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ON THE COMPOSITION OF THE EULER FUNCTION AND THE SUM OF DIVISORS FUNCTION

BY

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Abstract. Let $H(n) = \sigma(\phi(n))/\phi(\sigma(n))$, where $\phi(n)$ is Euler's function and $\sigma(n)$ stands for the sum of the positive divisors of n. We obtain the maximal and minimal orders of $H(n)$ as well as its average order, and we also prove two density theorems. In particular, we answer a question raised by Golomb.

1. Introduction. Let ϕ be Euler's function and let σ be the sum of divisors function. The composition of the functions σ and ϕ has been the object of several studies; see for instance Mąkowski and Schinzel [9], Pomerance [11], Sándor [12], Ford [2], Luca and Pomerance [8]. In 1993, Golomb [3] investigated the difference $\sigma(\phi(n)) - \phi(\sigma(n))$ showing that it is both positive and negative infinitely often, and asked what is the proportion of each.

In this paper, we answer this question of Golomb and more, by studying the behavior of the quotient

$$H(n) := \frac{\sigma(\phi(n))}{\phi(\sigma(n))}.$$

In particular, we obtain the maximal and minimal orders of $H(n)$, its average order, and we also prove two density theorems.

Given any positive real number x we write $\log x$ for the maximum between the natural logarithm of x and 1. If k is a positive integer, we write $\log_k x$ for the kth iteration of the function $\log x$. Throughout this paper, p, q and r stand for prime numbers, while γ stands for Euler's constant. We also use $\pi(x)$ for the number of primes up to x and $\omega(n)$ for the number of distinct prime factors of n.

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2. Main results

THEOREM 1. *The maximal order of $H(n)$ is $e^{2\gamma}\log_2^2 n$, that is,*

$$\limsup_{n\to\infty}\frac{H(n)}{\log_2^2 n}=e^{2\gamma}.$$

THEOREM 2. *There exists a positive constant δ such that the minimal order of $H(n)$ is $\delta/\log_2 n$, that is,*

$$\liminf_{n\to\infty} H(n)\log_2 n=\delta.$$

Moreover $\delta\in[(1/40)e^{-\gamma},2e^{-\gamma}]$.

THEOREM 3. *As $x\to\infty$,*

$$\frac{1}{x}\sum_{n\le x}H(n)=c_0\,e^{2\gamma}\log_3^2 x+O(\log_3^{3/2}x),$$

where

$$\begin{aligned}c_0&=\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\frac{\phi(n)}{\sigma(n)}\\&=\prod_p\left(1-\frac{3}{p(p+1)}+\frac{1}{p^2(p+1)}+\frac{(p-1)^3}{p^2}\sum_{i=3}^{\infty}\frac{1}{p^i-1}\right)\approx 0.4578.\end{aligned}$$

THEOREM 4. *For each number u, $0\le u\le 1$, the asymptotic density of the set of numbers n with*

$$H(n)>ue^{2\gamma}\log_3^2 n$$

exists, and this density function is strictly decreasing, varies continuously with u, and is 0 when $u=1$.

In particular, Theorem 4 shows that $\sigma(\phi(n))-\phi(\sigma(n))$ is positive for most n, thus providing an answer to Golomb's question.

THEOREM 5. *The set $\{H(1),H(2),H(3),\ldots\}$ is dense in $[0,\infty)$.*

3. Preliminary results

THEOREM A (Heath-Brown [6]). *Let k and a be coprime positive integers. Then there exists a prime number $p\equiv a \pmod k$ which satisfies $p=O(k^{11/2})$.*

REMARK. It has been shown by Alford, Granville and Pomerance [1] that for most values of k, one can replace the constant $11/2$ by $12/5+\varepsilon$ for any fixed $\varepsilon>0$. It can also be shown that if GRH holds, then the constant $11/2$ can be replaced by $2+\varepsilon$ for any fixed $\varepsilon>0$.

THEOREM B (Pomerance [11]). *There exists a constant $\kappa > 0$ such that, for all positive integers n,*

$$\frac{\sigma(\phi(n))}{n} > \kappa.$$

REMARK. This statement relates to a long standing conjecture of Mąkowski and Schinzel [9], which asserts that $\sigma(\phi(n))/n \geq 1/2$. Recently, Ford [2] has shown that $\kappa \geq 1/39.4$. Note also that the conjectured minimum $1/2$ is attained when n is twice the product of the first Fermat primes, such as $n = 2$, 6, 30, 510, 131070 and 8589934590.

LEMMA 1 (Mertens' theorem [11]). *The estimate*

$$\prod_{p \leq x}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right)$$

holds for large values of x.

LEMMA 2. $\displaystyle\liminf_{n \to \infty} \frac{\phi(n) \log_2 n}{n} = e^{-\gamma}$.

Proof. This result, which follows essentially from Mertens' theorem, was first obtained by Landau [7].

LEMMA 3. $\displaystyle\limsup_{n \to \infty} \frac{\sigma(n)}{n \log_2 n} = e^{\gamma}$.

Proof. This result also follows from Mertens' theorem and was first obtained by Gronwall [4].

LEMMA 4. *There exists a positive constant c_1 such that for large real numbers x, both $\phi(n)$ and $\sigma(n)$ are divisible by all prime powers $p^a < c_1 \log_2 x/\log_3 x$ for all positive integers $n < x$ with $O(x/\log_3^2 x)$ exceptions.*

Proof. The above result for the case of the function $\phi(n)$ is Lemma 2 in [8]. To prove the result for the function $\sigma(n)$, let m be an arbitrary positive integer and write

$$S(x, m) = \sum_{\substack{\log_2 x \leq q \leq x \\ m|(q+1)}} \frac{1}{q}.$$

From the Siegel–Walfisz theorem (see Theorem 5, Chapter II.8 in Tenenbaum [13]) and partial summation, it follows that there exist positive numbers c_1 and x_0 such that the inequality

$$S(x, m) \geq \frac{c_1 \log_2 x}{\phi(m)}$$

holds for $x > x_0$ and all $m \leq \log x$. Let $g(x) = c_1 \log_2 x/\log_3 x$. Using Brun's sieve, it follows that the set $\mathcal{N}_m$ of numbers $n \leq x$ which have no prime factor $q > \log_2 x$ congruent to -1 modulo m satisfies

$$\#\mathcal{N}_m < c_2 x \prod_{\substack{\log_2 x < q < \log x \\ m|(q+1)}} \left(1 - \frac{1}{q}\right) \le c_2 x \exp(-S(x,m)),$$

for some positive constant c_2. Assuming now that x_0 is chosen large enough so that $\log x > g(x)$ for all $x > x_0$, we see that if $m = p^a < g(x)$, then

$$\#\mathcal{N}_{p^a} < \frac{c_2 x}{\exp(S(x,p^a))} < \frac{c_2 x}{\exp(\log_3 x)} = \frac{c_2 x}{\log_2 x}.$$

