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WEAK MIXING OF A TRANSFORMATION SIMILAR TO PASCAL

BY

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Abstract. We construct a class of transformations similar to the Pascal transformation, except for the use of spacers, and show that these transformations are weakly mixing.

1. Introduction. The Pascal transformations arise as natural examples of adic transformations. Adic transformations were studied by Vershik as models for measure-preserving transformations [3]–[5]. Vershik conjectured that the Pascal transformations are weakly mixing, and while they are known to be totally ergodic this conjecture remains open [1], [2]. In this paper we define a class of transformations that are towers over the Pascal transformations, and show that they are weakly mixing (Theorem 1).

For each $0 < \alpha < 1$, define the Pascal transformation $S = S_\alpha$ ([1]) in the following way as a cutting and stacking transformation (see for example [2]):

We proceed by inductively defining a sequence of columns. Start by letting column $C_{0,0} = (B_{0,0}^{(0)})$ consist of one interval, called a *level*, of total mass 1 (we normalize the measure at the end). In the nth generation of columns we will have $n+1$ columns, where column m is denoted by $C_{n,m} = (B_{n,m}^{(0)}, \ldots, B_{n,m}^{(h_{n,m}-1)})$ for $0 \le m \le n$. The columns in generation $n+1$ are obtained from those in generation n in the following way. First cut level $B_{n,m}^{(i)}$ into sublevels $B_{n,m,0}^{(i)}, B_{n,m,1}^{(i)}$ with mass ratio $\alpha : 1-\alpha$. Then define

$$C_{n+1,m} = (B_{n,m,0}^{(0)}, \ldots, B_{n,m,0}^{(h_{n,m}-1)}, B_{n,m-1,1}^{(0)}, \ldots, B_{n,m-1,1}^{(h_{n,m-1}-1)}),$$

where levels $B_{n,m,i}^{(j)}$ with indices $m < 0$ or $m > n$ are ignored. We define the action of S on $B_{n,m}^{(i)}$ by sending it (using the standard translation of an interval to another of the same length) to $B_{n,m}^{(i+1)}$ when $i \neq h_{n,m}-1$. In the limit this defines a finite measure-preserving transformation S, known as the Pascal transformation.

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([1]) While S depends on α we do not write this explicitly.

Now we define a new transformation T_k, the *Pascal with spacers transformation*, in a similar way, only with additional *spacers* (an additional piece of our measure space of the correct total mass that is not part of any previous generation's column) placed on top of column $C_{n,m}$ if n is a multiple of k. Notice that all of the levels in column $C_{n,m}$ have mass $\alpha^{n-m}(1-\alpha)^m$ so T_k is measure-preserving. Notice also that the amount of mass added by spacers on generation n columns is 0 if k does not divide n and $\sum_{i=0}^{n} \alpha^i(1-\alpha)^{n-i} < (n+1)\max(\alpha, 1-\alpha)^n$ otherwise, hence T is defined on a space with finite mass. Therefore, we can instead consider the transformation T_k on the renormalized measure space so that the total mass of the space is 1. We shall prove the following theorem:

THEOREM 1. *The transformation T_k is weakly mixing if $k > 1$.*

We also note, though, that not all patterns of spacers are weakly mixing. In fact, we can show that T_1 is not weak mixing.

PROPOSITION 2. *T_1 is not weakly mixing.*

Proof. In T_1, all of the column heights are congruent to 2 modulo 3 and hence the function that assigns $e^{2\pi i(m+h)/3}$ to any point in the level $B_{n,m}^{(h)}$ is well defined and clearly has eigenvalue $e^{2\pi i/3}$. ∎

Notice that the above proof also shows that T_1 is not even totally ergodic.

