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CONDITIONS FOR p-SUPERSOLUBILITY AND p-NILPOTENCY OF FINITE SOLUBLE GROUPS

BY

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Abstract. Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. A subgroup H of G is called $\mathfrak{Z}$-permutably embedded in G if every Sylow subgroup of H is also a Sylow subgroup of some $\mathfrak{Z}$-permutable subgroup of G. By using this concept, we obtain some new criteria of p-supersolubility and p-nilpotency of a finite group.

1. Introduction. Throughout this paper, all groups considered are finite. Let G be a group. $\pi(G)$ denotes the set of all prime divisors of $|G|$. If π is a set of primes then an integer n is called a π-*number* if all its prime divisors belong to π, and π' is the complement of π in the set $\mathbb{P}$ of all primes.

Recall that $O_\pi(G)$ is the maximal normal π-subgroup of G, and $F(G)$, the *Fitting subgroup* of G, is the maximal nilpotent normal subgroup of G. Let p be a prime divisor of G. Then $F_p(G)$ is the maximal p-nilpotent normal subgroup of G. The reader is referred to Guo [8] or Robinson [17] for all unexplained terminology and notations.

A subgroup A of a group G is said to *permute* with a subgroup B if $AB = BA$. A subgroup H of G is called *quasinormal* [15] or *permutable* [6] in G if H permutes with all subgroups of G. If H permutes with all Sylow subgroups of G, then H is called *s-permutable* in G [11]. After [11, 5], permutability of subgroups was extensively studied (cf. [2, 4, 16]). More recently, in [19], by discussing weakly s-permutable subgroups of given order, some interesting results were obtained and many known results were generalized.

In [1], a set $\mathfrak{Z}$ is called a *complete set of Sylow subgroups* of G, or a *complete Sylow set* of G, if for each prime $p \in \pi(G)$, $\mathfrak{Z}$ contains exactly one Sylow p-subgroup of G, and a subgroup H of G is said to be $\mathfrak{Z}$-*permutable* in G if H permutes with every member of $\mathfrak{Z}$. By using the $\mathfrak{Z}$-permutability of some primary subgroups of given order, certain classes of groups were characterized (cf. [1, 14]). A subgroup U of a group G is called a *normally embedded subgroup* of G if every Sylow subgroup of U is also a Sylow sub-

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group of some normal subgroup of G (cf.[7]). If every Sylow subgroup of U is a Sylow subgroup of some permutable subgroup of G, then U is called a *permutably embedded subgroup* [3]. Along these lines, we will discuss subgroups in which every Sylow subgroup is also a Sylow subgroup of some $\mathfrak{Z}$-permutable subgroup of G and we introduce the following definition.

Definition 1.1. Let $\mathfrak{Z}$ be a complete set of Sylow subgroups of a group G. A subgroup H of G is called $\mathfrak{Z}$-*permutably embedded* in G if every Sylow subgroup of H is also a Sylow subgroup of some $\mathfrak{Z}$-permutable subgroup of G.

It is easy to see that all Sylow subgroups of a group G are $\mathfrak{Z}$-permutably embedded in G for any complete set $\mathfrak{Z}$ of Sylow subgroups of G. But if G is not nilpotent, then there must be some Sylow subgroups which are not $\mathfrak{Z}$-permutable in G. The following example shows that subgroups which are $\mathfrak{Z}$-permutably embedded but not $\mathfrak{Z}$-permutable in G are not necessarily Sylow subgroups, even in a soluble group.

Example 1.2. Let $G = S_4$ be the symmetric group of degree 4. Then G has a Sylow 2-subgroup $G_2 = \{(1), (12)(34), (13)(24), (14)(23), (14), (23), (1243), (1342)\}$ and a Sylow 3-subgroup $G_3 = \langle(123)\rangle$. Choose $\mathfrak{Z} = \{G_2, G_3\}$ and $H = \langle(12)\rangle$. Then H is not $\mathfrak{Z}$-permutable in G since it does not permute with G_2 in $\mathfrak{Z}$. Let $U = \langle(123), (12)\rangle \cong S_3$. Then it is easy to verify that U is $\mathfrak{Z}$-permutable in G, and H is a Sylow 2-subgroup of U. Hence H is $\mathfrak{Z}$-permutably embedded in G.

Let H be a subgroup of a group G. Then a subgroup T of G is called a *supplement* of H in G if $HT = G$. The role of supplements of some subgroups in a group was often studied together with the permutability of subgroups (cf. [9, 19]). In view of this, we shall prove the following main theorems of this paper:

Theorem A. *Let G be a p-soluble group and $\mathfrak{Z}$ a complete Sylow set of G. Assume that G has a normal subgroup N with p-supersoluble quotient. If every maximal subgroup of a Sylow p-subgroup of $F_p(N)$ either is $\mathfrak{Z}$-permutably embedded in G or has a p-supersoluble supplement in G, then G is p-supersoluble.*

Theorem B. *Let G be a p-soluble group and $\mathfrak{Z}$ a complete Sylow set of G. Assume that G has a normal subgroup N with p-supersoluble quotient and let P be a Sylow p-subgroup of $F_p(N)$. If every cyclic subgroup of order p or* 4 (*when P is a non-abelian* 2*-group*) *of P either is $\mathfrak{Z}$-permutably embedded in G or has a p-supersoluble supplement in G, then G is p-supersoluble.*

Note that if $(p-1, |G|) = 1$, then G is p-supersoluble if and only if it is p-nilpotent. In fact, if G is p-nilpotent, then G is certainly p-supersoluble.

