_N)^\alpha - c_1^-)^{-1}c_2^-\|_{L^\infty(\Omega, \mathbb{R}^m)}\}.$$

Since the solution does not blow up in the uniform norm, it must exist for all $t \geq 0$ and Assumption 1 is satisfied.

Corollary 2 now holds true and (17) has a global attractor \mathcal{A} . Additionally, (22) yields

$$\limsup_{t \rightarrow \infty} \|u(t, u_0)\|_{L^\infty(\Omega, \mathbb{R}^m)} \leq K.$$

REMARK 2. Note that $[L^p(\Omega, \mathbb{R}^m), D(-\Delta_N)]_\theta = H_{p,N}^{2\theta}$ except for $2\theta = 1 + 1/p$ (see [3], page 35), where $[\cdot, \cdot]_\theta$ denotes the complex interpolation functor of exponent $\theta \in (0, 1)$ (see [20]). From embeddings for Bessel potential spaces and embeddings I.2.9.6 and I.2.5.2 in [4] we obtain

$$X_p^\alpha \hookrightarrow [L^p(\Omega, \mathbb{R}^m), D(-\Delta_N)]_{\alpha^-} \hookrightarrow [L^q(\Omega, \mathbb{R}^m), D(-\Delta_N)]_{\beta^+} \hookrightarrow X_q^\beta,$$

for $\beta < \beta^+ < \alpha^- < \alpha$ and $1 < p \leq q < np/(n - 2(\alpha^- - \beta^+)p)$.

REMARK 3. The above considerations remain unchanged if instead of $-\Delta_N$ we consider any second order uniformly strongly elliptic operator in divergence form. Also the boundary condition can be changed to a more general form $b\partial u/\partial\nu = au$ provided that $a, b \geq 0$ and $a^2 + b^2 \neq 0$.

Consider now the fractional powers of a second order uniformly strongly elliptic operator which is not in divergence form. Let $a_{ij}, b_i, 1 \leq i, j \leq n, \Omega$ be sufficiently regular and L be the second order partial differential operator

$$-Lu = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i}.$$

As a consequence of a well known comparison result for parabolic equations (see [22], page 187), the analytic semigroup e^{-Lt} generated by the operator $-L$ with Dirichlet boundary condition is increasing. Therefore we can replace $(-\Delta_N)^\alpha$ in (17) by a finite sum of fractional powers of L .

4.3. Integral-pseudodifferential equations. Consider equations of the type

$$(23) \quad u_t(t, x) + (-\Delta_N)^\alpha u(t, x) = \int_{\Omega} G(x, y)u(t, y) dy + f(u).$$

As before Δ_N denotes the Neumann Laplacian in $L^p(\Omega, \mathbb{R}^m), p \in (1, \infty), \alpha \in (0, 1], \Omega$ is a bounded smooth domain in $\mathbb{R}^n, u \in \mathbb{R}^m, G : \Omega \times \Omega \rightarrow \mathbb{R}, G \geq 0,$ and $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is quasi-monotone increasing.

For local well posedness of (23) in X_p^β with $\beta < \alpha$ (and $X_p^\theta, 0 \leq \theta \leq 1,$ as in Section 4.2) we assume that f is locally Lipschitz and either $2\beta > n/p$ or (18) is satisfied. Let $A = (-\Delta_N)^\alpha$ and assume that G is such that the operator

$$v \xrightarrow{B} \int_{\Omega} G(\cdot, y)v(y) dy$$

is bounded in X_p (for these conditions on G and estimates on $\|B\|_{L(X_p)}$ in terms of G see [14], page 134).

The order relation in X_p is induced by the usual order in \mathbb{R}^m . We need first justify that the semigroup generated by $-A + B$ is increasing. Indeed, $-A$ generates an analytic, increasing semigroup of contractions on X_p . Since the integral kernel is nonnegative, e^{Bt} is increasing and the assumptions of Proposition 2 (see Appendix) are satisfied. Therefore, $e^{(-A+B)t}$ is increasing. Also, since B is bounded, $A - B$ is sectorial and the fractional power spaces corresponding to $A - B$ coincide with the fractional power spaces X_p^α of A (see [13], page 29).