Summing up the above inequalities over all the $O(g(x)/\log g(x))$ prime powers $p^a < g(x)$, we get

$$\sum_{p^a < g(x)} \#\mathcal{N}_{p^a} \ll \frac{x g(x)}{\log_2 x \log g(x)} \ll \frac{x}{\log_3^2 x}.$$

Finally, let $\mathcal{M}$ be the set of all positive integers $n \le x$ such that n is divisible by the square of a prime $q \ge \log_2 x$. Then

$$\#\mathcal{M} \le \sum_{q \ge \log_2 x} \frac{x}{q^2} \ll \frac{x}{\log_2 x \log_3 x} \ll \frac{x}{\log_3^2 x},$$

where we used the fact that

$$\sum_{p > z} \frac{1}{p^2} \ll \frac{1}{z \log z}. \tag{1}$$

Note now that if $n \le x$ is such that p^a does not divide $\sigma(n)$ for some $p^a < g(x)$, then either n is in $\mathcal{M}$ or n is in

$$\bigcup_{p^a < g(x)} \mathcal{N}_{p^a},$$

and by the above estimates both these sets are of cardinality $O(x/\log_3^2 x)$, thereby completing the proof of Lemma 4.

LEMMA 5. *Let x be a positive real number. Set*

$$h_\phi(n) = \sum_{\substack{p|\phi(n) \\ p > \log_2 x}} \frac{1}{p} \quad \textit{and} \quad h_\sigma(n) = \sum_{\substack{p|\sigma(n) \\ p > \log_2 x}} \frac{1}{p}.$$

Then

$$\sum_{n \le x} h_\phi(n) \ll \frac{x}{\log_3 x} \quad \textit{and} \quad \sum_{n \le x} h_\sigma(n) \ll \frac{x}{\log_3 x}. \tag{2}$$

Proof. Clearly we have

$$\sum_{\substack{n \le x \\ p|\phi(n)}} 1 \le \frac{x}{p^2} + \sum_{\substack{q \le x \\ p|(q-1)}} \frac{x}{q} \ll \frac{x}{p^2} + \frac{x \log_2 x}{p} \ll \frac{x \log_2 x}{p}.$$

It now follows that

$$\sum_{n\le x} h_\phi(n) = \sum_{p\le x}\frac{1}{p}\sum_{\substack{n\le x\\ p|\phi(n)}} 1 \ll x\log_2 x \sum_{p>\log_2 x}\frac{1}{p^2} \ll \frac{x}{\log_3 x},$$

where we used (1) with $z := \log_2 x$, thus establishing the first assertion in (2). We use a similar argument to establish the second assertion in (2). First of all, note that since $\omega(n) < \log x$ for all $n \le x$ provided x is large enough, it follows that

$$h_\sigma(n) \le \sum_{i\le\log x}\frac{1}{p_i} \ll \log_3 x,$$

where we used p_i to denote the ith prime number. Let $\mathcal{N}_1$ be the set of all positive integers $n \le x$ such that there exists a prime $q > \log_3^2 x$ whose square divides n. Then, by (1),

$$\#\mathcal{N}_1 \le \sum_{q>\log_3^2 x}\frac{x}{q^2} \ll \frac{x}{\log_3^2 x\log_4 x}.$$

Hence,

$$\sum_{n\in\mathcal{N}_1} h_\sigma(n) \ll \#\mathcal{N}_1\log_3 x \ll \frac{x}{\log_3 x\log_4 x}. \tag{3}$$

Now let $\mathcal{N}_2$ be the set of those $n \le x$ which are not in $\mathcal{N}_1$ and which are divisible by a prime power q^a, with $a = \lfloor c_3\log_4 x\rfloor + 2$, where $c_3 := 2/\log 2$. For a fixed prime number q, the number of such numbers n is $\le x/q^a$, and therefore

$$\#\mathcal{N}_2 \le \sum_{q\ge 2}\frac{x}{q^a} \le \frac{x}{2^a} + x\int_2^\infty \frac{dt}{t^a} \ll \frac{x}{2^a} \ll \frac{x}{\log_3^2 x},$$

which implies that

$$\sum_{n\in\mathcal{N}_2} h_\sigma(n) \ll \#\mathcal{N}_2\log_3 x \ll \frac{x}{\log_3 x}. \tag{4}$$

Finally, let $\mathcal{N}_3$ be the set of positive integers $n \le x$ which do not belong to either $\mathcal{N}_1$ or $\mathcal{N}_2$. If $n \in \mathcal{N}_3$ and $q^{\alpha_q} \,\|\, n$ with $\alpha_q > 1$, then $q < \log_3^2 x$ and $\alpha_q \ll \log_4 x$, so that $q^{\alpha_q} \le \exp(O(\log_4^2 x))$. Hence $\sigma(q^{\alpha_q}) < \exp(O(\log_4^2 x))$. In particular, for large x, we have $\sigma(q^{\alpha_q}) < \log_2 x$. Hence, if $n \in \mathcal{N}_3$ and $p > \log_2 x$ is a prime dividing $\sigma(n)$, then there exists a prime factor $q \,\|\, n$ of n such that $p \,|\, (q+1)$. Now the same argument used for the function h_ϕ tells us that if $p > \log_2 x$ is a fixed prime, then

$$\sum_{\substack{p|\sigma(n)\\ n\in\mathcal{N}_3}} 1 \ll \sum_{\substack{q\le x\\ p|(q+1)}}\frac{x}{q} \ll \frac{x\log_2 x}{p}.$$

Therefore

$$\sum_{n\in\mathcal{N}_3} h_\sigma(n) \le \sum_{p>\log_2 x} \frac{1}{p} \sum_{\substack{p\mid\sigma(n)\\ n\in\mathcal{N}_3}} 1 \ll \sum_{p\ge\log_2 x} \frac{x\log_2 x}{p^2} \ll \frac{x}{\log_3 x}. \tag{5}$$

The second estimate (2) then follows from estimates (3)–(5), and the proof of Lemma 5 is complete.

LEMMA 6. *As* $x\to\infty$,

$$\sum_{n\le x} \frac{\phi(n)}{\sigma(n)} = c_0 x + O(x^{3/4}),$$

where c_0 *is the constant appearing in the statement of Theorem* 3.