Conversely, if G is p-supersoluble and $(p-1, |G|) = 1$, then any pd-chief factor H/K of G is cyclic of order p and so $\mathrm{Aut}(H/K)$ is of order $p-1$. Since $G/C_G(H/K)$ is isomorphic to a subgroup of $\mathrm{Aut}(H/K)$, the order of $G/C_G(H/K)$ is a divisor of $p-1$. It follows from $(p-1, |G|) = 1$ that $G/C_G(H/K) = 1$. Hence every pd-chief factor H/K of G is central and so G is p-nilpotent. Considering this, if $(p-1, |G|) = 1$ in Theorem A or B, we can also obtain a criterion of p-nilpotency of G. Moreover, if $(p^2-1, |G|) = 1$, we can obtain

THEOREM C. *Let G be a p-soluble group and $\mathfrak{Z}$ a complete Sylow set of G. Assume that G has a normal subgroup N with p-nilpotent quotient and let P be a Sylow p-subgroup of $F_p(N)$. If $(p^2-1, |G|) = 1$ and every subgroup of order p^2 of P either is $\mathfrak{Z}$-permutable in G or has a p-nilpotent supplement in G, then G is p-nilpotent.*

2. Preliminaries. In this section, we shall give some elementary properties of $\mathfrak{Z}$-permutably embedded subgroups, and for the sake of easy reference, we shall also cite some useful known results from the literature.

Let N be a normal subgroup of G, and $\mathfrak{Z}$ a complete Sylow set of G. Following [1], we shall denote by $\mathfrak{Z}N$ the following set of subgroups of G:

$$\mathfrak{Z}N = \{G_pN \mid G_p \in \mathfrak{Z}\},$$

by $\mathfrak{Z}N/N$ the following set of subgroups of G/N:

$$\mathfrak{Z}N/N = \{G_pN/N \mid G_p \in \mathfrak{Z}\},$$

and by $\mathfrak{Z} \cap N$ the following set of subgroups of G:

$$\mathfrak{Z} \cap N = \{G_p \cap N \mid Gp \in \mathfrak{Z}\}.$$

Clearly, $\mathfrak{Z}N/N$ and $\mathfrak{Z}\cap N$ are complete Sylow sets of G/N and N respectively.

LEMMA 2.1. *Let H and K be subgroups of a group G, and $\mathfrak{Z}$ a complete set of Sylow subgroups of G. Then the following hold:*

(i) *if H is $\mathfrak{Z}$-permutably embedded in G, and K is permutable in G, then HK is $\mathfrak{Z}$-permutably embedded in G;*

(ii) *if H is $\mathfrak{Z}$-permutably embedded in G, and K is normal in G, then HK/K is $\mathfrak{Z}K/K$-permutably embedded in G/K;*

(iii) *if K is normal in G, and $K \subseteq H$, then H is $\mathfrak{Z}$-permutably embedded in G if and only if H/K is $\mathfrak{Z}K/K$-permutably embedded in G/K;*

(iv) *if H is $\mathfrak{Z}$-permutably embedded in G, and K is normal in G, then $H \cap K$ is $\mathfrak{Z}$-permutably embedded in G;*

(v) *if H is $\mathfrak{Z}$-permutably embedded in G, and K is subnormal in G, then $H \cap K$ is $\mathfrak{Z} \cap K$-permutably embedded in K.*

Proof. (i) For any $p \in \pi(H)$, let U be a $\mathfrak{Z}$-permutable subgroup of G with a Sylow p-subgroup P_1 which is also a Sylow p-subgroup of H. Then for any

$Q \in \mathfrak{Z}$, both U and K permute with Q and hence $UKQ = UQK = QUK$. Thus UK is $\mathfrak{Z}$-permutable in G. To complete the proof of (i), we show that some Sylow p-subgroup of UK is also a Sylow p-subgroup of HK. Let P be a Sylow p-subgroup of HK with $P_1 \subseteq P$ and P_2 a Sylow p-subgroup of K contained in P. Then

$$\frac{|HK|}{|P_1P_2|} = \frac{|H|\,|K|\,|P_1 \cap P_2|}{|H \cap K|\,|P_1|\,|P_2|}$$

is a p'-number and so $P = P_1P_2$. By the same argument, $|UK|/|P| = |UK|/|P_1P_2|$ is a p'-number and so P is a Sylow p-subgroup of UK. Thus (i) holds.

(ii) is a direct corollary of (i) and (iii).

(iii) Assume that H is $\mathfrak{Z}$-permutably embedded in G. For any $p \in \pi(H)$, let U be a $\mathfrak{Z}$-permutable subgroup of G with a Sylow p-subgroup P which is also a Sylow p-subgroup of H. Then $UQ = QU$ for any subgroup Q in $\mathfrak{Z}$ and it follows that

$$(UK/K)(QK/K) = UQK/K = QUK/K = (QK/K)(UK/K).$$

Thus UK/K is $\mathfrak{Z}K/K$-permutable in G/K. It is easy to see that PK/K is a Sylow p-subgroup of both H/K and U/K. By the definitions, we conclude that H/K is $\mathfrak{Z}K/K$-permutably embedded in G/K.

Conversely, assume that H/K is $\mathfrak{Z}K/K$-permutably embedded in G/K. For any $p \in \pi(H/K)$, let U/K be a $\mathfrak{Z}K/K$-permutable subgroup of G/K with a Sylow p-subgroup L/K which is also a Sylow p-subgroup of H/K. Then for any Sylow subgroup $QK/K \in \mathfrak{Z}K/K$, $(U/K)(QK/K) = (QK/K)(U/K)$, and thus $UQ = (UK)Q = U(QK) = (QK)U = QU$. Hence U is $\mathfrak{Z}$-permutable in G. Let P be a Sylow p-subgroup of L. Then $L = PK$ and $|L : P|$ is a p'-number and hence $|U : P| = |U : L|\,|L : P|$ is a p'-number. It follows that P is a Sylow p-subgroup of U. By the same argument, one can prove that P is also a Sylow p-subgroup of H. Thus H is $\mathfrak{Z}$-permutably embedded in G.

