COLLOQUIUM MATHEMATICUM

VOL. 90 2001 NO. 1

A NOTE ON CERTAIN SEMIGROUPS OF ALGEBRAIC NUMBERS

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Abstract. The cross number $\kappa(a)$ can be defined for any element a of a Krull monoid. The property $\kappa(a) = 1$ is important in the study of algebraic numbers with factorizations of distinct lengths. The arithmetic meaning of the weaker property, $\kappa(a) \in \mathbb{Z}$, is still unknown, but it does define a semigroup which may be interesting in its own right. This paper studies some arithmetic (divisor theory) and analytic (distribution of elements with a given norm) properties of that semigroup and a related semigroup of ideals.

1. Notation. In the first section we consider a Krull monoid M (written multiplicatively), i.e. a commutative cancellative semigroup with a unit, for which there exists a group epimorphism $v : (M) \to \coprod_{i\in I} \mathbb{Z}$ of (M) (the group of quotients of M) onto a free abelian group such that $M = \{x \in (M) : v_i(x) \geq 0 \text{ for all } i \in I\}$, as defined in [5] and [6]. The concept of a Krull monoid is equivalent to that of a semigroup with divisor theory, as shown in [6]. Let $\partial : M \to D$, where D is a free abelian semigroup, be a divisor theory for M with the class group written as $\mathrm{Cl}(M)$. We further assume that $\mathrm{Cl}(M)$ is finite and that there are infinitely many prime elements of D (prime divisors) in each class. The neutral element of $\mathrm{Cl}(M)$, the *principal class*, is denoted as $\mathcal{H}(M)$. If $\mathfrak{a} \in D$, then $[\mathfrak{a}]$ will denote the class of $\mathfrak{a}$ in the class group.

In the second section we apply the results obtained for general Krull monoids to an algebraic number field K with the ring R_K of algebraic integers and the semigroup $\mathcal{I}(R_K)$ of non-zero integral ideals. We do it in the obvious way by fixing $M = R_K^*$ (the multiplicative semigroup) and $\partial : R_K^* \to \mathcal{I}(R_K)$, $\partial(a) = (a)$. All of our previous assumptions on M are satisfied by R_K^* for arbitrary K. In this case H denotes the class group of K, h is the class number and $\mathcal{H}$ stands for the class of principal ideals. The set of non-zero prime ideals of R_K is written as $\mathcal{P}(R_K)$. The Dedekind zeta-function of K is denoted by ζ_K. We also adopt the standard shorthand notation $\mathrm{e}(x) = \exp(2\pi i x)$.

If $X \in \mathrm{Cl}(M)$ and $a \in M$ or $a \in D$, then $\Omega_X(a)$ denotes, as usual, the number of prime divisors of a in X. The *cross number* (cf. [4]) of elements

2000 *Mathematics Subject Classification*: 11R27, 11N45, 11M41.

of D and M is defined as

$$\kappa(\mathfrak{a}) = \sum_{X \in \mathrm{Cl}(M)} \frac{\Omega_X(\mathfrak{a})}{\operatorname{ord} X}$$

for $\mathfrak{a} \in D$, and

$$\kappa(a) = \kappa(\partial(a))$$

for $a \in M$. This quantity was also called *weight* in [3] and *Zaks–Skula function* in [1].

We will be concerned with the subset S_M of M defined as

$$S_M = \{a \in M : \kappa(a) \in \mathbb{Z}\}$$

and an analogous subset of D,

$$\mathcal{S}_M = \{\mathfrak{a} \in D : \kappa(\mathfrak{a}) \in \mathbb{Z}\}.$$

In particular, for $M = R_K^*$, we put $S_K = S_M$ and $\mathcal{S}_K = \mathcal{S}_M$. The condition $\kappa(a) \in \mathbb{Z}$ was considered by Śliwa for $M = R_K^*$ ([13], condition C_0). Its stronger version, (C) $\kappa(a) = 1$, is related to distinct lengths of factorizations of an element into irreducibles. For example, if A consists of those elements of M whose prime divisors all lie in a given set of classes, then all elements of A have unique factorization lengths if and only if each irreducible element in A satisfies (C). In this case $\kappa(a)$ gives the length of factorization of any $a \in A$ (cf. [12] and also [11] and [14]).

