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GENERALIZED WEIGHTED QUASI-ARITHMETIC MEANS AND THE KOLMOGOROV–NAGUMO THEOREM

BY

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Abstract. A generalization of the weighted quasi-arithmetic mean generated by continuous and increasing (decreasing) functions $f_1, \ldots, f_k : I \to \mathbb{R}$, $k \geq 2$, denoted by $A^{[f_1,\ldots,f_k]}$, is considered. Some properties of $A^{[f_1,\ldots,f_k]}$, including "associativity" assumed in the Kolmogorov–Nagumo theorem, are shown. Convex and affine functions involving this type of means are considered. Invariance of a quasi-arithmetic mean with respect to a special mean-type mapping built of generalized means is applied in solving a functional equation. For a sequence of continuous strictly increasing functions $f_j : I \to \mathbb{R}$, $j \in \mathbb{N}$, a mean $A^{[f_1,f_2,\ldots]} : \bigcup_{k=1}^{\infty} I^k \to I$ is introduced and it is observed that, except symmetry, it satisfies all conditions of the Kolmogorov–Nagumo theorem. A problem concerning a generalization of this result is formulated.

1. Introduction. Supposing that a function $f : I \to \mathbb{R}$ is continuous and strictly monotonic in a real interval I and $f_1, \ldots, f_k : I \to \mathbb{R}$, $k \geq 2$, are arbitrary functions, we show that a function $M : I^k \to \mathbb{R}$ defined by

$$M(x_1, \ldots, x_k) := f^{-1}\Big(\sum_{j=1}^{k} f_j(x_j)\Big)$$

is a mean if, and only if, $f = \sum_{j=1}^{k} f_j$ and, for each $i \in \{1, \ldots, k\}$, the function f_i is continuous, monotonic, and of the same type of monotonicity as f (Theorem 1, cf. also [7] where the case $k = 2$ is considered). The function $A^{[f_1,\ldots,f_k]} := M$ generalizes the weighted quasi-arithmetic mean (cf. for instance [1], [2], [4]). We show, in particular, that $A^{[f_1,\ldots,f_k]}$ is symmetric iff it is quasi-arithmetic, and, for each $i \in \{1, \ldots, k\}$ and all $x_1, \ldots, x_k \in I$, we have

$$A^{[f_1,\ldots,f_k]}(x_1, \ldots, x_k) = A^{[f_1,\ldots,f_k]}\Big(\underbrace{y, \ldots, y}_{i \text{ times}}, x_{i+1}, \ldots, x_k\Big),$$

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where $y = A^{[f_1,\ldots,f_i]}(x_1,\ldots,x_i)$; so the mean $A^{[f_1,\ldots,f_k]}$ inherits the characteristic "associativity" property of the classical quasi-arithmetic means (Theorem 2). In Section 3, the equality $A^{[g_1,\ldots,g_k]} = A^{[f_1,\ldots,f_k]}$ is examined. In Section 4 we consider functions which are convex, concave or affine with respect to the mean $A^{[f_1,\ldots,f_k]}$. Using the functional equation $h(\beta(x)+\delta(y)) = \gamma(x)+\eta(y)$ (Lemma 1), we find the form of affine functions with respect to $A^{[f_1,\ldots,f_k]}$. In Section 5 we remark that the question of comparability of the means $A^{[f_1,\ldots,f_k]}$ and $A^{[g_1,\ldots,g_k]}$ leads to a convexity-type inequality. In Section 6 we observe that the quasi-arithmetic mean $A^{[f]}$,

$$A^{[f]}(x_1,\ldots,x_k) = f^{-1}\Big(\frac{1}{n}\sum_{i=1}^{k} f(x_i)\Big), \qquad x_1,\ldots,x_k \in I,$$

with $f := f_1 + \cdots + f_k$, is invariant with respect to the mean-type mapping $\mathbf{M} : I^k \to I^k$ given by

$$\mathbf{M} = (A^{[f_1,\ldots,f_k]}, A^{[f_2,f_3,\ldots,f_k,f_1]}, \ldots, A^{[f_k,f_1,\ldots,f_{k-1}]}),$$

and we apply this fact in solving a functional equation.

In connection with the above mentioned "associativity" property, in the final Section 7, for a given sequence of continuous and strictly increasing functions $f_j : I \to \mathbb{R}$, $j \in \mathbb{N}$, we define a mean $A^{[f_1,f_2,\ldots]} : \bigcup_{k=1}^{\infty} I^k \to I$, and observe that, except symmetry, it satisfies all the assumptions of the celebrated theorem of Kolmogorov–Nagumo [3], [10] on a characterization of quasi-arithmetic means (Corollary 3). Based on this, we formulate a conjecture generalizing the Kolmogorov–Nagumo theorem.

2. Generalized quasi-arithmetic means, their properties, and some lemmas.

Let $I \subset \mathbb{R}$ be an arbitrary interval and $k \in \mathbb{N}$, $k \geq 2$. A function $M : I^k \to \mathbb{R}$ is called a *k-variable mean* in I if

$$\min(x_1,\ldots,x_k) \leq M(x_1,\ldots,x_k) \leq \max(x_1,\ldots,x_k), \qquad x_1,\ldots,x_k \in I;$$

if, moreover, each of these two inequalities becomes an equality only in the case when $x_1 = \cdots = x_k$, the mean M is called *strict*.