(iv) For any $p \in \pi(H)$, let U be a $\mathfrak{Z}$-permutable subgroup of G with a Sylow p-subgroup which is also a Sylow p-subgroup of H. Then $UQ = QU$ for any subgroup Q in $\mathfrak{Z}$.

We claim that $U \cap K$ is $\mathfrak{Z}$-permutable in G. It is sufficient to prove that $(U \cap K)Q = UQ \cap KQ$ for any $Q \in \mathfrak{Z}$. Clearly, $(U \cap K)Q \subseteq UQ \cap KQ = (U \cap KQ)Q$. Since

$$|U \cap KQ| = |U|\,|KQ|/|UKQ| = |U|\,|K|\,|Q|\,|UK \cap Q|/(|K \cap Q|\,|UK|\,|Q|)$$

and $|U \cap K| = |U|\,|K|/|UK|$, we see that $|U \cap KQ|/|U \cap K| = |UK \cap Q|/|K \cap Q|$ is a q-number, where q is the unique prime divisor of $|Q|$. It follows that $|UQ \cap KQ|/|(U \cap K)Q|$ is a q-number. But Q is a Sylow q-subgroup of G. So $|UQ \cap KQ|/|(U \cap K)Q|$ must be a q'-number. Hence $|(U \cap K)Q| = |UQ \cap KQ|$.

It follows that $(U \cap K)Q = UQ \cap KQ$ is a subgroup of G and hence $U \cap K$ is $\mathfrak{Z}$-permutable in G, proving our claim.

Let P be a Sylow p-subgroup of both H and U. Then $P \cap K = P \cap U \cap K$ is a Sylow p-subgroup of $U \cap K$ since $U \cap K$ is normal in U. Similarly, $P \cap K$ is also a Sylow p-subgroup of $H \cap K$, and this shows that $H \cap K$ is $\mathfrak{Z}$-permutably embedded in G.

(v) By induction, we may assume that K is normal in G. Let $H_1 = H \cap K$. By (iv), H_1 is $\mathfrak{Z}$-permutably embedded in G. For any $p \in \pi(H_1)$, let U be a $\mathfrak{Z}$-permutable subgroup of G with a Sylow p-subgroup which is also a Sylow p-subgroup of H_1; by the argument in (iv), we can also assume that $U \subseteq K$. Then for any $Q \in \mathfrak{Z}$, $UQ = QU$ and hence $U(Q \cap K) = UQ \cap K = (Q \cap K)U$. So U is $\mathfrak{Z} \cap K$-permutable in K. Therefore, $H_1 = H \cap K$ is $\mathfrak{Z} \cap K$-permutably embedded in K.

Lemma 2.2. *Let H be a p-subgroup of G, and L/K an abelian chief factor of G. Assume that $\mathfrak{Z}$ is a complete Sylow set of G, and H is $\mathfrak{Z}$-permutably embedded in G. If there is a Sylow p-subgroup G_p of G such that $H \cap L \trianglelefteq G_p$, then H either covers or avoids L/K.*

Proof. By Lemma 2.1, the hypotheses of the lemma still hold on G/K. So, by induction on $|G|$, we may assume that $K = 1$ and hence L is minimal normal in G. Since H is a $\mathfrak{Z}$-permutably embedded p-subgroup of G, there is a $\mathfrak{Z}$-permutable subgroup U of G with a Sylow p-subgroup H. Since L is abelian, L is primary. If L is not a p-group then clearly L is avoided by H. Assume L is a p-group. Let G_q be any element in $\mathfrak{Z}$. Then $UG_q = G_qU$. Assume G_q is not a p-group. Then H is also a Sylow p-subgroup of UG_q and so $H \cap L = L \cap UG_q \trianglelefteq UG_q$. By the choice of G_q, we see that $|G : N_G(H \cap L)|$ is a p-number. On the other hand, $H \cap L \trianglelefteq G_p$ for some Sylow p-subgroup G_p. So $H \cap L \trianglelefteq G$ and hence $H \cap L = L$ or 1 by the minimality of L. Thus H covers or avoids L, and the lemma follows.

Lemma 2.3 ([8, 1.8.1]). *Let N be a normal subgroup of a group G such that $N/N \cap \Phi(G)$ is p-nilpotent. Then N is also p-nilpotent.*

We use $\mathrm{Soc}(G)$ to denote the product of all minimal normal subgroups of G.

Lemma 2.4 ([18, 3.13]). *Let $N \leq K \leq \mathrm{Soc}(G)$ where $N, K \trianglelefteq G$. Then there is a normal subgroup T of G such that $K = N \times T$.*

Lemma 2.5 ([9, Lemma 3.1]). *Let N and L be normal subgroups of a group G such that P/L is a Sylow p-subgroup of NL/L, and M/L is a maximal subgroup of P/L. If P_p is a Sylow p-subgroup of $P \cap N$, then P_p is a Sylow p-subgroup of N such that $D = M \cap N \cap P_p$ is a maximal subgroup of P_p, and $M = LD$.*

Recall that the *generalized Fitting subgroup* $F^*(G)$ of a group G is the maximal quasinilpotent normal subgroup of G, and if G is soluble then $F^*(G) = F(G)$ is the Fitting subgroup of G. Let $\mathfrak{U}$ be the class of all supersoluble groups. Then the $\mathfrak{U}$-*hypercenter* $Z_\infty^{\mathfrak{U}}(G)$ of a group G is the maximal normal subgroup in which all G-chief factors are cyclic.