This paper describes some of the intrinsic properties of S_M and $\mathcal{S}_M$. In the case of an algebraic number field we also give asymptotic formulae for the number of elements of S_K and $\mathcal{S}_K$ whose norms do not exceed a given bound.

This research was supported by grant No. 2 PO3A 024 17 of the State Committee of Scientific Research (KBN) of Poland. The author wishes to express his gratitude to Professor Jerzy Kaczorowski for his guidance and help during the preparation of this paper.

2. Arithmetic characterization. The set S_M is a multiplicative subset of M and thus a commutative semigroup with cancellation law. For every $a \in M$ there exists an element $b \in S_M$ such that $a \mid b$. In fact, if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are elements of D with $\mathfrak{a}_2 \nmid \mathfrak{a}_1$, then we can find $b \in S_M$ such that $\mathfrak{a}_1 \mid \partial(b)$, but $\mathfrak{a}_2 \nmid \partial(b)$. Indeed, if $\mathfrak{a}_1 = \prod_{i=1}^q \mathfrak{p}_i^{\alpha_i}$, $\mathfrak{p}_i$ being prime, then let $\mathfrak{a}_3 = \prod_{i=1}^q \mathfrak{p}_i^{\alpha_i} \mathfrak{q}_i^{\alpha_i(\operatorname{ord}[\mathfrak{p}_i]-1)}$, where each $\mathfrak{q}_i$ is a prime element not dividing $\mathfrak{a}_2$, $\mathfrak{q}_i \in [\mathfrak{p}_i]$. Obviously $\mathfrak{a}_3$ is in the principal class, and any $b \in M$ such that $\partial(b) = \mathfrak{a}_3$ satisfies our assertion. If $a, b \in S_M$, then the relation $a \mid b$ in M or $\partial(a) \mid \partial(b)$ in D implies $a \mid b$ in S_M. It is easy to check that those conditions are necessary and sufficient for the semigroup homomorphism ∂ restricted to S_M to define a divisor theory for S_M (cf. [8]). Analogous remarks apply to the set $\mathcal{S}_M$.

It is convenient to introduce a semigroup homomorphism $f : D \to \mathbb{C}^*$,

$$f(\mathfrak{a}) = \mathrm{e}(\kappa(\mathfrak{a})).$$

Since $\mathcal{S}_M = \ker f$, the class group of $\mathcal{S}_M$ is given by $\mathrm{Cl}(\mathcal{S}_M) \cong \operatorname{im} f = \mu_m \cong C_m$, where $m = \max_{X\in \mathrm{Cl}(M)} \operatorname{ord} X$ is the exponent of $\mathrm{Cl}(M)$. Now we compute the quotient group M/S_M (or, equivalently, $(M)/(S_M)$) and the class group of S_M.

Theorem 1. *Let M be a Krull monoid with a finite class group* $\mathrm{Cl}(M) \cong C_{d_1} \oplus \ldots \oplus C_{d_k}$, $d_1 \mid d_2 \mid \ldots \mid d_k = m$, *having infinitely many prime divisors in each class. Additionally, let $d_0 = 1$. We have*

(i) $$M/S_M \cong \begin{cases} C_m & \text{if } 2\nmid \frac{m}{d_{k-1}}, \\ C_{m/2} & \text{otherwise.} \end{cases}$$

(ii) $$\mathrm{Cl}(S_M) \cong \begin{cases} \mathrm{Cl}(M) \oplus C_m & \text{if } 2\nmid \frac{m}{d_{k-1}}, \\ \mathrm{Cl}(M) \oplus C_{m/2} & \text{otherwise.} \end{cases}$$