THEOREM 1. *Let $I \subset \mathbb{R}$ be an interval, and $k \in \mathbb{N}$, $k \geq 2$. Suppose that a function $f : I \to \mathbb{R}$ is continuous and strictly monotonic, and $f_1,\ldots,f_k : I \to \mathbb{R}$ are arbitrary functions. Then the function $M : I^k \to \mathbb{R}$,*

$$M(x_1,\ldots,x_k) := f^{-1}\Big(\sum_{j=1}^{k} f_j(x_j)\Big), \tag{1}$$

is a mean if, and only if,

$$f = \sum_{j=1}^{k} f_j, \tag{2}$$

and, for each $i \in \{1,\dots,k\}$, the function f_i is continuous, monotonic, and of the same type of monotonicity as f; moreover, for each $i \in \{1,\dots,k\}$,

$$M(x_1,\dots,x_k) := f^{-1}\Big(\sum_{j=1,\, j\neq i}^{k} f_j(x_j) + f(x_i) - \sum_{j=1, j\neq i}^{k} f_j(x_i)\Big), \qquad x_1,\dots,x_k \in I, \tag{3}$$

and

$$M(x_1,\dots,x_k) := \Big(\sum_{j=1}^{k} f_j\Big)^{-1}\Big(\sum_{j=1}^{k} f_j(x_j)\Big), \qquad x_1,\dots,x_k \in I. \tag{4}$$

Proof. Since

$$(-f)^{-1}\Big(\sum_{j=1}^{k}(-f_j)(x_j)\Big) = f^{-1}\Big(\sum_{j=1}^{k} f_j(x_j)\Big), \qquad x_1,\dots,x_k \in I,$$

we can assume, without any loss of generality, that f is strictly increasing.

Assume that M defined by (1) is a mean in I.

From (1), taking $x_1 = \dots = x_k = x$ in the definition of a mean, we get

$$f^{-1}\Big(\sum_{j=1}^{k} f_j(x)\Big) = x, \qquad x \in I,$$

whence (2)–(4) hold true.

Fix $i \in \{1,\dots,k\}$ and take arbitrary $x, y \in I$, $x < y$. Since M is a mean, setting $x_j = x$ for $j \neq i$ and $x_i = y$ in (3), we get

$$x \le f^{-1}\Big(\sum_{j=1,\, j\neq i}^{k} f_j(x) + f(y) - \sum_{j=1,\, j\neq i}^{k} f_j(y)\Big) \le y,$$

whence, as f is increasing,

$$f(x) \le \sum_{j=1,\, j\neq i}^{k} f_j(x) + f(y) - \sum_{j=1,\, j\neq i}^{k} f_j(y) \le f(y). \tag{5}$$

By (1), from the first of these inequalities, we get

$$\sum_{j=1}^{k} f_j(x) \le \sum_{j=1,\, j\neq i}^{k} f_j(x) + \sum_{j=1}^{k} f_j(y) - \sum_{j=1,\, j\neq i}^{k} f_j(y),$$

which reduces to the inequality

$$f_i(x) \le f_i(y).$$

This proves that, for each $i \in \{1,\dots,k\}$, the function f_i is increasing. It follows that at any $t \in \operatorname{int} I$, the one-sided limits $f_i(t+)$ and $f_i(t-)$ exist.

Letting y tend to x in (5), by the continuity of f, we obtain

$$f(x) = \sum_{j=1,\, j\neq i}^{k} f_j(x) + f(x) - \sum_{j=1,\, j\neq i}^{k} f_j(x+),$$

that is,

$$\sum_{j=1,\, j\neq i}^{k} f_j(x) = \sum_{j=1,\, j\neq i}^{k} f_j(x+), \tag{6}$$

and this equality holds true for all $x \in \operatorname{int} I \cup \{\inf I\}$ if $\inf I \in I$.

Similarly, letting x tend to y in (5), we get

$$\sum_{j=1,\, j\neq i}^{k} f_j(y) = \sum_{j=1,\, j\neq i}^{k} f_j(y-) \tag{7}$$

for all $y \in \operatorname{int} I \cup \{\sup I\}$ if $\sup I \in I$.

By the continuity of f we have $f(t-) = f(t) = f(t+)$ for all $t \in \operatorname{int} I$; $f(t+) = f(t)$ if $t = \inf I \in I$, and $f(t-) = f(t)$ if $t = \sup I \in I$. Hence, for $t \in \operatorname{int} I$, we get

$$\sum_{j=1,\, j\neq i}^{k} f_j(t-) + f_i(t-) = \sum_{j=1,\, j\neq i}^{k} f_j(t) + f_i(t) = \sum_{j=1,\, j\neq i}^{k} f_j(t+) + f_i(t+),$$

whence, by (6) and (7),

$$f_i(t-) = f_i(t) = f_i(t+).$$

If $t = \inf I \in I$ then from the equality $f(t+) = f(t)$ and (6) we get $f_i(t+) = f_i(t)$. If $t = \sup I \in I$ then from the equality $f(t-) = f(t)$ and (7) we get $f_i(t-) = f_i(t)$. This proves that, for each $i \in \{1, \ldots, k\}$, the function f_i is continuous in I.

To prove the converse implication, assume that $f_1, \ldots, f_k : I \to \mathbb{R}$ are continuous, increasing, $f : I \to \mathbb{R}$ is strictly increasing and such that (2) holds true. Hence, for arbitrary $x_1, \ldots, x_k \in I$, putting

$$x = \min(x_1, \ldots, x_k), \qquad y = \max(x_1, \ldots, x_k),$$

we have

$$f(x) = \sum_{j=1}^{k} f_j(x) \le \sum_{j=1}^{k} f_j(x_j) \le \sum_{j=1}^{k} f_j(y) = f(y). \tag{8}$$

Since f is continuous, the number $\sum_{j=1}^{k} f_j(x_j)$ belongs to the range of f, and so the function M in (1) is correctly defined.