LEMMA 2.6 ([13, Lemma 2.17]). *Let G be a group and E a normal subgroup of G. If $F^*(E) \subseteq Z_\infty^{\mathfrak{U}}(G)$, then $E \subseteq Z_\infty^{\mathfrak{U}}(G)$.*

3. Proof of Theorem A. Assume that Theorem A is false and let G be a counterexample with minimal order.

We first show that the hypotheses of the theorem still hold on the quotient group G/Φ, where $\Phi = \Phi(G)$ is the Frattini subgroup of G. Consider $F/\Phi = F_p(N\Phi/\Phi)$. Then $F = F \cap N\Phi = (F \cap N)\Phi$. Since F/Φ is a p-nilpotent normal subgroup of G/Φ, F is a p-nilpotent normal subgroup of G by Lemma 2.3. Hence $F \cap N \leq F_p(N)$. On the other hand, because $F_p(N)/F_p(N) \cap \Phi \cong F_p(N)\Phi/\Phi \leq F_p(N\Phi/\Phi)$, we have $F_p(N) \subseteq F$. Consequently, $F \cap N = F_p(N)$ and therefore

$$F_p(N\Phi/\Phi) = F/\Phi = (F \cap N)\Phi/\Phi = F_p(N)\Phi/\Phi.$$

Now let P/Φ be a Sylow p-subgroup of F/Φ, let M/Φ be a maximal subgroup of P/Φ, and let P_p be a Sylow p-subgroup of $P \cap F_p(N)$. Then, by Lemma 2.5, P_p is a Sylow p-subgroup of $F_p(N)$, and $L = M \cap F_p(N) \cap P_p$ is a maximal subgroup of P_p. By our hypotheses, L either is $\mathfrak{Z}$-permutably embedded in G or has a p-supersoluble supplement T in G. By Lemma 2.5 again, we have $M = \Phi L$. If L is $\mathfrak{Z}$-permutably embedded in G, then, by Lemma 2.1, $L\Phi/\Phi$ is $\mathfrak{Z}\Phi/\Phi$-permutably embedded G/Φ, and so $M/\Phi = L\Phi/\Phi$ is $\mathfrak{Z}\Phi/\Phi$-permutably embedded in G/Φ. If L has a p-supersoluble supplement T in G, then $T\Phi/\Phi$ is also a p-supersoluble supplement of $L\Phi/\Phi$ in G/Φ. Thus, the group G/Φ has a normal subgroup $N\Phi/\Phi$ such that each maximal subgroup of every Sylow p-subgroup of $F_p(N\Phi/\Phi) = F_p(N)\Phi/\Phi$ either is $\mathfrak{Z}\Phi/\Phi$-permutably embedded in G/Φ or has a p-supersoluble supplement in G/Φ. Because $(G/\Phi)/(N\Phi/\Phi) \cong G/N\Phi \cong (G/N)/(N\Phi/N)$ is a p-supersoluble group, we see that the hypotheses of the theorem still hold on G/Φ.

If $\Phi \neq 1$, then $|G/\Phi| < |G|$, and so G/Φ is p-supersoluble by the choice of G. Hence G is p-supersoluble by [10, VI, 9.3]. This contradicts our assumption on G. Hence $\Phi(G) = 1$. Analogously, we can prove that $O_{p'}(G) = 1$ and hence

$$F_p(G) = O_p(G) = F(G) = R_1 \times \cdots \times R_t$$

by [6, A, 10.6], where $R_1, \ldots, R_t$ are the minimal normal subgroups of G. Clearly, $F_p(N) = F(N) = N \cap F(G)$ and hence $F(N)$ itself is its Sylow

p-subgroup. Let P be the Sylow p-subgroup of G contained in $\mathfrak{Z}$, and M_i be a maximal subgroup of R_i, $i = 1, \dots, t$, normal in P. Assume that for some index i, we have $|M_i| \neq 1$. Also, assume that $R_i \not\subseteq N$. Then NR_i/N is a p-chief factor of the p-supersoluble group G/N and so $|NR_i/N|$ is a prime. But $R_i \cong NR_i/N$, and we see that $M_i = 1$. This contradiction shows that $R_i \leq N$. By using Lemma 2.4, we see that $F(N) = R_i \times D$ for some normal subgroup D of G. Let $M = M_iD$. Since $|F(N) : M| = |R_i : M_i| = p$, M is a maximal subgroup of $F(N) = F_p(N)$. If M is $\mathfrak{Z}$-permutably embedded in G, then so is $M \cap R_i = M_i$ by Lemma 2.1(iv). Applying Lemma 2.2, we deduce that M_i covers or avoids R_i. But clearly $M_i < R_i$, so M_i avoids R_i. Thus $|M_i| = 1$ and $|R_i| = p$. This contradiction shows that M is not $\mathfrak{Z}$-permutably embedded in G, and hence by our hypotheses, M has a p-supersoluble supplement T in G. Since $R_i \cap T$ is normal in both T and $F(N)$, we have $R_i \cap T \trianglelefteq G = MT = F(N)T$.

If $R_i \cap T \neq 1$, then $R_i \subseteq T$ and hence $G = MT = DT$. So, $G/D = DT/D \cong T/T \cap D$ is a p-supersoluble group. This implies that $R_i \cong R_iD/D$ is a group of prime order. This is a contradiction.

Now, assume that $R_i \cap T = 1$. Without loss of generality, we can assume that $i = 1$. Recall that $F(N) = R_1 \times \cdots \times R_n = R_1 \times D$.