Next we summarize the comparison results for (23). Assume that $f^+, f^- : \mathbb{R}^m \rightarrow \mathbb{R}^m$ are locally Lipschitz continuous and consider the problems

$$(24) \quad \begin{aligned} u_t^+ + (-\Delta_N)^\alpha u^+ - Bu^+ &= f^+(u^+), \quad t > 0, x \in \Omega, \\ u^+(0) &= u_0^+ \in X_p^\beta, \end{aligned}$$

and

$$(25) \quad \begin{aligned} u_t^- + (-\Delta_N)^\alpha u^- - Bu^- &= f^-(u^-), \quad t > 0, x \in \Omega, \\ u^-(0) &= u_0^- \in X_p^\beta. \end{aligned}$$

As in Corollary 3 we have:

COROLLARY 4. *Assume that $2\beta p > n$, $f^+ \geq f^-$, and either f^+ or f^- is quasi-monotone increasing. If $u^+(t, u_0^+)$, $u^-(t, u_0^-)$ denote the solutions of (24) and (25) respectively and $u_0^+ \geq u_0^-$, then $u^+(t, u_0^+) \geq u^-(t, u_0^-)$ for as long as both solutions exist.*

Concentrating as before on the case $2\beta p \leq n$ assume G is such that $B \in L(X_q)$ for $p \leq q \leq q_1$ with $q_1 > n/(2\beta)$, f satisfies (18) and is quasi-monotone increasing.

For existence of an attractor we assume that the following dissipativeness condition is satisfied:

$$(26) \quad \limsup_{|u_j| \rightarrow \infty} f_j(u)/u_j < -\|B\|.$$

From (26) the condition (2) is satisfied with $c_1^\pm < -\|B\|$.

Now, for a result similar to Lemma 2, we proceed exactly as in Subsection 4.2 using the constant $K = \max\{\|((- \Delta_n)^\alpha - B - c_1^+)^{-1} c_2^+\|_{L^\infty(\Omega, \mathbb{R}^m)}, \|((- \Delta_n)^\alpha - B - c_1^-)^{-1} c_2^-\|_{L^\infty(\Omega, \mathbb{R}^m)}\}$.

Corollary 4 now implies that the semigroup associated with (23) has a global attractor \mathcal{A} and

$$\limsup_{t \rightarrow \infty} \|u(t, u_0)\|_{L^\infty(\Omega, \mathbb{R}^m)} \leq K.$$

5. Appendix. Here we give conditions enabling us to obtain many operators with increasing resolvent starting from known increasing resolvent operators. In particular we need to know in which cases the sum of increasing resolvent operators has increasing resolvent and that a root of an increasing resolvent operator has increasing resolvent.

5.1. Operators with increasing resolvent and increasing semigroups. The following well known result establishes the equivalence between the monotonicity of the resolvent of the generator of a C_0 -semigroup and the monotonicity of the semigroup itself.

PROPOSITION 1. *Let (X, \leq) be an ordered Banach space, and let A be a sectorial operator. Assume that there exists a $\lambda_0 \in \mathbb{R}$ such that $\operatorname{Re} \sigma(A) > \lambda_0$. Then $e^{-At} : X \rightarrow X$ is increasing for all $t \geq 0$ if and only if $(A + \lambda)^{-1}$ is increasing for every $\lambda > \lambda_0$.*

The corollary below plays an important role in the proof of most comparison results presented in the paper.

COROLLARY 5. Let (X, \leq) be an ordered Banach space, and let A be a sectorial operator. Assume that for every $\lambda > \lambda_0$, $(A + \lambda)^{-1}$ is increasing. Let $u_\eta(t, u_0)$ denote the solution of

$$\dot{u} + Au = \eta u, \quad u(0) = u_0.$$

If $0 \leq u_1 \leq u_2$ and $\lambda < \mu$, then $u_\lambda(t, u_1) \leq u_\lambda(t, u_2) \leq u_\mu(t, u_2)$ for every $t \geq 0$.

Our next result provides conditions under which the sum of operators with increasing resolvent is an operator with increasing resolvent.

THEOREM 7. Assume that $-A$ and $-B$ are generators of bounded C_0 -semigroups, A and B commute, $A + B$ is closed and densely defined with domain $D(A) \cap D(B)$, and $\lambda \in \rho(-A - B)$ for some $\lambda > 0$. Then $-A - B$ generates a bounded C_0 -semigroup satisfying $e^{-(A+B)t} = e^{-At}e^{-Bt}$.

Proof. For a moment let us change the norm in the Banach space X in such a way that $-A$ generates a C_0 -semigroup of contractions. Let $-A_\lambda = -\lambda A(\lambda + A)^{-1}$ and $-B_\lambda = -\lambda B(\lambda + B)^{-1}$. Then $\|e^{-A_\lambda t}\| \leq 1$ for all $\lambda > 0$ and since $e^{-A_\lambda t}x \rightarrow e^{-At}x$ and $e^{-B_\lambda s}x \rightarrow e^{-Bs}x$ for all $x \in X$, $s, t \geq 0$, we have

$$\lim_{\lambda \rightarrow \infty} e^{-A_\lambda t - B_\lambda s}x = \lim_{\lambda \rightarrow \infty} e^{-A_\lambda t}e^{-B_\lambda s}x = e^{-At}e^{-Bs}x.$$

Of course the above remains true in the original norm. Also, from a similar argument, we have

$$\lim_{\lambda \rightarrow \infty} e^{-B_\lambda t - A_\lambda s}x = \lim_{\lambda \rightarrow \infty} e^{-B_\lambda t}e^{-A_\lambda s}x = e^{-Bs}e^{-At}x,$$

showing that $e^{-At}e^{-Bs} = e^{-Bs}e^{-At}$.