Proof. (i) Obviously $M/S_M \cong f(\mathcal{H}(M)) < \mu_m$. If $2\nmid \frac{m}{d_{k-1}}$, then we choose classes X_1, X_2 such that $\operatorname{ord} X_1 = \operatorname{ord} X_1X_2 = m$ and $\operatorname{ord} X_2 = d_{k-1}$. Now, if $\mathfrak{p}_1$, $\mathfrak{p}_2$, $\mathfrak{p}_3$, $\mathfrak{p}_4$ are prime, $\mathfrak{p}_1 \in X_1$, $\mathfrak{p}_2 \in X_2$, $\mathfrak{p}_3 \in X_1X_2$, $\mathfrak{p}_4 \in X_1^{-1}$ and $r = (2md_{k-1} - m + d_{k-1})/(2d_{k-1})$ then $\mathfrak{p}_1^{r+1}\mathfrak{p}_2\mathfrak{p}_3^{m-1}\mathfrak{p}_4^r$ is in the principal class and $f(\mathfrak{p}_1^{r+1}\mathfrak{p}_2\mathfrak{p}_3^{m-1}\mathfrak{p}_4^r) = \mathrm{e}(1/m)$, so $f(\mathcal{H}(M)) = \mu_m$.

Suppose now that $2 \mid \frac{m}{d_{k-1}}$. Since $f(\mathfrak{p}\mathfrak{q}) = \mathrm{e}(2/m)$ for $\mathfrak{p} \in X$, $\mathfrak{q} \in X^{-1}$, $\operatorname{ord} X = m$, we have $\mu_{m/2} < f(\mathcal{H}(M))$. Let $F = \{X \in \mathrm{Cl}(M) : X^{m/2} = 1\}$. Clearly $\mathrm{Cl}(M)/F \cong C_2$, so if $\mathfrak{p}_1^{\alpha_1} \cdot \ldots \cdot \mathfrak{p}_l^{\alpha_l}$ is in the principal class with $\mathfrak{p}_i$ prime, $\mathfrak{p}_i \in X_i$, then $\sum_{X_i \notin F} \alpha_i \equiv 0 \pmod 2$. Note that $m/\operatorname{ord} X$ is even if and only if $X \in F$. Therefore

$$\sum_{i=1}^{l} \frac{m\alpha_i}{\operatorname{ord} X_i} \equiv \sum_{X_i \notin F} \frac{m\alpha_i}{\operatorname{ord} X_i} \equiv \sum_{X_i \notin F} \alpha_i \equiv 0 \pmod 2.$$

Hence $f(\mathcal{H}(M)) < \mu_{m/2}$ and consequently $f(\mathcal{H}(M)) = \mu_{m/2}$.

To prove (ii), let π_i, $i = 1, \ldots, k$, denote the projection of $\mathrm{Cl}(M)$ onto the summand C_{d_i} of $C_{d_1} \oplus \ldots \oplus C_{d_k}$ and let $\widetilde{f} : D \to C_m$,

$$\widetilde{f}(\mathfrak{a}) = m\kappa(\mathfrak{a}) \pmod m.$$

Put $G = C_{d_1} \oplus \ldots \oplus C_{d_k} \oplus C_m$. The homomorphism $\iota : D \to G$, $\iota(\mathfrak{a}) = (\pi_1([\mathfrak{a}]), \ldots, \pi_k([\mathfrak{a}]), f(\mathfrak{a}))$, induces an isomorphism of $\mathrm{Cl}(S_M)$ onto the subgroup $\operatorname{im} \iota$ of G. If $2\nmid \frac{m}{d_{k-1}}$, then by (i), $|\mathrm{Cl}(S_M)| = |G|$, so $\mathrm{Cl}(S_M) \cong G$. Similarly, if $2 \mid \frac{m}{d_{k-1}}$, then $(G : \operatorname{im} \iota) = 2$ and we show that $\operatorname{im} \iota = F = \{(a_1, \ldots, a_{k+1} \in G : a_k + a_{k+1} \equiv 0 \pmod 2)\}$. Indeed, for any prime $\mathfrak{p} \in D$

we have $2\nmid\pi_k[\mathfrak{p}]$ if and only if $2\nmid\widetilde{f}(\mathfrak{p})$, hence $\kappa(\mathfrak{p})\subset F$ and the result follows. ■

In 1960 Carlitz [2] showed that all elements of R_K^* have unique factorization lengths if and only if $h(K)\le 2$. The following corollary is essentially the Carlitz theorem.