From (8) we obtain

$$x = f^{-1}\Big(\sum_{j=1}^{k} f_j(x)\Big) \le f^{-1}\Big(\sum_{j=1}^{k} f_j(x_j)\Big) \le f^{-1}\Big(\sum_{j=1}^{k} f_j(y)\Big) \le y,$$

that is, $\min(x_1, \dots, x_k) \le M(x_1, \dots, x_k) \le \max(x_1, \dots, x_k)$. Thus M is a mean. This completes the proof. ■

According to Theorem 1, given continuous strictly monotonic functions $f_1, \dots, f_k : I \to \mathbb{R}$ of the same kind of monotonicity, the function $A^{[f_1,\dots,f_k]} : I^k \to I$,

$$A^{[f_1,\dots,f_k]}(x_1, \dots, x_k) := \Big(\sum_{j=1}^{k} f_j\Big)^{-1}\Big(\sum_{j=1}^{k} f_j(x_j)\Big), \qquad x_1, \dots, x_k \in I, \tag{9}$$

is a mean, and will be referred to as a (generalized) *weighted quasi-arithmetic mean with generators* $f_1, \dots, f_k$ (cf. [7], also [9] and [8]).

REMARK 1. Let $\varphi : I \to \mathbb{R}$ be a continuous and strictly monotonic, and fix $w_1, \dots, w_k \in (0,1)$ with $w_1 + \dots + w_k = 1$. Taking $f_j = w_j\varphi$ for $j = 1, \dots, k$, we get

$$A^{[f_1,\dots,f_k]}(x_1, \dots, x_k) = \varphi^{-1}\Big(\sum_{j=1}^{k} w_j\varphi(x_j)\Big),$$

that is, $A^{[f_1,\dots,f_k]}$ becomes a weighted quasi-arithmetic mean with generator φ and weights $w_1, \dots, w_k$. This justifies why $A^{[f_1,\dots,f_k]}$ is called a generalized weighted quasi-arithmetic mean [7].

Let us note some properties of the mean $A^{[f_1,\dots,f_k]}$.

THEOREM 2. *Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \ge 2$. Assume that $f_1, \dots, f_k : I \to \mathbb{R}$ are continuous, monotonic of the same type, and $f_1 + \dots + f_k$ is strictly monotonic. Then*

(i) $A^{[-f_1,\dots,-f_k]} = A^{[f_1,\dots,f_k]}$;

(ii) *the mean $A^{[f_1,\dots,f_k]}$ is increasing with respect to each variable;*

(iii) *for all $x_1, \dots, x_k \in I$, if $\min(x_1, \dots, x_k) < \max(x_1, \dots, x_k)$ then either*

$$\min(x_1, \dots, x_k) < A^{[f_1,\dots,f_k]}(x_1, \dots, x_k)$$

or

$$A^{[f_1,\dots,f_k]}(x_1, \dots, x_k) < \max(x_1, \dots, x_k);$$

(iv) *$A^{[f_1,\dots,f_k]}$ is strictly increasing with respect to each variable if, and only if, $f_1, \dots, f_k$ are strictly monotonic;*

(v) *$A^{[f_1,\dots,f_k]}$ is a strict mean iff it is strictly increasing with respect to each variable;*

(vi) $A^{[f_1,\ldots,f_k]}$ *is symmetric if, and only if, there is a function* $g : I \to \mathbb{R}$ *and* $c_j \in \mathbb{R}$ *such that* $f_j = g + c_j$ *for* $j = 1, \ldots, k$; *in particular*

$$A^{[f_1,\ldots,f_k]}(x_1, \ldots, x_k) = g^{-1}\left(\frac{1}{k}\sum_{j=1}^{k} g(x_j)\right), \qquad x_1, \ldots, x_k \in I,$$

i.e. $A^{[f_1,\ldots,f_k]}$ *coincides with the quasi-arithmetic mean* $A^{[g]}$ *generated by* g;

(vii) $A^{[f_1,\ldots,f_k]}$ *has the following associativity-type property: for each* $i \in \{1, \ldots, k\}$, *if the functions* $f_1 + \cdots + f_i$, $f_2 + \cdots + f_{i+1}, \ldots, f_{k-i+1} + \cdots + f_k$ *are strictly monotonic, then for all* $x_1, \ldots, x_k \in I$,

$$\begin{aligned}
&A^{[f_1,\ldots,f_k]}(x_1, \ldots, x_k)\\
&= A^{[f_1,\ldots,f_k]}\Big(\underbrace{A^{[f_1,\ldots,f_i]}(x_1, \ldots, x_i), \ldots, A^{[f_1,\ldots,f_i]}(x_1, \ldots, x_i)}_{i\ times}, x_{i+1}, \ldots, x_k\Big)\\
&= A^{[f_1,\ldots,f_k]}\Big(x_1, \underbrace{A^{[f_2,\ldots,f_{i+1}]}(x_2, \ldots, x_{i+1}), \ldots, A^{[f_2,\ldots,f_{i+1}]}(x_2, \ldots, x_{i+1})}_{i\ times}, x_{i+2}, \ldots, x_k\Big)\\
&= \cdots =\\
&A^{[f_1,\ldots,f_k]}\Big(x_1, \ldots, x_{k-i}, \underbrace{A^{[f_{k-i+1},\ldots,f_k]}(x_{k-i+1}, \ldots, x_k), \ldots, A^{[f_{k-i+1},\ldots,f_k]}(x_{k-i+1}, \ldots, x_k)}_{i\ times}\Big).
\end{aligned}$$

Proof. Properties (i)–(iv) are easy to verify.