Proof. Given any number s with $\Re(s)>1$ and letting $\zeta(s)$ stand for the Riemann zeta function, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\phi(n)/\sigma(n)}{n^s} &= \prod_p \left(1 + \frac{\frac{p-1}{p+1}}{p^s} + \frac{\frac{p(p-1)}{p^2+p+1}}{p^{2s}} + \frac{\frac{p^2(p-1)}{p^3+p^2+p+1}}{p^{3s}} + \cdots\right)\\
&= \zeta(s)\prod_p\left(1-\frac{1}{p^s}\right)\prod_p \left(1 + \frac{\frac{p-1}{p+1}}{p^s} + \frac{\frac{p(p-1)}{p^2+p+1}}{p^{2s}} + \frac{\frac{p^2(p-1)}{p^3+p^2+p+1}}{p^{3s}} + \cdots\right)\\
&= \zeta(s)\prod_p \left(1 + \frac{\frac{p-1}{p+1}-1}{p^s} + \frac{\frac{p(p-1)}{p^2+p+1}-\frac{p-1}{p+1}}{p^{2s}} + \frac{\frac{p^2(p-1)}{p^3+p^2+p+1}-\frac{p(p-1)}{p^2+p+1}}{p^{3s}} + \cdots\right)\\
&= \zeta(s)R(s),
\end{aligned}$$

say. Expanding the product $R(s)$ into a Dirichlet series, say

$$R(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

we find that it converges absolutely in the half-plane $\Re(s)\ge 3/4$. Setting $b_n=\phi(n)/\sigma(n)$, we have $b_n=\sum_{d\mid n} a_d$, and therefore

$$\begin{aligned}
\sum_{n\le x} b_n = \sum_{n\le x}\sum_{d\mid n} a_d = \sum_{n\le x} a_d\left[\frac{x}{d}\right] &= x\sum_{d\le x}\frac{a_d}{d} + O\Big(\sum_{d\le x}|a_d|\Big)\\
&= R(1)x + O\left(x\sum_{d>x}\frac{|a_d|}{d}\right) + O\Big(\sum_{d\le x}|a_d|\Big).
\end{aligned}$$

Since

$$\sum_{d\le x}|a_d| = \sum_{d\le x}\frac{|a_d|}{d^{3/4}}\, d^{3/4} = O(x^{3/4})$$

and

$$\sum_{d>x} \frac{|a_d|}{d} = \sum_{d>x} \frac{|a_d|}{d^{3/4}} \frac{1}{d^{1/4}} \le x^{-1/4} \sum_{d>x} \frac{|a_d|}{d^{3/4}} = O(x^{-1/4}),$$

it follows that

$$\sum_{n\le x} \frac{\phi(n)}{\sigma(n)} = R(1)x + O(x^{3/4}),$$

which completes the proof of Lemma 6, since $R(1) = c_0$.

LEMMA 7. *There exists a constant c_4 such that the set of positive integers $n \le x$ with $\omega(\phi(n)) > c_4 \log_2^2 x$ contains at most $O(x/\log_2^2 x)$ elements. The same holds when the ϕ function is replaced by the σ function.*

Proof. First let $\mathcal{D}_1$ be the set of all $n \le x$ such that $k = \omega(n) > 3e \log_2 x$. A well-known result of Hardy and Ramanujan (see [5]) asserts that

$$\#\{n \le x : \omega(n) = k\} \ll \frac{x}{\log x} \cdot \frac{1}{(k-1)!} \cdot (\log_2 x + O(1))^{k-1},$$

an inequality which together with Stirling's formula implies that

$$\#\{n \le x : \omega(n) = k\} \ll \frac{x}{\log x} \cdot \left(\frac{e \log_2 x + O(1)}{k-1}\right)^{k-1} < \frac{x}{\log x} \cdot \frac{1}{2^{k-1}}$$

since $k - 1 > 3e \log\log x - 1$ and x is assumed to be large. Thus,

$$\#\mathcal{D}_1 = \#\{n \le x : \omega(n) > 3e \log_2 x\} \ll \frac{x}{\log x} \sum_k \frac{1}{2^k} \ll \frac{x}{\log x} \ll \frac{x}{\log_2^2 x}.$$

Assume now that $\mathcal{D}_2$ is the set of all $n \le x$ which are divisible by the square of a prime $p > \log_2^2 x$. Then

$$\#\mathcal{D}_2 \le \sum_{p > \log_2^2 x} \frac{x}{p^2} \ll \frac{x}{\log_2^2 x}.$$

Let $\mathcal{D}_3$ be the set of those $n \le x$ which are divisible by a prime number p such that $\omega(p-1) \ge b := \lfloor e^2 \log_2 x \rfloor$. Then

$$\begin{aligned}\#\mathcal{D}_3 &\le \sum_{\substack{p\le x\\ \omega(p-1)\ge b}} \frac{x}{p} \le x \sum_{k\ge b} \frac{1}{k!} \Bigg(\sum_{q^a \le x} \frac{1}{q^a}\Bigg)^k \ll x \sum_{k\ge b} \left(\frac{e\log_2 x + O(1)}{k}\right)^k \\ &\ll x \sum_{k\ge b} \frac{1}{2^k} \ll \frac{x}{2^b} \ll \frac{x}{\log x} \ll \frac{x}{\log_2^2 x},\end{aligned}$$

where we used the facts that $e > 2$ and $2^{e^2} > e$. Let $\mathcal{D}_4$ be the set of those $n \le x$ which are divisible by a prime number p such that $\omega(p+1) \ge b$. The same argument as above shows that

$$\#\mathcal{D}_4 \ll \frac{x}{\log_2^2 x}.$$

Let $\mathcal{D}_5$ be the set of those $n \le x$ which do not belong to $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4$ and such that there exists a prime power $p^a \mid n$, where $a = \lfloor c_3 \log_3 x \rfloor$, where $c_3 = 2/\log 2$. By an argument similar to the one used in the proof of Lemma 5, we get

$$\#\mathcal{D}_5 \le x \sum_{p \ge 2} \frac{1}{p^a} \ll \frac{x}{2^a} + x \int_2^\infty \frac{dt}{t^a} \ll \frac{x}{2^a} \ll \frac{x}{\log_2^2 x}.$$

Put $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5$. Assume that $n \notin \mathcal{D}$. Writing m for the largest square-full divisor of n, we note that

$$\phi(m) \le m \le \sigma(m) \le m^2 \le \Big(\prod_{p \le \log_2^2 x} p \Big)^{c_3 \log_3 x} < \exp(2c_3 \log_2^2 x \log_3 x) =: T(x)$$

for large x by the prime number theorem. Hence,

$$\max\{\omega(\phi(m)), \omega(\sigma(m))\} \ll \frac{\log T(x)}{\log_2 T(x)} \ll \log_2^2 x.$$

Since clearly

$$\max\{\omega(\phi(n/m)), \omega(\sigma(n/m))\} \le 3eb \log_2 x < 3e^3 \log_2^2 x,$$

it follows that

$$\max\{\omega(\phi(n)), \omega(\sigma(n))\} \ll \log_2^2 x,$$

where we also used the obvious fact that m and n/m are coprime. Let c_4 be the constant implied in the last Vinogradov symbol above. Noticing that $\mathcal{D}$ contains $O(x/(\log_2 x)^2)$ elements, the conclusion of Lemma 7 follows.