If $n = 1$, then $M = M_1 < R_1$. Hence $G = MT = R_1T$. Since $R_1 \cap T = 1$, $|R_1| = |G : T| = |MT : T| \leq |M| < |R_1|$, a contradiction.

Suppose that $n = 2$ and $F(N) = R_1 \times R_2$. If $R_2 \leq T$, then $TM = TM_1 = G$, and hence $|G : T| \leq |M_1| < |R_1|$. But $R_1T = G$, and so $|G : T| = |R_1|$. This contradiction shows that $R_2 \not\leq T$. On the other hand, if $R_1R_2 \cap T = 1$, then, by $G = TM_1R_2 = TR_1R_2$, we have $|G : T| = |R_1|\,|R_2|$. But from $TM_1R_2 = G$, we derive that $|G : T| \leq |M_1|\,|R_2| < |R_1|\,|R_2|$, a contradiction.

Hence $\Delta = R_1R_2 \cap T \neq 1$. Since R_1R_2 is an abelian group, $\Delta = R_1R_2 \cap T$ is clearly a normal subgroup of G. Let R be a minimal normal subgroup of G contained in Δ. Since $R_1 \not\leq T$, $R_2 \not\leq T$ and $R \leq T$, we see that $R \neq R_1$, and $R \neq R_2$. Hence $F(N) = R_1R_2 = RR_2$, and so $F(N)/R_2 \cong R_1 \cong R$. Analogously, we can prove that $R_1 \cong R_2$. We also note that $\Delta = \Delta \cap R_1R_2 = \Delta \cap R_1R = R(\Delta \cap R_1) = R$. Hence

$$|G| = \frac{|T|\,|R_1R_2|}{|T \cap R_1R_2|} = \frac{|T|\,|R_1|\,|R_2|}{|R|} = |T|\,|R_1|,$$

and so $|G : T| = |R_1|$.

Let $E = R_1T$. Assume that $E \neq G$. Then $|G : T| = |E : T|\,|G : E| > |R_1|$, a contradiction. Hence $E = G = R_1T$, and so $G/R_1 \cong T/R_1 \cap T$ is a p-supersoluble group. But R_2R_1/R_1 is a minimal normal p-subgroup in G/R_1 and so $|R_2| = |R_1| = p$. This contradiction shows that $n \geq 3$.

Assume that $\Delta \nleq T$ for every minimal normal subgroup Δ of G contained in $R_2 \cdots R_n$. Then, evidently, $T \cap R_2 \cdots R_n = 1$, and hence $|G : TR_2 \cdots R_n| \leq |M_1|$. It is clear that $TR_2 \cdots R_n \neq G$. But $R_1TR_2 \cdots R_n = G$, and we have $|G : TR_2 \cdots R_n| = |R_1|$, a contradiction.

Hence, there is a minimal normal subgroup Δ_1 in G such that $\Delta_1 \leq T \cap R_2 \cdots R_n$. We note that since $\bigcap_{i=2}^n R_2 \cdots R_{i-1}R_{i+1} \cdots R_n = 1$, there exists an index i such that $R_2 \cdots R_{i-1}R_iR_{i+1} \cdots R_n = R_2 \cdots R_{i-1}\Delta_1R_{i+1} \cdots R_n$. Thus we may suppose, without loss of generality, that there is an index $2 \leq i < n$ such that $R_2, \dots, R_i \leq T$ and that for every minimal normal subgroup Δ_2 of G contained in $R_{i+1} \cdots R_n$, we have $\Delta_2 \nleq T$. This implies that $T \cap R_{i+1} \cdots R_n = 1$.

Now let $\Delta_3 = R_1R_{i+1} \cdots R_n \cap T$. Assume that $\Delta_3 = 1$. Then, since $G = TR_1 \cdots R_n = TR_1R_{i+1} \cdots R_n$, we have $|G : T| = |R_1|\,|R_{i+1}| \cdots |R_n|$. On the other hand, as $G = TM_1R_2 \cdots R_n = TM_1R_{i+1} \cdots R_n$, we have $|G : T| \leq |M_1|\,|R_{i+1}| \cdots |R_n|$, a contradiction. Consequently, $\Delta_3 \neq 1$.

Let L be a minimal normal subgroup of G contained in Δ_3. Since $L \leq T$, we have $L \nleq R_{i+1} \cdots R_n$. But $L \leq R_1R_{i+1} \cdots R_n$, therefore $LR_{i+1} \cdots R_n = R_1R_{i+1} \cdots R_n$, and so

$$\begin{aligned} G &= TR_1R_2 \cdots R_n = R_2 \cdots R_iTR_1R_{i+1} \cdots R_n = R_2 \cdots R_iTLR_{i+1} \cdots R_n \\ &= R_2 \cdots R_iTR_{i+1} \cdots R_n. \end{aligned}$$

Hence $G/R_2 \cdots R_n \cong T/(T \cap R_2 \cdots R_n) = T/T \cap D$ is a p-supersoluble group. This implies that $R_1 \cong R_1R_2 \cdots R_n/R_2 \cdots R_n$ is a group of prime order; however, this is a contradiction. Hence, every group R_i has a prime order for $i = 1, \dots, t$ and so $F^*(N) = F(N) = F_p(N) \subseteq Z_\infty^{\mathfrak{U}}(G)$, where $\mathfrak{U}$ is the formation of all supersoluble groups. Now, by Lemma 2.6, $N \subseteq Z_\infty^{\mathfrak{U}}(G)$. Therefore, G is p-supersoluble since G/N is. This contradiction completes the proof.

4. Proofs of Theorems B and C

Proof of Theorem B. Assume that the theorem is not true. Via the following steps, we shall prove the theorem assuming that G is a counterexample of minimal order.