To prove (v) suppose that $A^{[f_1,\ldots,f_k]}$ is strict. We may assume that $f_1, \ldots, f_k$ are increasing. Choose arbitrarily $i \in \{1, \ldots, k\}$, $x, y \in I$, $x < y$, and put

$$x_j = x \quad \text{for } j \in \{1, \ldots, k\} \setminus \{i\}, \qquad \text{and} \qquad x_i = y.$$

Since $A^{[f_1,\ldots,f_k]}$ is strict, we have

$$x = \min(x_1, \ldots, x_k) < A^{[f_1,\ldots,f_k]}(x_1, \ldots, x_k).$$

Hence, making use of (9) and the strict monotonicity of $\sum_{j=1}^{k} f_j$, we get

$$\left(\sum_{j=1}^{k} f_j\right)(x) < \sum_{j=1}^{k} f_j(x_j),$$

that is, $f_i(x) < f_i(y)$. Thus we have shown that, for every $i \in \{1, \ldots, k\}$, the function f_i is strictly increasing. Conversely, if $f_1, \ldots, f_k$ are strictly monotonic then, by (iv), the mean $A^{[f_1,\ldots,f_k]}$ is strict.

To prove (vi), assume that $A^{[f_1,\ldots,f_k]}$ is symmetric. Hence, for $i, j \in \{1, \ldots, k\}$, $i < j$, we have

$$A^{[f_1,\ldots,f_k]}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = A^{[f_1,\ldots,f_k]}(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k),$$

whence, taking $x_i = x$, $x_j = y$, from the definition of $A^{[f_1,\ldots,f_k]}$, we obtain

$$f_i(x) - f_j(x) = f_i(y) - f_j(y), \qquad x, y \in I,$$

which implies that $f_i - f_j$ is a constant function. Taking here $j = 1$ and putting $g := f_1$, $c_1 := 0$, we get

$$f_i(x) = g(x) + c_i, \qquad x \in I,\ i = 1, \ldots, k,$$

for some $c_2, \ldots, c_k \in \mathbb{R}$. Now from (9), setting $c := \sum_{j=1}^{k} c_j$, we have

$$\begin{aligned} A^{[f_1,\ldots,f_k]}(x_1, \ldots, x_k) &= \Big(\sum_{j=1}^{k} f_j\Big)^{-1}\Big(\sum_{j=1}^{k} f_j(x_j)\Big) \\ &= (kg + c)^{-1}\Big(\sum_{j=1}^{k} g(x_j) + c\Big) = g^{-1}\Big(\frac{1}{k}\sum_{j=1}^{k} g(x_j)\Big) \end{aligned}$$

for all $x_1, \ldots, x_k \in I$. The converse implication is easy to verify.

To show (vii), take $i \in \{1, \ldots, k\}$ and note that, by (4),

$$\begin{aligned} &A^{[f_1,\ldots,f_k]}(x_1, \ldots, x_k) = \Big(\sum_{j=1}^{k} f_j\Big)^{-1}\Big(\sum_{j=1}^{k} f_j(x_j)\Big) \\ &= \Big(\sum_{j=1}^{k} f_j\Big)^{-1}\Big(\Big(\sum_{j=1}^{i} f_j\Big) \circ \Big[\Big(\sum_{j=1}^{i} f_j\Big)^{-1}\Big(\sum_{j=1}^{i} f_j(x_j)\Big)\Big] + \sum_{j=i+1}^{k} f_j(x_j)\Big) \\ &= \Big(\sum_{j=1}^{k} f_j\Big)^{-1}\Big(\sum_{j=1}^{i} f_j \circ \Big[\Big(\sum_{j=1}^{i} f_j\Big)^{-1}\Big(\sum_{j=1}^{i} f_j(x_j)\Big)\Big] + \sum_{j=i+1}^{k} f_j(x_j)\Big) \\ &= A^{[f_1,\ldots,f_k]}\Big(\underbrace{A^{[f_1,\ldots,f_i]}(x_1, \ldots, x_i), \ldots, A^{[f_1,\ldots,f_i]}(x_1, \ldots, x_i)}_{i\ \text{times}}, x_{i+1}, \ldots, x_k\Big), \end{aligned}$$

and similarly we get the remaining equalities. ∎

In view of (i), we may assume from now on that $f_1, \ldots, f_k$ are increasing.

LEMMA 1. *Let $I, J \subset \mathbb{R}$ be intervals, $\beta, \gamma : I \to \mathbb{R}$ nonconstant continuous functions, and $\delta, \eta : J \to \mathbb{R}$ arbitrary functions. If $h : \beta(I) + \delta(J) \to \mathbb{R}$ satisfies the functional equation*

$$h(\beta(x) + \delta(y)) = \gamma(x) + \eta(y), \qquad x \in I,\ y \in J, \tag{10}$$

then there is a unique additive function $\alpha : \mathbb{R} \to \mathbb{R}$ and a unique $c \in \mathbb{R}$ such that

$$h(u) = \alpha(u) + c, \qquad u \in (\beta(I) + \delta(J)).$$

Moreover there is $b \in \mathbb{R}$ such that

$$\gamma(x) = \alpha(\beta(x)) - b, \quad x \in I; \qquad \eta(y) = \alpha(\delta(y)) + b + c, \quad y \in J.$$

Here $\beta(I) + \delta(J) := \{u + v : u \in \beta(I),\ v \in \delta(J)\}$.

Proof. Without any loss of generality we can assume that there are $x_0 \in \operatorname{int} I$ and $y_0 \in \operatorname{int} J$ such that $\beta(x_0) = 0$ and $\delta(y_0) = 0$. Indeed, in the opposite case we could fix any $x_0 \in \operatorname{int} I$ and $y_0 \in \operatorname{int} J$, define $\bar{\beta} : I \to \mathbb{R}$ by $\bar{\beta}(x) := \beta(x) - \beta(x_0)$, $\bar{\gamma} : J \to \mathbb{R}$ by $\bar{\gamma} : J(y) := \delta(y) - \delta(y_0)$, $\bar{h} : (\bar{\beta}(I) + \bar{\delta}(J)) + \beta(x_0) + \delta(y_0) \to \mathbb{R}$ and consider the functional equation

$$\bar{h}(\bar{\beta}(x) + \bar{\delta}(y)) = \gamma(x) + \eta(y), \qquad x \in I,\, y \in J,$$

that is equivalent to (10).