4. The maximal order of $H(n)$. We will show that for n sufficiently large,

$$H(n) \le (1 + o(1))\, e^{2\gamma} \log_2^2 n. \tag{6}$$

Then clearly the proof of Theorem 1 will follow if we can also show the following result.

Claim. *There exists an infinite sequence of integers n for which $H(n)$ is bounded below by $(1 + o(1))\, e^{2\gamma} \log_2^2 n$.*

To prove (6), first observe that it follows from Lemma 2 that

$$\sigma(\phi(n)) \le (1 + o(1)) e^{\gamma} \phi(n) \log_2 \phi(n) \le (1 + o(1)) e^{\gamma}\, n \log_2 n. \tag{7}$$

On the other hand, it follows from Lemma 1 that

$$\phi(n) \ge (1 + o(1))\, \frac{e^{-\gamma} n}{\log_2 n},$$

so that

$$\phi(\sigma(n)) \ge (1 + o(1))\, \frac{e^{-\gamma} \sigma(n)}{\log_2 \sigma(n)} \ge (1 + o(1)) e^{-\gamma}\, \frac{n}{\log_2 n}. \tag{8}$$

Combining (7) and (8), we obtain (6).

Hence, in order to complete the proof of Theorem 1, it remains to prove our Claim. So let x be a large integer, and let P and Q be the smallest primes such that

$$P \equiv 1 \pmod{M(x)} \quad \text{and} \quad Q \equiv -1 \pmod{M(x)},$$

where $M(x) = \mathrm{LCM}[1, 2, \ldots, x]$, and set

$$n = PQ.$$

From the prime number theorem, it is clear that

$$M(x) = e^{(1+o(1))x} < e^{2x},$$

say. Hence, from Theorem A, it follows that

$$P \ll e^{11x}, \quad Q \ll e^{11x}, \quad \text{so that} \quad n = PQ \ll e^{22x}.$$

Thus, $n < e^{23x}$ for large x. For this particular integer n, we have, since $\phi(n) = (P-1)(Q-1)$,

$$\begin{aligned}\frac{\sigma(\phi(n))}{\phi(n)} &= \prod_{p^{\alpha_p} \| (P-1)(Q-1)} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\alpha_p}}\right) \\ &\geq \prod_{p^{\alpha_p} \| (P-1)} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\alpha_p}}\right) \\ &\geq \prod_{p^{\beta_p} \leq x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\beta_p}}\right),\end{aligned}$$

where each exponent β_p is the unique positive integer satisfying $p^{\beta_p} \leq x < p^{\beta_p + 1}$. Therefore,

$$\frac{\sigma(\phi(n))}{\phi(n)} \geq \prod_{p \leq x} \left(1 + \frac{1}{p-1}\right) \prod_{p^{\beta_p} \leq x} \left(1 + O\left(\frac{1}{p^{\beta_p + 1}}\right)\right). \tag{9}$$

However,

$$\begin{aligned}\prod_{p^{\beta_p} \leq x} \left(1 + O\left(\frac{1}{p^{\beta_p+1}}\right)\right) &= \exp\left\{O\left(\sum_{p^{\beta_p} \leq x} \frac{1}{p^{\beta_p+1}}\right)\right\} = \exp\left\{O\left(\frac{\pi(x)}{x}\right)\right\} \\ &= 1 + O\left(\frac{1}{\log x}\right).\end{aligned}$$

Using this in (9), we deduce that, by Lemma 1,

$$\frac{\sigma(\phi(n))}{\phi(n)} \geq (1 + o(1)) \prod_{p \leq x} \frac{p}{p-1} = (1 + o(1)) e^{\gamma} \log x. \tag{10}$$

On the other hand, $\sigma(n) = (P+1)(Q+1)$, so that

$$\frac{\phi(\sigma(n))}{\sigma(n)} = \prod_{p|(P+1)(Q+1)} \left(1 - \frac{1}{p}\right) \le \prod_{p|(P+1)} \left(1 - \frac{1}{p}\right) \le \prod_{p|M(x)} \left(1 - \frac{1}{p}\right) = \prod_{p\le x}\left(1 - \frac{1}{p}\right) = (1+o(1))\,\frac{e^{-\gamma}}{\log x}, \tag{11}$$

where we used Lemma 1.

Gathering (10) and (11), we get

$$H(n)\cdot\frac{\sigma(n)}{\phi(n)} \ge (1+o(1))e^{2\gamma}\log^2 x. \tag{12}$$

Since by our choice of n, we have $\exp\{(1+o(1))x\} < n < \exp\{23x\}$, it follows that $(1+o(1))x < \log n < 23x$ and therefore $\log_2 n = \log x + O(1)$, which means that (12) can be replaced by

$$H(n)\cdot\frac{\sigma(n)}{\phi(n)} \ge (1+o(1))e^{2\gamma}\log_2^2 n. \tag{13}$$

Observing now that, for large x (that is, large P and Q),

$$\frac{\sigma(n)}{\phi(n)} = \frac{(P+1)(Q+1)}{(P-1)(Q-1)} = 1 + o(1),$$

we conclude that our Claim follows immediately from (13), since then by varying x one obtains infinitely many such integers n. The proof of Theorem 1 is thus complete.

5. The minimal order of $H(n)$. It follows from Theorem B and Lemma 3 that, for n sufficiently large,

$$\frac{\sigma(\phi(n))}{n} > \kappa \quad \text{and} \quad \frac{n}{\sigma(n)} \ge \frac{(1+o(1))e^{-\gamma}}{\log_2 n}.$$

Combining these with the trivial inequality $\sigma(n)/\phi(\sigma(n)) \ge 1$, we immediately get

$$H(n)\log_2 n = \frac{\sigma(\phi(n))}{n}\cdot\frac{\sigma(n)}{\phi(\sigma(n))}\cdot\frac{n}{\sigma(n)}\cdot\log_2 n \ge e^{-\gamma}\kappa. \tag{14}$$

To complete the proof of Theorem 2, we shall use an argument developed by Mąkowski and Schinzel in [9].

Let x be large and let $N(x) = \prod_{p<x} p$. Moreover let q be the smallest prime number exceeding $x\log x$, and choose $n = N(x)^{q-1}$, so that

$$\phi(n) = N(x)^{q-2}\phi(N(x)) = \prod_{p<x} p^{\alpha_p},$$

where $\alpha_p = q - 2 + \gamma_p$ and $\gamma_p \geq 0$ is such that $p^{\gamma_p} \parallel \phi(N(x))$. We then have

$$\sigma(\phi(n)) = \prod_{p<x} \sigma(p^{\alpha_p}) = \prod_{p<x} \frac{p^{\alpha_p+1} - 1}{p - 1}$$

and

$$\frac{\sigma(\phi(n))}{\phi(n)} = \prod_{p<x} \frac{p^{\alpha_p+1} - 1}{p^{\alpha_p}(p-1)} = (1 + o(1))e^{\gamma} \log x, \tag{15}$$

by Lemma 1.