COROLLARY 1. *For an algebraic number field K we have $S_K=R_K^*$ if and only if $h(K)\le 2$.*

3. Analytic properties of S_K and $\mathcal{S}_K$. Although generally S_K and $\mathcal{S}_K$ are not arithmetic semigroups (as defined in [7]), the next theorem shows that there is a degree of regularity in the distribution of their elements. Let m be as defined previously and let σ and t denote the real and imaginary parts of the complex variable s.

THEOREM 2. *For every algebraic number field K and every complex number s with $\operatorname{Re} s>1$, we have*

$$\sum_{\mathfrak{a}\in\mathcal{S}_K\cap\mathcal{H}}\frac{1}{N(\mathfrak{a})^s}=\frac{1}{|\mathrm{Cl}(S_K)|}\zeta_K(s)+\sum_{j=1}^{r}\frac{g_j(s)}{(s-1)^{w_j}}+g_{r+1}(s)$$

and

$$\sum_{\mathfrak{a}\in\mathcal{S}_K}\frac{1}{N(\mathfrak{a})^s}=\frac{1}{|\mathrm{Cl}(\mathcal{S}_K)|}\zeta_K(s)+\sum_{j=1}^{l}\frac{h_j(s)}{(s-1)^{z_j}}+h_{l+1}(s),$$

where w_j, $j=1,\ldots,r$, and z_j, $j=1,\ldots,l$, are complex numbers whose real parts are in the range $[0,1-\delta]$, δ being a constant depending only on H, $\delta>0$, and g_j, $j=1,\ldots,r+1$, h_j, $j=1,\ldots,l+1$, are complex functions with a regular, single-valued analytic continuation in the region

$$D=\left\{\sigma+it:2\ge\sigma>1-\frac{c_1}{\log(|t|+2)}\right\}$$

with a constant $c_1>0$ depending only on K. Moreover, in the same region, we have $g_j(s)=\mathrm{O}((|t|+2)^{\operatorname{Re} w_j}\log^{c_2}(|t|+3))$, $j=1,\ldots,r+1$, and $h_j(s)=\mathrm{O}((|t|+2)^{\operatorname{Re} z_j}\log^{c_2}(|t|+3))$, $j=1,\ldots,l+1$, with a constant $c_2>0$ depending only on K.

First, we introduce a family of functions (analogous to L-functions) suitable for our problem. Any character ψ of $\mathrm{Cl}(S_K)$ defines a completely multiplicative function on $\mathcal{I}(R_K)$. We refer to this function as ψ as well. Set

$$\begin{aligned}\mathcal{L}(s,\psi)&=\sum_{\mathfrak{a}\in\mathcal{I}(R_K)}\frac{\psi(\mathfrak{p})}{N(\mathfrak{a})^s}\qquad(\operatorname{Re} s>1)\\&=\prod_{\mathfrak{p}\in\mathcal{P}(R_K)}\frac{1}{1-\psi(\mathfrak{p})/N(\mathfrak{p})^s}.\end{aligned}$$

The function $\mathcal{L}(s,\psi)$ is holomorphic in the half-plane $\operatorname{Re} s > 1$. In case $\psi \in \widehat{H}$ we get $\mathcal{L}(s,\psi) = L(s,\psi)$, where $L(s,\psi)$ is a Hecke L-function (or Hecke zeta-function, cf. [10, p. 343]), but in general the properties of $\mathcal{L}(s,\psi)$ are not as nice as those of $L(s,\psi)$.

