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A FREE GROUP OF PIECEWISE LINEAR TRANSFORMATIONS

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Abstract. We prove the following conjecture of J. Mycielski: There exists a free nonabelian group of piecewise linear, orientation and area preserving transformations which acts on the punctured disk $\{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$ without fixed points.

1. Introduction. In the theory of paradoxical decompositions which goes back to Hausdorff, Banach and Tarski, and von Neumann (see the survey of M. Laczkovich [2] and also [5]) the following conjecture was open until recently:

(I) *The punctured disk* $D = \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$ *has a paradoxical decomposition relative to the group* $SL_2(\mathbb{R})$.

Likewise we have the more demanding conjecture:

(II) *There exists a partition of* D *into three sets* A, B, C *such that the six sets* $A, B, C, A \cup B, B \cup C, C \cup A$ *are equivalent to each other by finite decomposition relative to the group* $SL_2(\mathbb{R})$.

Recall that sets $A, B \subset X$ are *equivalent by finite decomposition* (or *equidecomposable*) *relative to a group* G acting on X if there exist finite partitions $\{A_i\}_{i=1}^k$ and $\{B_i\}_{i=1}^k$ of A and B respectively and $g_1, \dots, g_k \in G$ such that $g_i(A_i) = B_i$ for each $1 \le i \le k$. The set $E \subset X$ is *paradoxical relative to* G if E contains disjoint subsets A and B and each of them is equidecomposable to E relative to G.

Conjecture (I) was proved by M. Laczkovich [1]. (II) presents additional difficulties, and it will be proved in the present paper. In fact it is known (see [4] and Corollary 4.12 in [5]) that with the use of the Axiom of Choice, affirmative answers to (I), (II) and many similar conjectures follow from the following theorem:

THEOREM 1.1. *There exists a free nonabelian group* F *of permutations acting on the punctured disk* $D = \{(x,y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \le r^2\}$ *such that if* $f \in F \setminus \{e\}$ *and* $x \in D$ *then* $f(x) \neq x$, *and for every* $f \in F$ *there exists a*

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finite partition $D = D_1 \cup \dots \cup D_n$, where the sets D_i belong to the Boolean algebra generated by sets open in D, and there exist $\varphi_1, \dots, \varphi_n \in SL_2(\mathbb{R})$ such that $f{\restriction}D_i = \varphi_i{\restriction}D_i$ for $i = 1, \dots, n$.

J. Mycielski [3] proved a similar theorem for the hyperbolic plane and in [4] he conjectured the above one and outlined a possible approach to the proof. In the present paper we will show that indeed his approach can be realized. Our proof does not use the Axiom of Choice. Let us also mention that the proof of conjecture (I) in [1] is based on the fact that the action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2 \setminus \{(0,0)\}$ is locally commutative. However this fact does not suffice to prove conjecture (II).

2. Preliminaries. In this section we recall the material from Mycielski [4] that is relevant to our proof. All unexplained terminology can be found in [4].

In what follows, unless otherwise stated, linear transformations are represented by a matrix relative to the standard basis.

LEMMA 2.1. *If $\mathbf{A} \in SL_2(\mathbb{R})$ and $\mathrm{tr}(\mathbf{A}) \neq 2$, then $\mathbf{A}(x) \neq x$ for all $x \in \mathbb{R}^2 \setminus \{(0,0)\}$.*

Proof. It is enough to observe that $\mathrm{tr}(\mathbf{A}) \neq 2$ implies $\det(\mathbf{A} - \mathbf{I}) \neq 0$. Then apply this to the equation $\mathbf{A}x = x$ to obtain $x = (0,0)$. ∎

LEMMA 2.2. *For any $\varphi \in SL_2(\mathbb{R})$ there exists a rotation $\rho_\varphi \in SO_2(\mathbb{R})$ such that*

$$D \setminus \varphi(D) = \rho_\varphi(D \setminus \varphi^{-1}(D)).$$

Proof. This follows since the ellipses $\varphi^{-1}(D)$ and $\varphi(D)$ are congruent. ∎

Applying Lemma 2.2 for all $\varphi \in SL_2(\mathbb{R})$ we can define a piecewise linear transformation $\widehat{\varphi} : D \to D$ such that

$$\widehat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } x \in D \cap \varphi^{-1}(D), \\ \rho_\varphi(x) & \text{if } x \in D \setminus \varphi^{-1}(D). \end{cases}$$