On the other hand, again by Lemma 1,

$$\frac{\phi(n)}{n} = \prod_{p<x} \left(1 - \frac{1}{p}\right) = (1 + o(1)) \frac{e^{-\gamma}}{\log x}. \tag{16}$$

Combining (15) and (16), we obtain

$$\frac{\sigma(\phi(n))}{n} = \frac{\sigma(\phi(n))}{\phi(n)} \cdot \frac{\phi(n)}{n} = 1 + o(1). \tag{17}$$

We now examine the expression

$$\sigma(n) = \prod_{p<x} \frac{p^q - 1}{p - 1}. \tag{18}$$

Fix a prime $p < x$ and set

$$\frac{p^q - 1}{p - 1} = r_1^{\beta_1} \cdots r_t^{\beta_t},$$

where, for each $i = 1, \ldots, t$, $r_i = r_i(p)$ is a prime and $\beta_i = \beta_i(p)$ a positive integer. We then have $p^q \equiv 1 \pmod{r_i}$ for each positive integer $i \leq t$, and by Fermat's Little Theorem it follows easily that $r_i \equiv 1 \pmod{q}$ (for if not, then from $p^q \equiv 1 \pmod{r_i}$ it would follow that $p \equiv 1 \pmod{r_i}$, which would lead to the conclusion that $(p^q - 1)/(p - 1)$ and $p - 1$ have a common factor $r_i > 1$, which is impossible because $(p^q - 1)/(p - 1)$ is congruent modulo $p - 1$ to the prime $q > p - 1$). Hence

$$x^q > p^q > \frac{p^q - 1}{p - 1} > q^t,$$

which, since $q > x \log x$, implies that

$$t < \frac{q \log x}{\log q} < q. \tag{19}$$

From this it follows that

$$\frac{\phi\left(\frac{p^q-1}{p-1}\right)}{\frac{p^q-1}{p-1}} = \prod_{i=1}^{t} \left(1 - \frac{1}{r_i}\right) \geq \exp\left\{-2 \sum_{i=1}^{t} \frac{1}{r_i}\right\}, \tag{20}$$

where we used the fact that $1 - z > e^{-2z}$ for all z in $(0, 1/4)$. Since it follows from (19) that there are at most q such primes r_i in the arithmetic progression 1 mod q, we have

$$\sum_{i=1}^{t} \frac{1}{r_i} \le \frac{1}{q\cdot 1} + \frac{1}{q\cdot 2} + \cdots + \frac{1}{q\cdot q} < \frac{2\log q}{q},$$

which, combined with (20), yields

$$\frac{\phi\left(\frac{p^q-1}{p-1}\right)}{\frac{p^q-1}{p-1}} \ge \exp\left\{-\frac{4\log q}{q}\right\}.$$

It follows that

$$1 \ge \frac{\phi(\sigma(n))}{\sigma(n)} \ge \prod_{\substack{r \mid \frac{p^q-1}{p-1} \\ \text{for some } p<x}} \left(1 - \frac{1}{r}\right) \ge \exp\left\{-4\,\frac{\pi(x)\log q}{q}\right\} = 1 + o(1),$$

since we have chosen $q > x\log x$ and since $\pi(x) \ll x/\log x$. We have thus established that

$$\frac{\phi(\sigma(n))}{\sigma(n)} = 1 + o(1).$$

It now follows from Lemma 1 that

$$\begin{aligned} \frac{\phi(\sigma(n))}{n} &= \frac{\phi(\sigma(n))}{\sigma(n)} \cdot \frac{\sigma(n)}{n} = (1+o(1)) \prod_{p<x}\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^{q-1}}\right) \\ &= (1+o(1)) \prod_{p<x}\left\{\left(1 + \frac{1}{p-1}\right)\left(1 + O\left(\frac{1}{p^q}\right)\right)\right\} \\ &= (1+o(1)) e^{\gamma} (\log x) \exp\left(O\left(\sum_{p<x} \frac{1}{p^q}\right)\right) \\ &= (1+o(1)) e^{\gamma} (\log x) \exp\left(O\left(\frac{\pi(x)}{2^{x\log x}}\right)\right) = (1+o(1)) e^{\gamma} \log x. \end{aligned} \tag{21}$$

Combining (17) and (21), we get

$$H(n) = \frac{\sigma(\phi(n))}{n} \cdot \frac{n}{\phi(\sigma(n))} = (1+o(1))\,\frac{e^{-\gamma}}{\log x}. \tag{22}$$

It remains to estimate the size of n. Recall that, by our choice of n and q, we have

$$n = \left(\prod_{p<x} p\right)^{q-1} = \exp\{(1+o(1))xq\} = \exp\{(1+o(1))x^2\log x\},$$

so that $(1+o(1))x^2\log x = \log n$, from which we easily obtain

$$x = (1+o(1))\sqrt{\frac{2\log n}{\log_2 n}},$$

which yields

$$\log x = \frac{1}{2}\,(1+o(1))\log_2 n.$$

Substituting this in (22), we obtain

$$H(n) = (1+o(1))\,\frac{e^{-\gamma}}{\frac{1}{2}\log_2 n},$$

from which we may conclude that there exist infinitely many integers n such that

$$H(n)\log_2 n = (1+o(1))2e^{-\gamma}.$$

The proof of Theorem 2 is completed by combining this last result with (14) and taking into account the remark following the statement of Theorem B concerning the improved lower bound for κ.