Theorem 1(ii) shows that ψ can be represented as a product $\psi = \psi_1\psi_2$, where ψ_1 is a character of H and ψ_2 is a character of $\mathcal{I}(R_K)/\mathcal{S}_K$. If H is as in the first case of the theorem (i.e. $2 \nmid \frac{m}{d_{k-1}}$), then the choice of ψ_1 and ψ_2 is unique. In the second case ($2 \mid \frac{m}{d_{k-1}}$) we have exactly two choices, because $\mathrm{Cl}(S_K)$ is identified with a subgroup of index 2 of $H \times \mathcal{I}(R_K)/\mathcal{S}_K$, so ψ has 2 extensions to $H \times \mathcal{I}(R_K)/\mathcal{S}_K$. We substitute $\psi_2(\mathfrak{a}) = f(\mathfrak{a})^k$ for some $k \in \{0,\ldots,m-1\}$. Now, for $\operatorname{Re} s > 1$,

$$\begin{aligned}
\mathcal{L}(s,\psi) &= \prod_{X\in H}\prod_{\mathfrak{p}\in X}\frac{1}{1-\psi_1(X)\,\mathrm{e}(k/\operatorname{ord} X)/N(\mathfrak{p})^s}\\
&= \exp\Bigg(\sum_{X\in H}\sum_{\mathfrak{p}\in X}\frac{\psi_1(X)\,\mathrm{e}(k/\operatorname{ord} X)}{N(\mathfrak{p})^s}\Bigg)\cdot \xi_1(s)\\
&= \exp\Bigg(\sum_{X\in H}\psi_1(X)\,\mathrm{e}\bigg(\frac{k}{\operatorname{ord} X}\bigg)\sum_{\mathfrak{p}\in X}\frac{1}{N(\mathfrak{p})^s}\Bigg)\cdot \xi_1(s)\\
&= \exp\Bigg(\sum_{\chi\in\widehat{H}}\bigg(\frac{1}{h}\sum_{X\in H}\mathrm{e}\bigg(\frac{k}{\operatorname{ord} X}\bigg)\psi_1(X)\overline{\chi(X)}\bigg)\log L(s,\chi)\Bigg)\cdot\xi_2(s)\\
&= \exp\Bigg(\sum_{\chi\in\widehat{H}} a(k,\psi_1\overline{\chi})\log L(s,\chi)\Bigg)\cdot\xi_2(s),
\end{aligned}$$

where

$$a(k,\chi) = \frac{1}{h}\sum_{X\in H}\mathrm{e}\bigg(\frac{k}{\operatorname{ord} X}\bigg)\chi(X)$$

for all $\chi \in \widehat{H}$ and ξ_1,ξ_2 are Dirichlet series with abscissas of absolute convergence $\le 1/2$.

We can see that $\mathcal{L}(s,\psi)$ is essentially a product of complex powers of Hecke L-functions. The rest of the proof consists of two lemmas.

LEMMA 1. *With k, χ and $a(k,\chi)$ defined as above, we have $a(k,\chi) = 1$ if $f^k\chi = \chi_0$ (the trivial character) and* $\operatorname{Re} a(k,\chi) \in [-1,1)$ *otherwise. Moreover,* $\operatorname{Re} a(k,\chi) - \lfloor \operatorname{Re} a(k,\chi)\rfloor < 1-\delta$ *for some constant $\delta > 0$ depending only on H.*

Proof. To obtain the equality in the first case, observe that if $X \in H$ and $\mathfrak{p} \in X$ is a prime ideal, then $\mathrm{e}(k/\operatorname{ord} X)\chi(X) = f^k\chi(\mathfrak{p}) = 1$. On the other hand, if $f^k\chi \ne \chi_0$, then we can find a prime ideal $\mathfrak{p}$ such that $f^k\chi(\mathfrak{p}) \ne 1$. We have $\mathrm{e}(k/\operatorname{ord}[\mathfrak{p}])\chi([\mathfrak{p}]) \ne 1$ and each $\mathrm{e}(k/\operatorname{ord} X)\chi(X)$ is an mth root of unity, so $-1 \le \operatorname{Re} a(k,\chi) < 1$. The last assertion is obvious, since the number of possible values of $a(k,\chi)$ is finite for any H. ∎