(1) $O_{p'}(G) = 1$ *and so* $F_p(N) = F(N) = P$ *is a* p*-group.*

Since N is normal in G, $O_{p'}(N) \subseteq O_{p'}(G)$ and so

$$F_p(NO_{p'}(G))/O_{p'}(G) = F_p(N)O_{p'}(G)/O_{p'}(G) = PO_{p'}(G)/O_{p'}(G).$$

Let $H/O_{p'}(G)$ be a cyclic subgroup of $PO_{p'}(G)/O_{p'}(G)$ of order p or 4 (when $PO_{p'}(G)/O_{p'}(G) \cong P$ is a nonabelian 2-group). Then $H = (H \cap P)O_{p'}(G)$ and $H \cap P$ is cyclic of order p or 4. By hypotheses, $H \cap P$ either is $\mathfrak{Z}$-permutably embedded in G or has a p-supersoluble supplement T in G.

If $H \cap P$ is $\mathfrak{Z}$-permutably embedded in G, then by Lemma 2.1, $H/O_{p'}(G) = (H \cap P)O_{p'}(G)/O_{p'}(G)$ is $\mathfrak{Z}O_{p'}(G)/O_{p'}(G)$-permutably embedded in $G/O_{p'}(G)$; if $H \cap P$ has a p-supersoluble supplement T in G, then, clearly, $O_{p'}(G) \subseteq T$ and $T/O_{p'}(G)$ is a p-supersoluble supplement of $H/O_{p'}(G)$ in $G/O_{p'}(G)$.

Assume $O_{p'}(G) \neq 1$. Then $|G/O_{p'}(G)| < |G|$ and hence $G/O_{p'}(G)$ is p-supersoluble by the choice of G. But this induces that G is p-supersoluble, a contradiction. Hence $O_{p'}(G) = 1$ and so $O_{p'}(N) = 1$. Therefore, $F_p(N) = PO_{p'}(N) = P = F(N)$, proving (1).

(2) *Let L be a minimal normal subgroup of G. Then L is of order p.*

If $L \nsubseteq N$ then LN/N is a minimal normal subgroup of a p-supersoluble group G/N. Since $O_{p'}(G) = 1$, $L \cong LN/N$ is cyclic of order p. Assume that $L \subseteq N$. Recall that N is p-soluble. Again as $O_{p'}(G) = 1$, L is a p-group. Assume G_p is the Sylow p-subgroup of G lying in $\mathfrak{Z}$. Then $L \trianglelefteq G_p$ and $L \cap Z(G_p) \neq 1$. Choose x to be an element of order p in $L \cap Z(G_p)$ and let $H = \langle x \rangle$. Then $H \trianglelefteq G_p$. Also, by the hypotheses, H either is $\mathfrak{Z}$-permutably embedded in G or has a p-supersoluble supplement T in G.

Assume that H has a p-supersoluble supplement T in G. Then $G = HT = LT$. Since L is minimal normal in G, $L \subseteq T$ or $L \cap T = 1$. If $L \subseteq T$ then $G = LT = T$ is p-supersoluble, a contradiction.I f $L \cap T = 1$, then $|L| = |G : T| = |HT : T| \leq |H| \leq |L|$. So $L = H$ is cyclic.

Assume that H is $\mathfrak{Z}$-permutably embedded in G. Then H covers or avoids L by Lemma 2.2. But $H \cap L = H \neq 1$ by the choice of H, so H covers L. This means that $L = H$ is cyclic of order p.

(3) *Every G-chief factor L/K in P is of prime order.*

Assume that there exists a G-chief factor L/K in P which is not of prime order. Then by (2), $K \neq 1$. Choose a G-chief factor L/K in P such that $|L/K|$ is not a prime but $|X/Y|$ is a prime for all chief factors X/Y of G with $|X| < |L|$.

Let $W = \bigcap_{X \subseteq K} C_G(X/Y)$, where X/Y is a G-chief factor. Then, by [6, A, (12.3)], all elements in W of p'-order act trivially on K since they act trivially on each G-chief factor of K. Let $C = C_G(K)$.

Assume $L \nsubseteq C$. If $L \subseteq KC$, then $L \cap C/K \cap C \cong L/K$ is a chief factor of G. By the choice of L/K, $|L/K| = |L \cap C/K \cap C|$ is a prime, a contradiction. If $L \nsubseteq KC$, then it is easy to see that $LC/K = L/K \times KC/K$ and therefore, all p'-elements in C act trivially on L/K. It follows that all p'-elements in W act trivially on L/K. Hence $W \subseteq C_G(L/K)$. Since $G/W = G/\bigcap_{X \subseteq K} C_G(X/Y)$ is an abelian group of exponent dividing $p-1$ and $W \subseteq C_G(L/K)$, $G/C_G(L/K)$ is an abelian group of exponent dividing

$p-1$. Since L/K is G-irreducible, L/K is of prime order by [20, I, Lemma 1.3], a contradiction.

Now assume that $L \subseteq C$. Then $K \subseteq Z(L)$. Let a, b be elements of order p in L. Suppose $p > 2$ or P is abelian. Then $(ab)^p = a^p b^p [b, a]^{p(p-1)/2} = 1$. Hence the product of elements of order p is still of order p and therefore $\Omega = \{a \in L \mid a^p = 1\}$ is a subgroup of L.

If $\Omega \subseteq K$, then all elements of W with p'-order act trivially on every element of L of order p since they act trivially on K. It follows from [10, IV, Satz 5.12] that all elements in W of p'-order act trivially on L. Thus $W \subseteq C_G(L/K)$ and, as in the above argument, L/K is of prime order, a contradiction.