Setting $y = y_0$ and then $x = x_0$ in (10) we get

$$h(\beta(x)) = \gamma(x) + \eta(y_0), \quad x \in I; \qquad h(\delta(y)) = \gamma(x_0) + \eta(y), \quad y \in J.$$

whence, from (10),

$$h(\beta(x) + \delta(y)) = h(\beta(x)) + h(\delta(y)) - c, \qquad x \in I,\, y \in J,$$

where $c := \eta(y_0) + \gamma(x_0)$. Setting $H := h - c$, we get

$$H(\beta(x) + \delta(y)) = H(\beta(x)) + H(\delta(y)), \qquad x \in I,\, y \in J,$$

whence

$$H(u + v) = H(u) + H(v), \qquad u \in \beta(I),\, v \in \delta(J),$$

so H is additive in a nontrivial interval containing 0. Clearly there exists a unique additive function $\alpha : \mathbb{R} \to \mathbb{R}$ that is an extension of H. From the definition of H we get $h = \alpha + c$. Setting $h = \alpha + c$ in (10) and making use of the additivity of α, we obtain

$$\alpha(\beta(x)) - \gamma(x) = \eta(y) - \alpha(\delta(y)) - c, \qquad x \in I,\, y \in J,$$

whence there is $b \in \mathbb{R}$ such that $\alpha(\beta(x)) - \gamma(x) = b$ for all $x \in I$, and $\eta(y) - \alpha(\delta(y)) - c = b$ for all $y \in J$. This completes the proof. ∎

The following result is a reformulation of Theorem 2 in [7].

LEMMA 2. *Let $I \subset \mathbb{R}$ be an interval and let $f, g, F, G : I \to \mathbb{R}$ be continuous, increasing and such that $f + F$ and $g + G$ are strictly increasing.* Then $A^{[g,G]} = A^{[f,F]}$ *if, and only if, there exist $a, b, c \in \mathbb{R}$, $a \neq 0$, such that*

$$g(x) = af(x) + b, \qquad G(x) = aF(x) + c, \qquad x \in I. \tag{11}$$

3. Equality of generalized weighted quasi-arithmetic means

THEOREM 3. *Let $I \subset \mathbb{R}$ be an interval, $k \in \mathbb{N}$, $k \geq 2$, and let $f_1, \ldots, f_k$, $g_1, \ldots, g_k : I \to \mathbb{R}$ be continuous, increasing such that $f_1 + \cdots + f_k$ and $g_1 + \cdots + g_k$ are strictly increasing.* Then

$$A^{[g_1,\ldots,g_k]} = A^{[f_1,\ldots,f_k]} \tag{12}$$

if, and only if, there exist $a, b_1, \ldots, b_k \in \mathbb{R}$, $a \neq 0$, such that

$$g_j(x) = af_j(x) + b_j, \qquad x \in I,\, j = 1, \ldots, k. \tag{13}$$

Proof. Assume that (12) holds true for $k = 2$. Setting $f := f_1$, $F := f_2$, $g := g_1$, $G := g_2$, we hence get $A^{[g,G]} = A^{[f,F]}$ and, in view of Lemma 2, there are $a, b, c \in \mathbb{R}$, $a \neq 0$, such that (11) holds true. Setting $b_1 := b$, and

$b_2 := c$ we obtain (13) for $k = 2$. Thus, in the case $k = 2$, equality (12) implies (13).

Assume that (12) holds true for $k \in \mathbb{N}$, $k > 2$. Choosing arbitrarily $i \in \{1, \dots, k\}$, we can write (12) in the following form: for all $x_1, \dots, x_k \in I$,

$$\Big(g_i + \sum_{j=1,\, j\neq i}^{k} g_j\Big)^{-1}\Big(g_i(x_i) + \sum_{j=1,\, j\neq i}^{k} g_j(x_j)\Big) = \Big(f_i + \sum_{j=1,\, j\neq i}^{k} f_j\Big)^{-1}\Big(f_i(x_i) + \sum_{j=1,\, j\neq i}^{k} f_j(x_j)\Big).$$

Taking $x_i = x$, $x_j = y$ for all $j \in \{1, \dots, k\} \setminus \{i\}$, for $x, y \in I$, and setting

$$F_i := \sum_{j=1,\, j\neq i}^{k} f_j, \qquad G_i := \sum_{j=1,\, j\neq i}^{k} g_j,$$

we hence get

$$(g_i + G_i)^{-1}(g_i(x) + G_i(y)) = (f_i + F_i)^{-1}(f_i(x) + F_i(y)), \qquad x, y \in I,$$

that is, $A^{[g_i, G_i]} = A^{[f_i, F_i]}$. Applying Lemma 2 we conclude that for each $i \in \{1, \dots, k\}$ there are $a_i, b_i, c_i, \in \mathbb{R}$, $a_i \neq 0$, $i \in \{1, \dots, k\}$, such that

$$g_i(x) = a_i f_i(x) + b_i, \qquad \sum_{j=1,\, j\neq i}^{k} g_j(x) = a_i\Big(\sum_{j=1,\, j\neq i}^{k} f_j(x)\Big) + c_i, \qquad x \in I.$$

Adding these equalities we get

$$\sum_{j=1}^{k} g_j(x) = a_i\Big(\sum_{j=1}^{k} f_j(x)\Big) + b_i + c_i, \qquad x \in I.$$