LEMMA 2. *If ψ is a character of $\mathrm{Cl}(S_K)$, $\psi \ne \chi_0$, then, for $\operatorname{Re} s > 1$, $\mathcal{L}(s,\psi) = g(s)/(s-1)^w$, where $g(s)$ is a regular complex function defined in D and w is a complex number with $0 \le \operatorname{Re} w \le 1-\delta$ for some constant $\delta > 0$ depending only on H. Moreover, $g(s) = \mathrm{O}((|t|+2)^{\operatorname{Re} w} \log^{c_2}(|t|+3))$ in D for some constant c_2 depending only on K.*

Proof. By [10, pp. 356 and 372], $\log L(s,\chi)$ has a regular analytic continuation in D for all $\chi \in \widehat{H} \setminus \{\chi_0\}$, so

$$\exp\Big(\sum_{\chi\in\widehat{H}\setminus\{\chi_0\}} a(k,\psi_1\overline{\chi}) \log L(s,\chi)\Big) \cdot \xi_2(s)$$

is regular in D. Let $z = a(k,\psi_1)$. The remaining factor $\exp(a(k,\psi_1)\log\zeta_K(s))$ can be written either as $(s-1)^z\zeta_K^z(s)/(s-1)^z$, in case $\operatorname{Re} z \ge 0$, or as $(s-1)^{z+1}\zeta_K^z(s)/(s-1)^{z+1}$, in case $\operatorname{Re} z < 0$, and we put $w = z$ or $w = z+1$ accordingly. The function $(s-1)^w\zeta_K^z(s)$ is regular in D (again by [10, pp. 356 and 372]) and we have $0 \le \operatorname{Re} w \le 1-\delta$ taking δ from Lemma 1. The final upper bound is evident from the property

$$\log L(s,\chi) = \mathrm{O}(\log\log(|t| + e^e)), \qquad s \in D,\ |t| > 1,\ \chi \in \widehat{H},$$

quoted in [9]. ∎

Now it is enough to note that for $\operatorname{Re} s > 1$,

$$\sum_{\mathfrak{a}\in\mathcal{S}_K\cap\mathcal{H}} \frac{1}{N(\mathfrak{a})^s} = \frac{1}{mh}\sum_{k=0}^{m-1}\sum_{\chi\in\widehat{H}} \mathcal{L}(s, f^k\chi)$$

and

$$\sum_{\mathfrak{a}\in\mathcal{S}_K} \frac{1}{N(\mathfrak{a})^s} = \frac{1}{m}\sum_{k=0}^{m-1} \mathcal{L}(s, f^k).$$

By Lemma 2 our theorem is proved.

COROLLARY 2. *For an algebraic number field K let $S_K(x)$ denote the number of non-associated elements of S_K whose norms do not exceed x and let $\mathcal{S}_K(x)$ denote the number of ideals in $\mathcal{S}_K$ whose norms do not exceed x, $x > 0$. Then*

$$S_K(x) = \frac{cx}{|\mathrm{Cl}(S_K)|} + \mathrm{O}\left(\frac{x}{(\log x)^\delta}\right)$$

and

$$\mathcal{S}_K(x) = \frac{cx}{|\mathrm{Cl}(\mathcal{S}_K)|} + \mathrm{O}\left(\frac{x}{(\log x)^\delta}\right),$$

where $c = \operatorname{res}_{s=1}\zeta_K(s)$ and $\delta > 0$ is a constant depending only on H.

Proof. We are going to use the *Main Lemma* from [9], a Tauberian-type theorem which gives an upper bound for the error term. For the statement of the Main Lemma we refer the reader to [9]. Here we use it only in its simplest form ($q = 0$) and disregard most of the terms of the estimate. By Theorem 2 both of our functions fulfill the assumptions of case II of the Lemma and we get, for S_K,

$$S_K(x) = \frac{cx}{|\mathrm{Cl}(S_K)|}\left(1 + \sum_{j=1}^{r} \frac{Q_j(\log\log x)}{(\log x)^{1-w_j}}\right) + \mathrm{O}\left(\frac{x(\log\log x)^{c_3}}{\log x}\right)$$

where each Q_j is a complex polynomial with coefficients depending on K. The result for $\mathcal{S}_K$ is analogous. Taking δ from Theorem 2 we arrive at the desired conclusion. ■

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Received 9 November 2000;
revised 22 February 2001 (3982)