The transformation T_k can be expressed symbolically as follows: the space, X, is the subset of $\{0,1\}^\omega \times \mathbb{Z}$ consisting of elements of the form $((0^a1^b0S), n)$ where S is some string of 1's and 0's, $b > 0$, and $0 \le n \le (a+b)/k+1$. The measure on X is defined on the cylinder sets, $[0^a1^b0S, n] = \{((0^a1^b0SS'), n) : S' \in \{0,1\}^\omega\}$, where $b > 0$, S is any finite string and $0 \le n \le (a+b)/k+1$, by $\mu[0^a1^b0S, n] = \alpha^x(1-\alpha)^y$, where x and y are the numbers of 0's and 1's respectively in the string 0^a1^b0S. The transformation T_k acts on X by

$$T_k(((0^a1^b0S), n)) = \begin{cases} ((0^a1^b0S), n+1) & \text{if } n \le (a+b)/k, \\ ((1^{b-1}0^{a+1}1S), 0) & \text{otherwise.} \end{cases}$$

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2. Ergodicity

LEMMA 3. *T_k is ergodic.*

Proof. Notice that the induced map of T_k on the complement of the spacers in X is S. Since for any point $x \in X$, $T_k^{-n}(x)$ is in this set for some n, the result follows from the well known ergodicity of S. ■

3. Some machinery involving column heights and copy heights. We will use the convention that $h_{n,m} = 0$ if $m < 0$ or $m > n$.

LEMMA 4. *For* $n \geq 0$,

$$h_{n+1,m} = \begin{cases} h_{n,m} + h_{n,m-1} & \text{if } n+1 \not\equiv 0 \pmod{k}, \\ h_{n,m} + h_{n,m-1} + 1 & \text{if } n+1 \equiv 0 \pmod{k}. \end{cases}$$

Proof. This follows immediately from the construction of the columns. ■

DEFINITION 1. If I is a level in some column $C_{r,s}$, and $0 \leq m \leq n$ with $n \geq r$, let $P_{n,m}(I)$ denote the set of copies of I in column $C_{n,m}$. In other words, $P_{n,m}(I)$ is the set of levels, J, in $C_{n,m}$ so that $J \subset I$.

DEFINITION 2. If I is a level in column $C_{n,m}$ where $I = B_{n,m}^{(H)}$, then let $h(I) = H$.

DEFINITION 3. If I is a level in $C_{r,s}$, $n, m \in \mathbb{Z}$ with $n \geq m \geq 0$, and $\lambda \in \mathbb{C}$, let

$$S_{n,m}(I,\lambda) = \sum_{I' \in P_{n,m}(I)} \mu(I')\lambda^{h(I')}.$$

If $m \notin [0,n]$, let $S_{n,m}(I,\lambda) = 0$.

The idea of the proof of Theorem 1 will be to assume for the sake of contradiction that T_k has an eigenfunction, f, with eigenvalue $\lambda \neq 1$. We will then look at some interval I on which f is nearly constant. We will consider the values of f on the generation-N copies of I. If two copies of I are both in column $C_{N,M}$, their f-values are proportional to λ raised to the power of their heights; therefore the integral of f over $P_{N,M}(I)$ is bounded above by $|S_{N,M}(I,\lambda)|$. We will produce a contradiction by proving that $\lim_{N\to\infty} \sum_M |S_{N,M}(I,\lambda)| = 0$. To do this we will use the following lemmas:

LEMMA 5. *If I is a level of a column of generation at most n, then*

$$S_{n+1,m}(I,\lambda) = \alpha S_{n,m}(I,\lambda) + (1-\alpha)\lambda^{h_{n,m}} S_{n,m-1}(I,\lambda).$$

Proof. Let the heights of the copies of I in $C_{n,m}$ be H_i for $1 \leq i \leq k_1$. Let the heights of the copies of I in $C_{n,m-1}$ be G_i for $1 \leq i \leq k_2$. Then the heights of the copies of I in $C_{n+1,m}$ are H_i and $G_j + h_{n,m}$. Therefore,

$$\begin{aligned} S_{n+1,m}(I,\lambda) &= \alpha^{n+1-m}(1-\alpha)^m\Big(\sum_{i=1}^{k_1}\lambda^{H_i} + \sum_{i=1}^{k_2}\lambda^{G_i+h_{n,m}}\Big) \\ &= \alpha\sum_{i=1}^{k_1}\alpha^{n-m}(1-\alpha)^m\lambda^{H_i} \\ &\quad + (1-\alpha)\lambda^{h_{n,m}}\sum_{i=1}^{k_2}\alpha^{n+1-m}(1-\alpha)^{m-1}\lambda^{G_i} \\ &= \alpha S_{n,m}(I,\lambda) + (1-\alpha)\lambda^{h_{n,m}}S_{n,m-1}(I,\lambda). \ ■ \end{aligned}$$