Let $\kappa_0 > 1$ be a real number such that $D \subset \rho_1\varphi(D) \cup \rho_2\varphi(D) \cup \rho_3\varphi(D)$ for some rotations $\rho_1, \rho_2, \rho_3 \in SO_2(\mathbb{R})$ and φ represented by the matrix

$$\mathbf{A}_0 = \begin{pmatrix} \kappa_0 & 0 \\ 0 & 1/\kappa_0 \end{pmatrix}.$$

We observe that there exist three orthonormal, oriented bases with the same orientation as the standard basis, such that if $\varphi_1, \varphi_2, \varphi_3 \in SL_2(\mathbb{R})$ are represented by a matrix

$$\mathbf{A} = \begin{pmatrix} \kappa & 0 \\ 0 & 1/\kappa \end{pmatrix}$$

relative to these bases, where $1 < \kappa < \kappa_0$, then the composition $\widehat{\varphi}_1\widehat{\varphi}_2\widehat{\varphi}_3$ has the following property:

(P) For every $x \in D$, $\widehat{\varphi}_1\widehat{\varphi}_2\widehat{\varphi}_3(x) = fgh(x)$, where
$$(f, g, h) \in \{\varphi_1, \rho\} \times \{\varphi_2, \rho\} \times \{\varphi_3, \rho\} \setminus \{(\rho, \rho, \rho)\}$$
and ρ is the counter-clockwise rotation through $\pi/2$.

REMARK 2.3. Clearly, if each of the three bases defined above is rotated counter-clockwise by the same angle, then the transformations $\varphi_4, \varphi_5, \varphi_6 \in SL_2(\mathbb{R})$ represented by the matrix $\mathbf{A}$ relative to these bases are such that the map $\widehat{\varphi}_4\widehat{\varphi}_5\widehat{\varphi}_6$ also has the property (P).

Finally, the above implies that Theorem 1.1 reduces to the following statement:

THEOREM 2.4. *There exist triples* $(\varphi_1, \varphi_2, \varphi_3), (\varphi_4, \varphi_5, \varphi_6) \in (SL_2(\mathbb{R}))^3$ *and a real number* κ *with* $1 < \kappa < \kappa_0$ *such that the pair of transformations* $\widehat{\varphi}_1\widehat{\varphi}_2\widehat{\varphi}_3, \widehat{\varphi}_4\widehat{\varphi}_5\widehat{\varphi}_6 : D \to D$ *generate a free group as required in Theorem* 1.1.

REMARK 2.5. In the standard basis, each φ_i has a matrix of the form $\mathbf{S}_{\varphi_i}^{-1}\mathbf{A}\mathbf{S}_{\varphi_i}$, where $\mathbf{S}_{\varphi_i}$ is an orthogonal matrix.

3. Proof of Theorem 2.4. Choose $\varphi_1, \ldots, \varphi_6 \in SL_2(\mathbb{R})$ represented by the matrix $\mathbf{A} = \left(\begin{smallmatrix} \kappa & 0 \\ 0 & 1/\kappa \end{smallmatrix}\right)$ in some six, pairwise different, orthonormal bases $\mathcal{B}_1, \ldots, \mathcal{B}_6$ with the same orientation as the standard basis. Assume also that $\widehat{\varphi}_1\widehat{\varphi}_2\widehat{\varphi}_3$ and $\widehat{\varphi}_4\widehat{\varphi}_5\widehat{\varphi}_6$ have the property (P). Further let D be the punctured disk and $\mathbf{R}$ be the matrix corresponding to the rotation ρ, defined in the property (P). Denote by Ψ the set consisting of the elements $\widehat{\varphi}_1\widehat{\varphi}_2\widehat{\varphi}_3, \widehat{\varphi}_4\widehat{\varphi}_5\widehat{\varphi}_6$ and their inverses.

LEMMA 3.1. *Let* w *be a nontrivial, irreducible composition of* l *elements from* Ψ. *Then there exists a finite partition* $\{D_i\}_{i=1}^k$ *of* D *such that the sets* D_i *belong to the Boolean algebra generated by sets open in* D *and* w *restricted to any* D_i *has matrix of the form* $\mathbf{P}_1\mathbf{X}_1 \ldots \mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1}$, *where* $n \le 3l$, $\mathbf{P}_i$ *is an orthogonal matrix and* $\mathbf{X}_i \in \{\mathbf{A}, \mathbf{A}^{-1}\}$ *for* $i = 1, \ldots, n$. *Moreover for each* $i \le n+1$, *either*

(i) $\mathbf{P}_i = \mathbf{R}^{k_1}\mathbf{S}_{\varphi_r}^{-1}$ *for* $r \le 6$ *and* $k_1 \in \{0, 1, 2, 3\}$, *or*