6. The mean value of $H(n)$. We use the method developed in [8]. Let $M_0(x)$ be the least common multiple of all prime powers $p^a < g(x)$, where $g(x) = c_1\log_2 x/\log_3 x$ and c_1 is the constant of Lemma 4. Moreover, let $\mathcal{A} = \mathcal{A}(x) = \{n : \sqrt{x} < n \le x \text{ and } M_0(n) \mid \gcd(\phi(n), \sigma(n))\}$. Then

$$\frac{\sigma(\phi(n))}{\phi(n)} \ge e^{\gamma}\log_3 x\left(1+O\left(\frac{1}{\log_3 x}\right)\right) \qquad (n\in\mathcal{A}). \tag{23}$$

(This follows from inequality (37) in [8].) Using the same method and then applying Lemma 1, we get

$$\begin{aligned}\frac{\phi(\sigma(n))}{\sigma(n)} \le \frac{\phi(M_0(n))}{M_0(n)} &= \prod_{p<g(x)}\left(1-\frac{1}{p}\right)\\ &\le \frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{1}{\log_3 x}\right)\right) \qquad (n\in\mathcal{A}).\end{aligned} \tag{24}$$

Combining (23) and (24) yields

$$\begin{aligned}H(n) &\ge \frac{\phi(n)}{\sigma(n)}\,\frac{e^{\gamma}\log_3 x}{e^{-\gamma}/\log_3 x}\left(1+O\left(\frac{1}{\log_3 x}\right)\right)\\ &= \frac{\phi(n)}{\sigma(n)}\,e^{2\gamma}\log_3^2 x\left(1+O\left(\frac{1}{\log_3 x}\right)\right)\end{aligned} \tag{25}$$

for $n\in\mathcal{A}$. It follows that

$$\sum_{n\le x} H(n) \ge \sum_{\substack{n\le x\\ n\in\mathcal{A}}} H(n) \ge e^{2\gamma}\log_3^2 x\left(1+O\left(\frac{1}{\log_3 x}\right)\right)\sum_{\substack{n\le x\\ n\in\mathcal{A}}}\frac{\phi(n)}{\sigma(n)}. \tag{26}$$

Now using Lemma 4 to estimate the size of $[1,x]\setminus\mathcal{A}$ and using the fact that $\phi(n)\le\sigma(n)$ for all n, we get, by Lemma 6,

$$\sum_{\substack{n\le x\\ n\in\mathcal{A}}}\frac{\phi(n)}{\sigma(n)}\ge\sum_{n\le x}\frac{\phi(n)}{\sigma(n)}-(\lfloor x\rfloor-\#\mathcal{A})=\sum_{n\le x}\frac{\phi(n)}{\sigma(n)}+O\left(\frac{x}{\log_3^2 x}\right)$$
$$=c_0\,x+O\left(\frac{x}{\log_3^2 x}\right).$$

Combining this with (26) yields

$$\sum_{n\le x}H(n)\ge c_0\,e^{2\gamma}x\log_3^2 x+O(x\log_3 x). \tag{27}$$

It remains to obtain the corresponding upper bound for $\sum_{n\le x}H(n)$. To do so, we first observe that we only need to consider those integers $n\in[\sqrt{x},x]$, since it follows from Theorem 1 that

$$\sum_{n\le\sqrt{x}}H(n)=O(\sqrt{x}\log_2^2 x). \tag{28}$$

Consider now the set

$$\mathcal{B}=\mathcal{B}(x)=\left\{n:\sqrt{x}<n\le x,\ h_\phi(n)<\frac{1}{\sqrt{\log_3 x}},\ h_\sigma(n)<\frac{1}{\sqrt{\log_3 x}}\right\},$$

and given a positive integer $n\in\mathcal{B}$, write $\phi(n)=n_1\cdot n_2$, where

$$n_1=\prod_{\substack{p^{\alpha_p}\|\phi(n)\\ p\le\log_2 x}}p^{\alpha_p}\quad\text{and}\quad n_2=\prod_{\substack{p^{\alpha_p}\|\phi(n)\\ p>\log_2 x}}p^{\alpha_p},$$

so that, by Lemma 1,

$$\begin{aligned}\frac{\sigma(\phi(n))}{\phi(n)}&=\prod_{p|n_1}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{\alpha_p}}\right)\cdot\prod_{p|n_2}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{\alpha_p}}\right)\\&\le(e^\gamma\log_3 x+O(1))\cdot\exp(O(h_\phi(n)))\\&=(e^\gamma\log_3 x+O(1))\cdot\exp\left\{O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right\}[4pt]\\&=e^\gamma\log_3 x+O(\sqrt{\log_3 x})\qquad(n\in\mathcal{B}).\end{aligned}\tag{29}$$

On the other hand, given $n\in\mathcal{B}$ and writing $\sigma(n)=m_1\cdot m_2$, where

$$m_1=\prod_{\substack{p^{\alpha_p}\|\sigma(n)\\ p\le\log_2 x}}p^{\alpha_p}\quad\text{and}\quad m_2=\prod_{\substack{p^{\alpha_p}\|\sigma(n)\\ p>\log_2 x}}p^{\alpha_p},$$

we get, by a similar argument,

$$\begin{aligned}(30)\qquad \frac{\phi(\sigma(n))}{\sigma(n)} &= \frac{\phi(m_1)}{m_1}\cdot\frac{\phi(m_2)}{m_2}\\ &\geq \frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{1}{\log_3 x}\right)\right)\cdot\left(1+O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right)\\ &= \frac{e^{-\gamma}}{\log_3 x}\left(1+O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right)\qquad (n\in\mathcal{B}).\end{aligned}$$

Gathering (29) and (30), we obtain

$$(31)\qquad H(n)\leq \frac{\phi(n)}{\sigma(n)}e^{2\gamma}\log_3^2 x\left(1+O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right)\qquad (n\in\mathcal{B}),$$

from which it follows that

$$\begin{aligned}(32)\qquad \sum_{\substack{n\leq x\\ n\in B}} H(n) &\leq e^{2\gamma}\log_3^2 x\left(1+O\left(\frac{1}{\sqrt{\log_3 x}}\right)\right)\sum_{\substack{n\leq x\\ n\in B}}\frac{\phi(n)}{\sigma(n)}\\ &\leq e^{2\gamma}c_0 x\log_3^2 x + O(x\log_3^{3/2} x).\end{aligned}$$

It remains to consider the contribution of those integers $n\in[\sqrt{x},x]$ which do not belong to the set $\mathcal{B}$. The set of these numbers is contained in $\mathcal{C}_\phi\cup\mathcal{C}_\sigma$, where, given $f\in\{\phi,\sigma\}$, we write $\mathcal{C}_f$ for the set of those numbers $n\in[\sqrt{x},x]$ such that $h_f(n)\geq 1/\sqrt{\log_3 x}$. Lemma 5 shows that

$$\frac{x}{\log_3 x}\gg\sum_{n\in C_f} h_f(n)\geq\frac{\#\mathcal{C}_f}{\sqrt{\log_3 x}},$$

so that

$$(33)\qquad \#\mathcal{C}_f\ll x/\sqrt{\log_3 x}\qquad \text{for } f=\phi \text{ and } f=\sigma.$$

We now call upon Lemma 7. Let $\mathcal{D}$ be the exceptional set mentioned in that lemma. Since by Theorem 1, $H(n)\ll\log_2^2 n$, it follows that

$$(34)\qquad \sum_{n\in\mathcal{D}} H(n)=O(x).$$

We now let $\mathcal{E}$ be the set of those $n\leq x$ which are not in $\mathcal{D}$. Thus, by Lemma 7, if $n\in\mathcal{E}$, then $\omega(\phi(n))$ and $\omega(\sigma(n))$ are both $O(\log_2^2 x)$. In particular, for large x, we have

$$\max\{h_\phi(n),h_\sigma(n)\}\leq\sum_{\log_2 x<p<\log_2^3 x}\frac{1}{p}\ll 1.$$