It follows that a_i does not depend on $i \in \{1, \dots, k\}$. Thus, setting $a := a_1$, we obtain

$$g_i(x) = a f_i(x) + b_i, \qquad x \in I,\ i = 1, \dots, k.$$

Since the converse implication is easy to verify, the proof is complete. ■

4. Convexity and affinity with respect to generalized weighted quasi-arithmetic means

Definition 1. Let $f_1, \dots, f_k : I \to \mathbb{R}$, $k \geq 2$, be continuous, of the same type of monotonicity and such that $\sum_{j=1}^{k} f_j$ is strictly monotonic in the interval I. Let J be a subinterval of I. We say that a function $\varphi : J \to I$ is $A^{[f_1, \dots, f_k]}$*-convex* if

$$\varphi\big(A^{[f_1, \dots, f_k]}(x_1, \dots, x_k)\big) \leq A^{[f_1, \dots, f_k]}(\varphi(x_1), \dots, \varphi(x_k)), \qquad x_1, \dots, x_k \in J;$$

$A^{[f_1,\dots,f_k]}$-*concave* if the converse inequality is satisfied; and $A^{[f_1,\dots,f_k]}$-*affine* if the equality is fulfilled.

For $k = 2$, setting here $f = f_1$, $g = f_2$, and making use of (9), we see that the $A^{[f,g]}$-convexity of φ reduces to the inequality

$$\varphi\big((f+g)^{-1}(f(x)+g(y))\big) \le (f+g)^{-1}\big(f(\varphi(x))+g(\varphi(y))\big), \quad x, y \in J, \tag{14}$$

(the $A^{[f,g]}$-concavity, to the converse inequality, and $A^{[f,g]}$-affinity to equality).

Remark 2. Let $I = \mathbb{R}$, $J \subset \mathbb{R}$, and $t \in (0,1)$. Taking $f(x) = tx$, $g(x) = (1-t)x$ for $x \in \mathbb{R}$, in (14) we get

$$\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y), \quad x, y \in J,$$

so $A^{[f,g]}$-convexity generalizes the classical t-convexity of $\varphi : J \to \mathbb{R}$. In particular, for $t = 1/2$ we get Jensen convexity.

Taking $J \subset (0,\infty)$, $f(x) = t\log x$, $g(x) = (1-t)\log x$ for $x > 0$ in (14) we get

$$\varphi(x^t y^{1-t}) \le [\varphi(x)]^t[\varphi(y)]^{1-t}, \quad x, y \in J,$$

so $A^{[f,g]}$-convexity generalizes the geometrical t-convexity of $\varphi : J \to (0,\infty)$. For $t = 1/2$ we get Jensen geometrical convexity:

$$\varphi(\sqrt{xy}) \le \sqrt{\varphi(x)\varphi(y)}, \quad x, y \in J.$$

Theorem 4. *Let I and $J \subset I$ be intervals. Suppose that $f, g : I \to \mathbb{R}$ are increasing and such that $f + g$ is continuous and strictly increasing. A function $\varphi : J \to I$ is $A^{[f,g]}$-affine, that is,*

$$\varphi\big((f+g)^{-1}(f(x)+g(y))\big) = (f+g)^{-1}\big(f(\varphi(x))+g(\varphi(y))\big), \quad x, y \in J, \tag{15}$$

if, and only if, there is an additive function $\alpha : \mathbb{R}\to\mathbb{R}$ and $b, c \in \mathbb{R}$ such that

$$\varphi = (f+g)^{-1} \circ [\alpha \circ (f+g) + c]$$

and

$$f \circ \varphi = \alpha \circ f - b, \quad g \circ \varphi = \alpha \circ g + b + c.$$

Proof. Assume that $\varphi : J \to I$ is $A^{[f,g]}$-affine. From (15) we get

$$(f+g) \circ \varphi\big((f+g)^{-1}(f(x)+g(y))\big) = f(\varphi(x)) + g(\varphi(y)), \quad x, y \in J,$$

Applying Lemma 1 with $h := (f+g)\circ\varphi\circ(f+g)^{-1}$, $\beta := f$, $\delta := g$, $\gamma := f\circ\varphi$ and $\eta := g \circ \varphi$, we obtain $(f+g) \circ \varphi \circ (f+g)^{-1} = \alpha + c$ for some additive function $\alpha : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$, whence

$$\varphi = (f+g)^{-1} \circ [\alpha \circ (f+g) + c].$$

From the "moreover" part of Lemma 1 we get

$$f \circ \varphi = \alpha \circ f - b, \quad g \circ \varphi = \alpha \circ g + b + c$$

for some $b \in \mathbb{R}$. The converse implication is easy to verify. ∎

Hence, by induction, we obtain

THEOREM 5. *Let I and $J \subset I$ be intervals. Suppose that $f_1, \dots, f_k : I \to \mathbb{R}$, $k \ge 2$, are increasing and $f_1 + \dots + f_k$ is continuous and strictly increasing. A function $\varphi : J \to I$ is $A^{[f_1,\dots,f_k]}$-affine if, and only if, there is an additive function $\alpha : \mathbb{R} \to \mathbb{R}$ and $b_1, \dots, b_k, c \in \mathbb{R}$ such that*

$$\varphi = \Big(\sum_{j=1}^{k} f_j\Big)^{-1} \circ \Big[\alpha \circ \Big(\sum_{j=1}^{k} f_j\Big)^{-1} + c\Big],$$

$$f_j \circ \varphi = \alpha \circ f_j + b_j, \qquad j = 1, \dots, k,$$

$$\sum_{j=1}^{k} b_j = c.$$

5. Comparability of generalized weighted quasi-arithmetic means

REMARK 3. Let $I \subset \mathbb{R}$ be an interval and $k \in \mathbb{N}$, $k \ge 2$. Assume that $f_j, g_j : I \to \mathbb{R}$, $j = 1, \dots, k$, are continuous, increasing and such that the functions $f := f_1 + \dots + f_k$ and $g := g_1 + \dots + g_k$ are strictly increasing. If moreover $f_1, \dots, f_k$ are strictly increasing, then

$$A^{[f_1,\dots,f_k]} \le A^{[g_1,\dots,g_k]}$$

if, and only if,

$$g \circ f^{-1}\Big(\sum_{j=1}^{k} u_j\Big) \le \sum_{j=1}^{k} g_j \circ f_j^{-1}(u_j), \qquad u_j \in f_j(I),\ j = 1, \dots, k. \tag{16}$$