Let us now show that $T(t) = e^{-At}e^{-Bt}$ is a bounded C_0 -semigroup with generator $-(A + B)$. First we observe that strong continuity at $t = 0$ and boundedness are clear and from

$$\begin{aligned} T(t+s) &= e^{-A(t+s)}e^{-B(t+s)} = e^{-At}e^{-As}e^{-Bt}e^{-Bs} = e^{-At}e^{-Bt}e^{-As}e^{-Bs} \\ &= T(t)T(s) \end{aligned}$$

we see that $T(t)$ is a semigroup. It remains to show that $-(A + B)$ is the generator of $T(t)$.

If $x \in D(A) \cap D(B) = D(A + B)$, then

$$\begin{aligned} T(t)x - x &= \lim_{\lambda \rightarrow \infty} (e^{-tA_\lambda}e^{-tB_\lambda}x - x) \\ &= \lim_{\lambda \rightarrow \infty} (e^{-A_\lambda t}e^{-B_\lambda t}x - e^{-B_\lambda t}x + e^{-B_\lambda t}x - x) \\ &= \lim_{\lambda \rightarrow \infty} \int_0^t e^{-A_\lambda s}e^{-B_\lambda t}(-A_\lambda x) ds + \lim_{\lambda \rightarrow \infty} \int_0^t e^{-B_\lambda s}(-B_\lambda x) ds \\ &= \int_0^t e^{-As}e^{-Bt}(-Ax) ds + \int_0^t T(s)(-Bx) ds. \end{aligned}$$

Now

$$\begin{aligned} \frac{1}{t}(T(t)x - x) &= \int_0^t e^{-As} e^{-Bt}(-Ax) ds + \int_0^t T(s)(-Bx) ds \\ &\rightarrow -(A+B)x \quad \text{as } t \rightarrow 0^+, \end{aligned}$$

for any $x \in D(A) \cap D(B) = D(A+B)$. Therefore the generator $-C$ of $T(t)$ must be an extension of $-(A+B)$. Let λ be a real number in the resolvent of $A+B$ and in the resolvent of the generator of $T(t)$. Then

$$X = (\lambda + (A+B))D(A+B) = (\lambda + C)D(C),$$

hence $A+B = C$ and the proof is complete. ■

COROLLARY 6. *If $-A$, $-B$, $-(A+B)$ are generators of C_0 -semigroups, A and B commute and have increasing resolvents, then $-(A+B)$ has increasing resolvent.*

The above results are closely related to the following results (see [21, 9]).

PROPOSITION 2. *Assume that $-A$ and $-B$ are generators of C_0 -semigroups, $D(A) \cap D(B)$ is dense in X and*

$$\|(e^{-At} e^{-Bt})^n\| \leq M e^{\omega t}, \quad n = 1, 2, \dots,$$

for some constants $M \geq 1$ and $\omega \geq 0$. If for some λ with $\operatorname{Re} \lambda > \omega$ the range of $\lambda I + A + B$ is dense in X , then the closure of $-(A+B)$ is the generator of a C_0 -semigroup $T(t)$ satisfying $\|T(t)\| \leq M e^{\omega t}$, $t \geq 0$. Furthermore,

$$T(t)x = \lim_{n \rightarrow \infty} (e^{-A(t/n)} e^{-B(t/n)})^n x, \quad x \in X,$$

uniformly on bounded subsets of \mathbb{R}^+ .

PROPOSITION 3. *If $-A$, $-B$, and $-(A+B)$ generate C_0 -semigroups, $\|e^{-(A+B)t}\| \leq M e^{\omega t}$, $t \geq 0$, and*

$$\|[(I + tA)^{-1}(I + tB)^{-1}]^n\| \leq M e^{\omega t}, \quad n = 1, 2, \dots,$$

then

$$e^{-(A+B)t}x = \lim_{n \rightarrow \infty} \left[\left(I + \frac{t}{n}A \right)^{-1} \left(I + \frac{t}{n}B \right)^{-1} \right]^n x, \quad x \in X.$$

COROLLARY 7. *If either the assumptions of Proposition 2 or the assumptions of Proposition 3 are satisfied, then*

$$e^{-(A+B)t} \geq 0, \quad t \geq 0,$$

or, equivalently, $(\lambda + A + B)^{-1}$ is increasing for $\lambda > \omega$.