LEMMA 6. *If $k>1$, $\lambda = e^{i\theta}$ where $\theta\in(-\pi,\pi]$, $n\equiv -2 \pmod k$, and I is a level of generation at most n, then for the transformation T_k,*

$$\sum_{m=0}^{n+4}|S_{n+4,m}(I,\lambda)| \le \sum_{m=0}^{n}|S_{n,m}(I,\lambda)|(1-(2-2\cos(\theta/6))\alpha^2(1-\alpha)^2);$$

moreover, regardless of the value of n,

$$\sum_{m=0}^{n+1}|S_{n+1,m}(I,\lambda)| \le \sum_{m=0}^{n}|S_{n,m}(I,\lambda)|.$$

Proof. Lemma 5 implies that

$$S_{n+1,m}(I,\lambda) \le \alpha|S_{n,m}(I,\lambda)| + (1-\alpha)|S_{n,m-1}(I,\lambda)|.$$

Our latter result follows from summing this over m.

The idea of the proof will be to use Lemma 5 repeatedly to relate the value of $S_{n+4,m}(I,\lambda)$ to the values of $S_{n,m-i}(I,\lambda)$. In particular, let P be the set of paths of length 4 starting from $(n+4,m)$ and taking steps of the form (x,y) to $(x-1,y)$ or $(x-1,y-1)$. For such a path, p, let $e(p)$ be the second coordinate of the end of the path. Let $h(p)$ be the sum of $h_{n',m'}$ over pairs (n',m') so that p takes a step from $(n'+1,m')$ to $(n',m'-1)$. By repeated use of Lemma 5, we have

$$S_{n+4,m}(I,\lambda) = \sum_{p\in P}\lambda^{h(p)}\alpha^{e(p)+4-m}(1-\alpha)^{m-e(p)}S_{n,e(p)}(I,\lambda). \tag{1}$$

For $n\equiv -2 \pmod k$, we wish to show that

$$\begin{aligned} |S_{n+4,m}(I,\lambda)| \le{}& \alpha^4|S_{n,m}(I,\lambda)| + 4\alpha^3(1-\alpha)|S_{n,m-1}(I\lambda)| \\ &+ \alpha^2(1-\alpha)^2(6-(2-2\cos(\theta/6)))|S_{n,m-2}(I,\lambda)| \\ &+ 4\alpha(1-\alpha)^3|S_{n,m-3}(I,\lambda)| + (1-\alpha)^4|S_{n,m-4}(I,\lambda)|, \end{aligned}$$

and our result will follow from summing over m. We will do this by showing that there exist two paths $p_1, p_2\in P$ with $\lambda^{h(p_1)}$ far from $\lambda^{h(p_2)}$, and using (1). In particular, if $\lambda^{h(p_1)-h(p_2)}$ is at least as far from 1 as $e^{i\theta/3}$, our result will follow.