EXAMPLE 1. Let $\varphi, \psi : I \to \mathbb{R}$ be continuous and strictly increasing. Taking in this remark $f_j = \varphi$, $g_j = \psi$ for $j = 1, \dots, k$, we find that $A^{[\varphi]} \le A^{[\psi]}$, that is,

$$\varphi^{-1}\left(\frac{\varphi(x_1) + \dots + \varphi(x_k)}{n}\right) \le \psi^{-1}\left(\frac{\psi(x_1) + \dots + \psi(x_k)}{n}\right), \quad x_1, \dots, x_k \in I,$$

if, and only if,

$$\psi \circ \varphi^{-1}\left(\frac{u_1 + \dots + u_k}{k}\right) \le \frac{\psi \circ \varphi^{-1}(u_1) + \dots + \psi \circ \varphi^{-1}(u_k)}{k}, \qquad u_1, \dots, u_k \in \varphi(I).$$

Similarly, taking $f_j = t_j\varphi$, $g_j = t_j\psi$, $t_j > 0$ for $j = 1, \dots, k$, $t_1 + \dots + t_k = 1$, we infer that

$$\varphi^{-1}\Big(\sum_{j=1}^{k} t_j\varphi(x_j)\Big) \le \psi^{-1}\Big(\sum_{j=1}^{k} t_j\psi(x_j)\Big), \qquad x_1, \dots, x_k \in I,$$

if, and only if, for all $u_1, \dots, u_k \in \varphi(I)$,

$$\psi \circ \varphi^{-1}(t_1 u_1 + \dots + t_k u_k) \le t_1 \psi \circ \varphi^{-1}(u_1) + \dots + t_k \psi \circ \varphi^{-1}(u_k).$$

Thus inequality (16) is related to convexity.

6. Invariance of means and application in solving a functional equation

REMARK 4. Let $f_1, \dots, f_k : I \to \mathbb{R}$ be continuous, increasing with $f := f_1 + \dots + f_k$ strictly increasing. The quasi-arithmetic mean $A^{[f]}$,

$$A^{[f]}(x_1, \dots, x_k) = f^{-1}\biggl(\frac{1}{n}\sum_{i=1}^{k} f(x_i)\biggr), \qquad x_1, \dots, x_k \in I,$$

is invariant with respect to the mean-type mapping $\mathbf{M} : I^k \to I^k$ defined by

$$\mathbf{M} = (A^{[f_1,\dots,f_k]}, A^{[f_2,f_3,\dots,f_k,f_1]}, \dots, A^{[f_k,f_1,\dots,f_{k-1}]}), \tag{17}$$

that is, $A^{[f]} \circ \mathbf{M} = A^{[f]}$.

Indeed, for all $x_1, \dots, x_k \in I$, we have

$$\begin{aligned}
nf\bigl(A^{[f]} \circ \mathbf{M}^{[f]}(x_1, \dots, x_k)\bigr) &= \sum_{i=1}^{k} f\bigl(A^{[f_i,f_{i+1},\dots,f_k,f_1,\dots,f_{k-i-1}]}(x_1, \dots, x_k)\bigr) \\
&= \sum_{i=1}^{k} (f_i(x_1) + f_{i+1}(x_2) + \dots + f_k(x_{i-1}) + f_1(x_i) + \dots + f_{k-i-1}(x_k)) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} f_j(x_i) = \sum_{j=1}^{k}\sum_{i=1}^{k} f_j(x_i) = \sum_{j=1}^{k}\Bigl(\sum_{i=1}^{k} f_j\Bigr)(x_i) = \sum_{j=1}^{k} f(x_i),
\end{aligned}$$

whence the invariance follows.

Theorem 2(iii) implies that if $\min(x_1, \dots, x_k) < \max(x_1, \dots, x_k)$, then

$$\max \mathbf{M}(x_1, \dots, x_k) - \min \mathbf{M}(x_1, \dots, x_k) < \max(x_1, \dots, x_k) - \min(x_1, \dots, x_k),$$

for all $x_1, \dots, x_k \in I$. Hence, applying [6, Theorem 1] (cf. also [5]) we obtain

COROLLARY 1. *The sequence $(\mathbf{M}^n)_{n\in\mathbb{N}}$ of iterates of the mean-type mapping $\mathbf{M} : I^k \to I^k$ given by* (17) *converges uniformly on compact subsets of I^k to the mean-type mapping $\mathbf{K} = (K_1, \dots, K_k)$ such that $K_1 = \dots = K_k = A^{[f]}$.*

EXAMPLE 2. The functions $f_1, f_2 : (0, \infty) \to (0, \infty)$ given by $f_1(x) = e^x - x$, $f_2(x) = x$, are increasing, $f_1 + f_2 = \exp$ is strictly increasing, and we have

$$A^{[f_1,f_2]}(x, y) = \log(e^x - x + y), \qquad A^{[f_2,f_1]}(x, y) = \log(x + e^y - y), \qquad x, y > 0.$$

According to Remark 4, the quasi-arithmetic mean

$$A^{[f_1+f_2]}(x,y) = \log\left(\frac{e^x+e^y}{2}\right), \qquad x,y>0,$$

is invariant with respect to the mapping $(A^{[f_1,f_2]}, A^{[f_2,f_1]})$ and, in view of Corollary 1,

$$\lim_{n\to\infty}(A^{[f_1,f_2]}, A^{[f_2,f_1]})^n = (A^{[f_1+f_2]}, A^{[f_1+f_2]}) \quad \text{ in } (0,\infty)^2.$$