Hence, writing $\phi(n)=n_1\cdot n_2$ and $\sigma(m)=m_1\cdot m_2$ as previously, we find that for $n \in \mathcal{E}$,

$$\begin{aligned}\frac{\sigma(\phi(n))}{\phi(n)} &= \prod_{p^{\alpha_p}\|n_1}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{\alpha_p}}\right)\prod_{p^{\alpha_p}\|n_2}\left(1+\frac{1}{p}+\cdots+\frac{1}{p^{\alpha_p}}\right)\\ &\le \prod_{p\le \log_2 x}\left(1+\frac{1}{p-1}\right)\exp(O(h_\phi(n)) \ll \log_3 x\end{aligned}$$

and

$$\begin{aligned}\frac{\phi(\sigma(n))}{\sigma(n)} &= \prod_{p\|m_1}\left(1-\frac{1}{p}\right)\prod_{p\|m_2}\left(1-\frac{1}{p}\right)\\ &\ge \prod_{p\le \log_2 x}\left(1-\frac{1}{p}\right)\exp(-h_\sigma(n)) \gg \log_3 x,\end{aligned}$$

from which we may conclude that $H(n) \ll \log_3^2 x$ for all $n \in \mathcal{E}$. Finally, recall that by (33), the set of those $n \in \mathcal{C}_\phi \cup \mathcal{C}_\sigma$ is of cardinality at most $O(x/\sqrt{\log_3 x})$, and therefore that

$$\sum_{n\in(\mathcal{C}_\phi\cup\mathcal{C}_\sigma)\cap\mathcal{E}} H(n) \le \max_{n\in\mathcal{E}}\{H(n)\}\cdot \#(\mathcal{C}_\phi\cup\mathcal{C}_\sigma) \ll x\log_3^{3/2} x,$$

which together with (28), (32) and (34) shows that

$$\sum_{n\le x} H(n) \le e^{2\gamma}c_0 x\log_3^2 x + O(x\log_3^{3/2} x). \tag{35}$$

Combining (27) and (35) completes the proof of Theorem 3.

7. The first density theorem for $H(n)$. Here, we follow essentially an argument used in [8]. In view of (25) and (31), both inequalities

$$H(n) \ge (1+o(1))\,\frac{\phi(n)}{\sigma(n)}\,e^{2\gamma}\log_3^2 n,$$

$$H(n) \le (1+o(1))\,\frac{\phi(n)}{\sigma(n)}\,e^{2\gamma}\log_3^2 n$$

hold on a set of density 1. Therefore, on a set of density 1,

$$H(n) = (1+o(1))e^{2\gamma}\log_3^2 n\,\frac{\phi(n)}{\sigma(n)}.$$

Since $\phi(n)/\sigma(n)$ has a continuous distribution function (see Exercises 2 and 3 of Chapter III.2 in Tenenbaum [13]), the proof of Theorem 4 is complete.

8. The second density theorem for $H(n)$. Fix $\delta \in (0,\infty)$ and let x be a very large positive real number. We shall now construct a finite set $\mathcal{R}$

of primes larger than x^{x^2} with

$$\prod_{r\in\mathcal{R}}\left(1+\frac{1}{r}\right)\in\left(\frac{e^{\gamma}\delta\log x}{2}-1,\frac{e^{\gamma}\delta\log x}{2}+1\right).$$

To construct $\mathcal{R}$, let $r_1<r_2<\cdots$ be all the primes $>x^{x^2}$ and let k be the largest positive integer such that

$$\prod_{i=1}^{k}\left(1+\frac{1}{r_i}\right)\le\frac{e^{\gamma}\delta\log x}{2}.$$

Observe that by the maximality of k and the fact that

$$r_{k+1}\ge r_1>x^{x^2}>\frac{e^{\gamma}\delta\log x}{2}$$

for all x sufficiently large, we get

$$\prod_{i=1}^{k+1}\left(1+\frac{1}{r_i}\right)\in\left(\frac{e^{\gamma}\delta\log x}{2},\frac{e^{\gamma}\delta\log x}{2}+1\right).$$

Hence, we can take $\mathcal{R}=\{r_i:i=1,\ldots,k+1\}$. Note that since

$$\prod_{i=1}^{k+1}\left(1+\frac{1}{r_i}\right)=\exp(\log_2 r_{k+1}-\log_2 r_1+o(1))>\exp(\log_2 r_{k+1}-3\log x),$$

it follows that $r_{k+1}<e^{x^4}$ for large x, for if not, then $r_k\ge e^{x^4}/2$, in which case

$$\prod_{i=1}^{k}\left(1+\frac{1}{r_i}\right)>\exp(\log_2 r_k-\log_2 r_1+o(1))>\exp(\log x)=x>\frac{e^{\gamma}\delta\log x}{2}+1,$$

which contradicts the definition of k.

We now let y be a parameter that depends on x with $z:=\log_2 y>r_{k+1}$. This inequality is fulfilled if we choose $\log_2 y>e^{x^4}$, which in turn holds if $\log_3 y>x^4$. Then let $\mathcal{P}$ be the set of all primes $p\le y$ such that $p\equiv 13 \pmod{72}$, $p\equiv 1+r_i \pmod{r_i^2}$ for all $i=1,\ldots,k+1$, and both $p-1$ and $p+1$ are coprime to all primes $r\le z$ which are ≥ 5 and do not belong to $\mathcal{R}$. Observe that the above conditions certainly put p in an arithmetic progression $a \pmod{b}$, where

$$b=72\prod_{i=1}^{k+1}r_i^2,$$

and $a\equiv 13 \pmod{72}$ and $a\equiv 1+r_i \pmod{r_i^2}$ for $i=1,\ldots,k+1$.

Now let

$$T:=\prod_{\substack{5\le r\le z\\ r\notin\mathcal{R}}}r,$$

and, for each $d \mid T$, let

$$\mathcal{A}(d) := \{a_d \pmod{bd} : d \mid a_d^2 - 1 \text{ and } a_d \equiv a \pmod{b}\},$$

so that $\#\mathcal{A}(d) = 2^{\omega(d)}$.