Consider two paths $p_1, p_2 \in P$ from $(n+4, m)$ to $(n, m-2)$ that each pass through (a, b) and $(a-2, b-1)$, identical except that p_1 passes through $(a-1, b-1)$ and p_2 passes through $(a-1, b)$. Then $h(p_1) - h(p_2) = h_{a-1,b} - h_{a-2,b}$. If we let $p_{i,1}, p_{i,2}$ for $i = 1, 2, 3$ be such pairs of paths with (a, b) equal to $(n+3, m-1)$, $(n+3, m)$ and $(n+4, m-1)$ respectively then

$$\begin{aligned} h(p_{1,1}) - h(p_{1,2}) &= h_{n+2,m-1} - h_{n+1,m-1} = h_{n+1,m-2}, \\ h(p_{2,1}) - h(p_{2,2}) &= h_{n+2,m-2} - h_{n+1,m-2} = h_{n+1,m-3}, \\ h(p_{3,1}) - h(p_{3,2}) &= h_{n+3,m-1} - h_{n+2,m-1} = h_{n+2,m-2} + 1. \end{aligned}$$

Since the last of these is one more than the sum of the first two, letting

$$E_i = \lambda^{h(p_{i,1})-h(p_{i,2})},$$

we see that $E_1^{-1} E_2^{-1} E_3 = \lambda$, and hence one of the E_i is at least as far from 1 as $e^{i\theta/3}$. ∎

4. Weak mixing

Proof of Theorem 1. Suppose for the sake of contradiction that T_k has an eigenvalue of $\lambda = e^{i\theta}$, where $\theta \in (-\pi, \pi]$ and $\theta \neq 0$. Let $f : X \to \mathbb{C}$ be the associated eigenfunction. Since T_k is ergodic by Lemma 3, and since $|f|$ is T_k-invariant, we may assume that $|f| = 1$ a.e. We may also assume that $\log(f)/i \in (-\pi/3, \pi/3)$ on a set, G, of positive measure. Let I be a level of stage n with n odd, which is at least $\frac{3}{4}$-full of G (i.e. $\mu(I \cap G) \geq \frac{3}{4}\mu(I)$). We have

$$\int_I \Re f(x)\, dx \geq \frac{1}{2} \cdot \frac{3}{4}\, \mu(I) - 1 \cdot \frac{1}{4}\, \mu(I) = \frac{1}{8}\, \mu(I).$$

Therefore,

$$\Big| \int_I f(x)\, dx \Big| \geq \frac{1}{8}\, \mu(I).$$

We have

$$\sum_m |S_{n,m}(I, \lambda)| = \mu(I)$$

since if I is a level of stage n, then $S_{n,m}(I, \lambda)$ is $\mu(I)\lambda^{h(I)}$ if I is a level in $C_{n,m}$, and 0 otherwise. Therefore, by Lemma 6,

$$\sum_m |S_{n+a\max(k,4),m}(I, \lambda)| \leq \mu(I)(1 - (2 - 2\cos(\theta/6))\alpha^2(1-\alpha)^2)^a.$$

Notice that for any integers N and M, and for a level J in column $C_{N,M}$, which has bottom level J',

$$\int_J f(x)\,dx = \lambda^{h(J)} \int_{J'} f(x)\,dx$$

since $J = T_k^{h(J)}(J')$. Therefore,

$$\Big|\int_{\bigcup\{I' : I' \in P_{N,M}(I)\}} f(x)\,dx\Big| = \Big|\Big(\int_{J'} f(x)\,dx\Big)\Big(\sum_{I' \in P_{N,M}(I)} \lambda^{h(I')}\Big)\Big| \le |S_{N,M}(I,\lambda)|.$$

Hence

$$\begin{aligned}\frac{1}{8}\,\mu(I) &\le \Big|\int_I f(x)\,dx\Big| = \Big|\sum_m \int_{\bigcup\{I' : I' \in P_{n+a\max(4,k),m}(I)\}} f(x)\,dx\Big| \\ &\le \sum_m \Big|\int_{\bigcup\{I' : I' \in P_{n+a\max(4,k),m}(I)\}} f(x)\,dx\Big| \le \sum_m |S_{n+a\max(4,k),m}(I,\lambda)| \\ &\le \mu(I)(1-(2-2\cos(\theta/6))\alpha^2(1-\alpha)^2)^a,\end{aligned}$$

which does not hold for sufficiently large values of a. Hence we have a contradiction. Therefore, T_k has no eigenvalues other than 1, and is ergodic. Therefore T_k is weakly mixing. ∎

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