Corollary 1 allows us to solve a functional equation. Namely, we have the following

Theorem 6. *Let $I\subset\mathbb{R}$ be an interval and $f_1,\dots,f_k : I\to\mathbb{R}$ be continuous, increasing with $f := f_1+\dots+f_k$ strictly increasing. Assume that $F : I^k\to\mathbb{R}$ is continuous on the diagonal $\{(x_1,\dots,x_k) : x_1=\dots=x_k\in I\}$. Then F satisfies the functional equation*

$$F\circ(A^{[f_1,\dots,f_k]}, A^{[f_2,f_3,\dots,f_k,f_1]},\dots,A^{[f_k,f_1,\dots,f_{k-1}]}) = F \tag{18}$$

if, and only if, $F=\varphi\circ A^{[f]}$ where $\varphi : I\to\mathbb{R}$ is an arbitrary continuous function.

Proof. Suppose that $F : I^k\to\mathbb{R}$ is continuous on the diagonal of I^k and satisfies (18), that is, $F\circ\mathbf{M}=F$, where $\mathbf{M}$ is given by (17). By induction we get

$$F = F\circ\mathbf{M}^n, \qquad n\in\mathbb{N}.$$

Letting $n\to\infty$, and making use of Corollary 1 and the continuity of F on the diagonal of I^k, we get

$$F(x_1,\dots,x_k) = F\big(A^{[f]}(x_1,\dots,x_k), A^{[f]}(x_1,\dots,x_k),\dots,A^{[f]}(x_1,\dots,x_k)\big)$$

for all $(x_1,\dots,x_k)\in I^k$. Hence, setting $\varphi(x) := F(x,\dots,x)$ for $x\in I$, we obtain

$$F(x_1,\dots,x_k) = \varphi(A^{[f]}(x_1,\dots,x_k)), \qquad x_1,\dots,x_k\in I.$$

Since it is easy to verify that any function of this form satisfies (18), the proof is complete. ∎

From Example 2, applying Theorem 6, we obtain

Corollary 2. *A function $F : (0,\infty)^2\to(0,\infty)$ that is continuous on the set $\{(x,x) : x>0\}$ satisfies the functional equation*

$$F\big(\log(e^x-x+y), \log(x+e^y-y)\big) = F(x,y), \qquad x,y>0,$$

if, and only if, $F(x,y)=\varphi(e^x+e^y)$ where $\varphi : (0,\infty)\to\mathbb{R}$ is an arbitrary continuous function.

7. A conjecture generalizing the Kolmogorov–Nagumo theorem. From Theorem 2(vii) & (vi) we obtain the following

COROLLARY 3. *Let $I \subset \mathbb{R}$ be an interval and $f_j : I \to \mathbb{R}$, $j \in \mathbb{N}$, be a sequence of continuous and strictly increasing functions. Then $A^{[f_1,f_2,\dots]} : \bigcup_{k=1}^{\infty} I^k \to I$ given by*

$$A^{[f_1,f_2,\dots]}(x_1,\dots,x_k) := A^{[f_1,\dots,f_k]}(x_1,\dots,x_k), \quad (x_1,\dots,x_k) \in I^k,\ k \in \mathbb{N},$$

is an "associative" mean in $\bigcup_{k=1}^{\infty} I^k$, that is, for all $n, r, k_1, \dots, k_r \in \mathbb{N}$, $k_1 < \dots < k_r = n$, and $x_1, \dots, x_n \in I$, we have

$$M(x_1,\dots,x_n) = M\Big(\underbrace{M_1,\dots,M_1}_{k_1 \text{ times}}, \underbrace{M_2,\dots,M_2}_{k_2-k_1 \text{ times}}, \dots, \underbrace{M_r,\dots,M_r}_{n-k_{r-1} \text{ times}}\Big), \tag{19}$$

where $M := A^{[f_1,f_2,\dots]}$ and

$$M_i := A^{[f_{k_{i-1}+1},\dots,f_{k_i}]}(x_{k_{i-1}+1},\dots,x_{k_i}), \quad i = 1,\dots,r \quad (k_0 := 0).$$

Moreover, the mean $A^{[f_1,f_2,\dots]}$ is symmetric if, and only if, there is a continuous and strictly increasing function $f : I \to \mathbb{R}$ such that $A^{[f_1,f_2,\dots]}$ is the quasi-arithmetic mean $A^{[f,f,\dots]}$ given by

$$A^{[f,f,\dots]}(x_1,\dots,x_k) := f^{-1}\left(\frac{f(x_1)+\dots+f(x_k)}{n}\right),$$
$$(x_1,\dots,x_k) \in I^k,\ k \in \mathbb{N}.$$

Recall that according to the celebrated result, obtained independently by Kolmogorov [3] and Nagumo [10], the quasi-arithmetic mean $A^{[f,\dots]} : \bigcup_{k=1}^{\infty} I^k \to I$ is the only continuous, strictly increasing, symmetric and "associative" mean.

This corollary shows that there are a lot of associative quasi-arithmetic means which are not symmetric.

Assume that $I \subset \mathbb{R}$ is an interval and $M : \bigcup_{k=1}^{\infty} I^k \to I$ is a mean that is continuous, strictly increasing (with respect to each variable) and such that for all $n, r, k_1, \dots, k_r \in \mathbb{N}$, $k_1 < \dots < k_r = n$, and $x_1, \dots, x_n \in I$, equality (19) holds true. We conjecture that then *there exists a sequence of continuous and strictly increasing functions $f_j : I \to \mathbb{R}$, $j \in \mathbb{N}$, such that $M = A^{[f_1,f_2,\dots]}$*.

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