If $\Omega \not\subseteq K$, then $L = \Omega K$. Choose an element a in $\Omega \setminus K$ such that $\langle a\rangle K/K \subseteq L/K \cap Z(G_p/K)$. Let $H = \langle a\rangle$. If H has a p-supersoluble supplement T in G, then HK/K has a p-supersoluble supplement TK/K in G/K. Thus $G/K = (HK/K)(TK/K) = (L/K)(TK/K)$. Since L/K is minimal normal in G/K and is abelian, either $L/K \cap TK/K = 1$ or $L/K \subseteq TK/K$ and $TK/K = G/K$.

If $L/K \cap UK/K = 1$, then $|L/K| = |G/K : TK/K| = |HTK/K : TK/K| \leq |H| = p$. It follows that L/K is cyclic of order p, which contradicts the choice of L/K.

If $L/K \subseteq TK/K = G/K$, then L/K is cyclic since L/K is minimal normal p-subgroup of G/K and $G/K = TK/K \cong T/T\cap K$ is p-supersoluble. Assume that H is $\mathfrak{Z}$-permutably embedded in G. Then H covers or avoids L/K by Lemma 2.2. Clearly, H does not avoid L/K by the choice of H. Hence H covers L/K and so $L/K = (H \cap L)K/K = HK/K$ is cyclic, a contradiction. This implies that every G-chief factor in P is cyclic. By a similar argument, we can show that every G-chief factor in P is cyclic when P is a nonabelian 2-group. Hence (3) holds.

(4) *Final contradiction.*

It follows directly from (3) that $F(N) = P \subseteq Z_\infty^{\mathfrak{U}}(G)$ and hence $N \subseteq Z_\infty^{\mathfrak{U}}(G)$ by Lemma 2.6. Therefore, G is p-supersoluble since G/N is. This is the final contradiction proving the theorem.

Proof of Theorem C. Assume that the theorem is not true and let G be a counterexample of minimal order. Then

(1) $O_{p'}(G) = 1$ *and* $F_p(N) = P = F(N)$.

This can be proved in the same way as step (1) in the proof of Theorem B.

(2) *Let R be a minimal normal subgroup of G contained in N. Then R is cyclic of order p and $R \subseteq Z(G)$.*

By (1), R is a p-group. Assume that $|R| > p^2$. Then R has a proper subgroup H of order p^2 and H is normal in some Sylow p-subgroup of G. By the hypotheses, H either is $\mathfrak{Z}$-permutably embedded in G or has a p-nilpotent supplement T in G. If H is $\mathfrak{Z}$-permutably embedded in G, then H covers or avoids R by Lemma 2.2. This is impossible by the choice of H. So H has a p-nilpotent supplement T in G. Thus $G = HT = RT$. Since R is an abelian normal subgroup of G, we have $R \cap T \trianglelefteq G$. Then the minimality of R shows that $R \cap T = R$ or 1. If $R \cap T = R$ then $R \subseteq T$ and $G = T$ is p-nilpotent, contrary to the choice of G. If $R \cap T = 1$, then $|R| = |G : T| = |HT : T| \leq |H| < |R|$, a contradiction.

Assume that $|R| = p^2$. Then $|\mathrm{Aut}\, R| = (p^2 - 1)(p^2 - p)$. Since $G/C_G(R)$ is isomorphic to some subgroup of $\mathrm{Aut}\, R$ and $(|G|, p^2 - 1) = 1$, $G/C_G(R)$ is a p-group. Now, applying [8, Lemma 1.7.11], we find that $G/C_G(R) = 1$ and so $R \subseteq Z(G)$. Thus $|R| = p$. The inclusion $R \subseteq Z(G)$ can be obtained directly from $(|G|, p^2 - 1) = 1$ and $|R| = p$.

(3) *$p = 2$ and P is nonabelian.*

If $\Phi(N) = 1$ then $F(N) = \mathrm{Soc}(N)$ is a product of some minimal normal subgroups of G. By (1), we have $F(N) \subseteq Z_\infty^{\mathfrak{U}}(G)$. Hence, by Lemma 2.6, G is p-supersoluble since G/N is p-nilpotent. But since $(|G|, p^2 - 1) = 1$, G p-supersoluble means G is p-nilpotent. Assume $\Phi(N) \neq 1$ and choose R to be a minimal normal subgroup of G contained in $\Phi(N)$. Then $F_p(N/R) = F(N/R) = F(N)/R$. For any subgroup H/R of order p in $F(N)/R$, H is of order p^2. By the hypotheses, H either is $\mathfrak{Z}$-permutably embedded in G or has a p-nilpotent supplement T in G. If H is $\mathfrak{Z}$-permutably embedded in G, then by Lemma 2.1, H/R is $\mathfrak{Z}R/R$-permutably embedded in G/R. If H has a p-nilpotent supplement T in G, then TR/R is a p-nilpotent supplement of H/R in G/R. Thus, if $p > 2$ or P is abelian, then the hypotheses of Theorem B hold and hence G is p-supersoluble. Therefore, G is p-nilpotent because $(|G|, p^2 - 1) = 1$. Thus $p = 2$ and P is nonabelian.