By the principle of inclusion and exclusion, the cardinality of the set $\mathcal{P}$ of primes is none other than

$$\sum_{d \mid T} \mu(d) \sum_{a_d \in \mathcal{A}(d)} \pi(y; a_d, bd),$$

where, as usual, $\pi(y; s, t)$ stands for the number of primes $p \le y$ satisfying $p \equiv s \pmod{t}$. Observing that $bT \ll \prod_{r \le z} r^2 \le e^{2(1+o(1))z} < e^{3z} < y^{1/3}$, we get, by the Bombieri–Vinogradov theorem,

$$\mathcal{P} = \frac{\pi(y)}{\phi(b)} \prod_{\substack{5 \le r \le z \\ r \notin \mathcal{R}}} \left(1 - \frac{2}{r-1}\right) + O\left(\frac{2^{\pi(z)} y}{\log^{10} y}\right).$$

Since

$$2^{\pi(z)} = \exp(O(\log_2 y / \log_3 y)) = (\log y)^{o(1)},$$

while

$$\phi(b) \ll \prod_{r \le z} r^2 < \exp(3 \log_2 y) = \log^3 y,$$

and since

$$\prod_{r \le z} \left(1 - \frac{2}{r-1}\right) \gg \exp\left\{-\sum_{r \le z} \frac{2}{r-1}\right\} \gg \exp\{-2 \log\log z\} = \frac{1}{\log^2 z} \ge \frac{1}{\log y},$$

it follows that

$$\mathcal{P} = \frac{\pi(y)}{\phi(b)} \prod_{\substack{5 \le r \le z \\ r \notin \mathcal{R}}} \left(1 - \frac{2}{r-1}\right) + O\left(\frac{y}{\log^9 y}\right) \gg \frac{\pi(y)}{\log^4 y} \gg \frac{y}{\log^5 y}.$$

Finally, let $\mathcal{P}'$ be the subset of those primes $p \in \mathcal{P}$ such that neither $\omega(p-1)$ nor $\omega(p+1)$ is larger than $e^2 \log_2 y$. From the estimates due to Hardy and Ramanujan (see [5] and the proof of Lemma 7), we know that

$$\begin{aligned}
\#\{n \le y : \omega(n) > e^2 \log_2 y\} &\ll \frac{y}{\log y} \sum_{k > e^2 \log_2 y} \frac{1}{(k-1)!} (\log_2 y + O(1))^k \\
&\ll \frac{y}{\log y} \sum_{k > e^2 \log_2 y} \left(\frac{e \log_2 y + O(1)}{k}\right)^k \\
&\ll \frac{y}{\log y} \cdot \frac{1}{2^{e^2 \log_2 y}} = o\left(\frac{y}{\log^5 y}\right),
\end{aligned}$$

because $e^2 \log 2 + 1 > 5$. Thus,

$$\#\mathcal{P}' \gg \frac{y}{\log^5 y}.$$

In particular, $\mathcal{P}'$ is non-empty for large y. Select P in $\mathcal{P}'$ and let $n = N(x)P$, where $N(x) = \prod_{p<x} p$. Then $\phi(n) = 12\phi(N(x)) \cdot (P-1)/12$, $\sigma(n) = 2\sigma(N(x)) \cdot (P+1)/2$, and $(P-1)/12$ is coprime to $4\phi(N(x))$, while $(P+1)/2$ is coprime to $2\sigma(N(x))$. The arguments from the proof of Theorem 2 now immediately show that

$$\begin{aligned}\frac{\sigma(\phi(n))}{\phi(n)} &= \frac{\sigma(12N(x))}{12N(x)} \cdot \frac{\sigma((P-1)/12)}{(P-1)/12} \\ &= e^{\gamma} \log x\left(1 + O\left(\frac{1}{\log x}\right)\right) \prod_{r\in\mathcal{R}}\left(1+\frac{1}{r}\right) \prod_{\substack{r^{\alpha_r} \| P-1 \\ r>z}} \left(1 + \frac{1}{r} + \cdots + \frac{1}{r^{\alpha_r}}\right) \\ &= e^{\gamma} \log_2 x\left(1 + O\left(\frac{1}{\log x}\right)\right) \cdot \frac{e^{\gamma}\delta \log x}{2} \cdot \exp\left(O\left(\sum_{\substack{r|(P-1) \\ r>z}} \frac{1}{r}\right)\right).\end{aligned}$$

Noting that $P-1$ has no more than $e^2 \log_2 y$ prime factors, it follows easily that

$$\begin{aligned}\sum_{\substack{r|(P-1) \\ r>z}} \frac{1}{r} &\le \sum_{\log_2 y < r < \log_2 y \log_3^2 y} \frac{1}{r} \ll \log\left(\frac{\log_3 y + 2\log_4 y}{\log_3 y}\right) \\ &\ll \frac{\log_4 y}{\log_3 y} \ll \frac{\log x}{x^4} \ll \frac{1}{\log x}.\end{aligned}$$

This means that

$$\frac{\sigma(\phi(n))}{\phi(n)} = \frac{e^{\gamma}\delta}{2} \log^2 x\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

By similar arguments, we get

$$\frac{\phi(n)}{n} = \prod_{r\le x}\left(1 - \frac{1}{r}\right) \cdot \frac{P-1}{P} = \frac{e^{-\gamma}}{\log x}\left(1 + O\left(\frac{1}{\log x}\right)\right).$$

It follows that

$$\frac{\sigma(\phi(n))}{n} = \frac{e^{\gamma}\delta}{2} \log x\left(1 + O\left(\frac{1}{\log x}\right)\right). \tag{36}$$

As we obtained (17) in the proof of Theorem 2, we also get, handling the case of $P+1$ as we did for $P-1$,

$$
\begin{aligned}
\frac{\phi(\sigma(n))}{\sigma(n)} &= \frac{1}{2}\frac{\phi(\sigma(N(x)))}{\sigma(N(x))}\cdot\frac{\phi((P+1)/2)}{(P+1)/2} \\
&= \frac{1}{2}\left(1+O\left(\frac{1}{\log x}\right)\right)\cdot\prod_{\substack{r\mid(P+1)\\ r>z}}\left(1-\frac{1}{r}\right) \\
&= \frac{1}{2}\left(1+O\left(\frac{1}{\log x}\right)\right).
\end{aligned}
\tag{37}
$$

Finally,

$$
\frac{\sigma(n)}{n} = \frac{\sigma(N(x))}{N(x)}\cdot\frac{P+1}{P} = e^{\gamma}\log x\left(1+O\left(\frac{1}{\log x}\right)\right). \tag{38}
$$

Gathering (37) and (38) yields

$$
\frac{\phi(\sigma(n))}{n} = \frac{e^{\gamma}\log x}{2}\left(1+O\left(\frac{1}{\log x}\right)\right). \tag{39}
$$

Combining (36) and (39), we obtain

$$
H(n) = \frac{e^{\gamma}\delta\log x}{2}\cdot\frac{2}{e^{\gamma}\log x}\left(1+O\left(\frac{1}{\log x}\right)\right) = \delta\left(1+O\left(\frac{1}{\log x}\right)\right).
$$

Since x is arbitrary, we see that δ is a cluster point of $\{H(n)\}_{n\geq 1}$, as claimed.

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