For a proof of Propositions 2 and 3 see [18], §3.5. Corollary 7 provides tools to show that the resolvent of a sum of increasing resolvent operators is increasing, without the hypothesis that the operators commute. These will deal well with the case when the operators involved are dissipative. For

some cases when the operators are not dissipative it may be more suitable to use Theorem 7.

The increasing resolvent property is preserved when we change the norm of the space to an equivalent one. This leads us to infer that it may be useful to know when one can change the norm of the space to an equivalent one in such a way as to make A and B simultaneously dissipative. For this we refer to [18], §1.5. The conditions on A and B that enable us to make such a change of norm are similar to the conditions in Propositions 2 and 3.

These results should contribute to enlarge the class of increasing resolvent operators. Our next result, which is a slight extension of Theorem 1.3.2 of [11], aims at simplifying the verification that some operators have increasing resolvents.

PROPOSITION 4. *Let H be an ordered Hilbert space and C its positive cone. Let $A : D(A) \subset H \rightarrow H$ be a self-adjoint positive semi-definite operator, that is, $\langle Au, u \rangle \geq 0$ for all $u \in D(A)$. Assume that H has a dense subset D such that:*

- $(A + \alpha)^{-1}D \subset D$,
- for each $d \in D$ we can define $|d| = \sup\{d, -d\} \in D \cap C$ such that $\|d\| = \||d|\|$ (then $d \in D$ is in C if and only if $d = |d|$),
- $\langle |d|, g \rangle \geq |\langle d, g \rangle|$ for all $d \in D$ and $g \in C$.

Consider the following assertions:

- (i) If $u \in D(A^{1/2})$ then $|u| \in D(A^{1/2})$ and

$$\langle A^{1/2}|u|, A^{1/2}|u| \rangle \leq \langle A^{1/2}u, A^{1/2}u \rangle.$$
- (ii) $(A + \lambda)^{-1}$ is increasing for all $\lambda > 0$.

Then (i) implies (ii).

5.2. Fractional powers and operators with increasing resolvent. Probably the most complete description of fractional powers of positive operators was given by H. Komatsu in a number of papers written in the middle 60's. In particular in [15] we find the following definition of an operator A of type $(\omega, M(\theta))$, $0 \leq \omega \leq \pi$:

DEFINITION 4. A densely defined closed linear operator A such that the resolvent of $-A$ contains the sector $|\arg \lambda| < \pi - \omega$ and

$$\sup_{|\arg \lambda| = \theta} \|\lambda(\lambda + A)^{-1}\| \leq M(\theta) < \infty$$

for $0 \leq \theta < \pi - \omega$ is called an operator of type $(\omega, M(\theta))$.

When $\omega < \pi/2$ the above notion coincides with that of a positive sectorial operator as in [13]. Further in [15, p. 319], we find a formula describing the resolvent of fractional powers of A through the resolvent of A . Namely, if A

is of type $(\omega, M(\theta))$ and $0 < \alpha < \pi/\omega$, then every $\lambda > 0$ is in the resolvent set $\rho((-A)^\alpha)$, and for $\alpha \in (0, 1)$,

$$(27) \quad (\lambda I + (-A)^\alpha)^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\tau^\alpha}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} (\tau I + A)^{-1} d\tau.$$

From the above formula it is evident that the resolvent of $(-A)^\alpha$ is increasing for positive λ whenever so is the resolvent of A .

This result has been used in [10], when $-A$ is the Dirichlet Laplacian in a bounded smooth domain, to obtain an “integration by parts formula” for fractional powers of operators. This formula was important to obtain energy estimates that ensured the existence of attractors. Here instead of energy estimates we have used comparison results.

5.3. Consistence of orderings in fractional power spaces. Finally we define an ordering in fractional power spaces of an ordered Banach space and state a result that shows the consistence of this ordering relation (see [6]).

Let (X, \leq) be an ordered Banach space and A be a sectorial operator in X . In X^α , for $\alpha > 0$, we consider the ordering induced by X . The positive cone in X^α is denoted by C_α .

PROPOSITION 5. *With the ordering induced by X , X^α is an ordered Banach space for any $\alpha \geq 0$. If $\alpha > \beta \geq 0$, then the inclusion $i_{\alpha,\beta} : X^\alpha \rightarrow X^\beta$ is increasing. Moreover, $C_\alpha \subset C_\beta$, $C_\beta \cap X^\alpha = C_\alpha$ and $\overline{C}_\alpha^{X^\beta} = C_\beta$.*

This shows the consistence of “ $f \geq 0$ ”, independently of the space X^α in which f lies. Therefore, we do not need to distinguish them.

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