(4) *Let H be a subgroup of P of order* 4*. If H is cyclic or $R \subseteq H$, where R is as in* (2)*, then H is $\mathfrak{Z}$-permutably embedded in G.*

By our hypotheses, H either is $\mathfrak{Z}$-permutably embedded in G or has a p-nilpotent supplement in G. Assume (4) is not true and let T be a p-nilpotent supplement of H in G. If H is cyclic, then $H = \langle x \rangle$ for some $x \in P$ of order 4. Clearly, $T \neq G$. Let M be a maximal subgroup of G contained T. Since $x^2 \in \Phi(P) \subseteq \Phi(G)$, $x^2 \in M$ and hence $H \cap M = \langle x^2 \rangle$. It follows that $|G : M| = |HT : M| = |HM : M| = 2$. Thus M is normal in G. Again as $H \cap M = \langle x^2 \rangle \subseteq \Phi(G)$, the group $M/\Phi(G) = T\Phi(G)/\Phi(G) \cong T/T \cap \Phi(G)$ is p-nilpotent and so is M by [8, Lemma 1.8.1]. But $O_{p'}(M) \subseteq O_{p'}(G) = 1$ by (1), so M is a p-group and hence so is G since $|G : M| = p = 2$, a contradiction. If $R \subseteq H$ then $R \subseteq Z(G)$ and by the same argument as above, we can also obtain a contradiction. Thus (4) holds.

(5) *Assume that $q \neq p$ is a prime divisor of $|G|$ and let Q be a Sylow q-subgroup of G in $\mathfrak{Z}$. Then $Q \subseteq C_G(x)$ for any element x of order 2 or 4 in P.*

Assume $|x| = 4$ and let $H = \langle x \rangle$. Then, by (4), H is $\mathfrak{Z}$-permutably embedded in G. Let U be a $\mathfrak{Z}$-permutable subgroup of G with a Sylow 2-subgroup H. Then $UQ = QU$ is a subgroup of G. Clearly H is also a Sylow 2-subgroup of UQ. Since H is cyclic, UQ is 2-nilpotent and hence $2'$-closed. On the other hand, since $H = P \cap UQ \trianglelefteq UQ$, we see that UQ is 2-closed. Thus $H \subseteq Z(UQ)$ and $Q \subseteq C_G(x)$.

Assume $|x| = 2$. Let $H = R\langle x \rangle$. By a similar argument we also obtain $Q \subseteq C_G(x)$ and thus (5) holds.

(6) *Final contradiction.*

Let $\Omega = \langle x \mid |x| = 2 \text{ or } 4 \rangle$. Then Q acts trivially on Ω by (5) and hence acts trivially on P by [10, IV, Satz 5.12]. Thus $Q \subseteq C_G(P)$. Since G/N is p-nilpotent, G/N is p'-closed. Let M/N be the normal p-complement of G/N. Then $M \trianglelefteq G$ and $Q \subseteq M$. We claim that $F_p(M) = F(M) = F(N) = P$. Since $O_{p'}(M) \subseteq O_{p'}(G) = 1$, we have $O_{p'}(M) = 1$ and so $F_p(M) = F(M) = O_p(M)$. It follows that $F_p(M) \subseteq N$ since M/N is a p'-group. Therefore, $F_p(M) = F(M) \subseteq F(N) = F_p(N) \subseteq F_p(M)$ and our claim holds. But this implies that $Q \subseteq M \cap C_G(P) = C_M(P) \subseteq P = F_p(N)$, a contradiction. This shows that G is a p-group and so it is nilpotent, contrary to the choice of G. Therefore Theorem C holds.

5. Some corollaries. In this section, we give some corollaries which can be obtained from our theorems.

Let p be a prime and G a group. As we know, if $(p-1, |G|) = 1$, then G is p-nilpotent if and only if G is p-supersoluble. The following corollaries can be obtained directly from Theorems A and B:

COROLLARY 5.1. *Let G be a p-soluble group and $\mathfrak{Z}$ a complete Sylow set of G. Assume that G has a normal subgroup N with p-nilpotent quotient. If $(p-1, |G|) = 1$ and every maximal subgroup of a Sylow p-subgroup of $F_p(N)$ either is $\mathfrak{Z}$-permutable in G or has a p-nilpotent supplement in G, then G is p-nilpotent.*

COROLLARY 5.2. *Let G be a p-soluble group and $\mathfrak{Z}$ a complete Sylow set of G. Assume that G has a normal subgroup N with p-nilpotent quotient. If $(p-1, |G|) = 1$ and every subgroup of order p or 4 (when $p = 2$ and a Sylow p-subgroup of $F_p(N)$ is nonabelian) of $F_p(N)$ either is $\mathfrak{Z}$-permutably embedded in G or has a p-nilpotent supplement in G, then G is p-nilpotent.*

By similar arguments to the proofs of Theorems A and B, we can obtain respectively:

COROLLARY 5.3. *Let G be a soluble group and $\mathfrak{Z}$ a complete Sylow set of G. Then G is supersoluble if and only if every maximal subgroup of every Sylow subgroup of $F(G)$ either is $\mathfrak{Z}$-permutable in G or has a supersoluble supplement in G.*

COROLLARY 5.4. *Let G be a soluble group and $\mathfrak{Z}$ a complete Sylow set of G. Then G is supersoluble if and only if every cyclic subgroup of prime order or of order 4 of every Sylow subgroup of $F(G)$ either is $\mathfrak{Z}$-permutable in G or has a supersoluble supplement in G.*

Some known results can also be deduced from our theorems.

COROLLARY 5.5 ([16]). *Let G be a soluble group. If all maximal subgroups of Sylow subgroups of $F(G)$ are normal in G, then G is supersoluble.*

COROLLARY 5.6 ([2]). *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. Suppose that all maximal subgroups of any Sylow subgroup of $F(E)$ are s-permutable in G. Then G is supersoluble.*

COROLLARY 5.7 ([4]). *Let G be a soluble group with a normal subgroup E such that G/E is supersoluble. If all maximal subgroups of Sylow subgroups of $F(E)$ are S-quasinormally embedded in G, then G is supersoluble.*

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