(ii) $\mathbf{P}_i = \mathbf{S}_{\varphi_r}\mathbf{R}^{k_1}\mathbf{S}_{\varphi_s}^{-1}$ *for* $r, s \le 6$ *and* $k_1 \in \{0, 1, 2, 3\}$, *or*

(iii) $\mathbf{P}_i = \mathbf{S}_{\varphi_r}\mathbf{R}^{k_1}$ *for* $r \le 6$ *and* $k_1 \in \{0, 1, 2, 3\}$,

where $\mathbf{P}_i$ *satisfies* (i) *(resp.* (iii)*) if and only if* $i = 1$ *(resp.* $i = n+1$*).*

Proof. By the property (P), for any $x \in D$ and any w, $w(x)$ equals the value of some composition of elements from Ψ.

The order of the elements in the composition determines the shape of the matrices $\mathbf{P}_i$ and also the form of the partition $\{D_i\}_{i=1}^k$. ∎

Let S be the set of all entries of the matrices $\mathbf{S}_{\varphi_1}, \ldots, \mathbf{S}_{\varphi_6}$ (see Remark 2.5).

LEMMA 3.2. *Let* $\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n$ *be a matrix such that* $\mathbf{X}_i \in \{\mathbf{A}, \mathbf{A}^{-1}\}$ *and* $\mathbf{P}_i$ *is an orthogonal matrix with entries from* S *for* $1 \le i \le n$ *and* $n = 1, 2, \dots$. *Then each entry of* $\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n$ *is a function of the form*

$$Q(\kappa) = a_{-n}\kappa^{-n} + \dots + a_{-1}\kappa^{-1} + a_0 + a_1\kappa + \dots + a_n\kappa^n,$$

where $a_i \in \mathbb{Q}(S)$, *the field generated by the set* S.

Moreover:

(i) *If* $\mathbf{X}_n = \mathbf{A}$ *then* $\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n$ *is of the form*

$$\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{A} = \begin{pmatrix} a^{(1)}_{-n}\kappa^{-(n-2)} + \dots + a^{(1)}_n\kappa^n & a^{(2)}_{-n}\kappa^{-n} + \dots + a^{(2)}_n\kappa^{n-2} \\ a^{(3)}_{-n}\kappa^{-(n-2)} + \dots + a^{(3)}_n\kappa^n & a^{(4)}_{-n}\kappa^{-n} + \dots + a^{(4)}_n\kappa^{n-2} \end{pmatrix}.$$

(ii) *If* $\mathbf{X}_n = \mathbf{A}^{-1}$ *then* $\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n$ *is of the form*

$$\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{A}^{-1} = \begin{pmatrix} b^{(1)}_{-n}\kappa^{-n} + \dots + b^{(1)}_n\kappa^{n-2} & b^{(2)}_{-n}\kappa^{-(n-2)} + \dots + b^{(2)}_n\kappa^n \\ b^{(3)}_{-n}\kappa^{-n} + \dots + b^{(3)}_n\kappa^{n-2} & b^{(4)}_{-n}\kappa^{-(n-2)} + \dots + b^{(4)}_n\kappa^n \end{pmatrix}.$$

Proof. By induction on the number of factors $\mathbf{P}_i\mathbf{X}_i$. We observe that the highest term of each entry of $\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1}$ has the form $a_n\kappa^n$ and the lowest $a_{-n}\kappa^{-n}$. Then multiplying the matrix by $\mathbf{A}$ (resp. $\mathbf{A}^{-1}$) is multiplying the first column by κ (resp. $1/\kappa$) and the second by $1/\kappa$ (resp. κ). ■

LEMMA 3.3. *Let* $\widehat{\mathbf{P}}_1, \dots, \widehat{\mathbf{P}}_n$ *be some orthogonal matrices such that each of them can be expressed by the conditions* (i)–(iii) *of Lemma* 3.1. *If the rotation angles* α_i *of* $\mathbf{S}_{\varphi_i}$, $i = 1, \dots, 6$, *are such that* $\alpha_i - \alpha_j \notin \{k\pi/2 : k \in \mathbb{Z}\}$ *and* $\alpha_i \notin \{k\pi/2 : k \in \mathbb{Z}\}$ *for any* $i \ne j$, *then no entry of* $\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1\dots\widehat{\mathbf{P}}_n\widehat{\mathbf{X}}_n$, *where* $\widehat{\mathbf{X}}_i \in \{\mathbf{A}, \mathbf{A}^{-1}\}$, *is a constant function in* κ, *for any positive integer* $n \ge 2$.

Proof. We consider two cases.

CASE 1: $\widehat{\mathbf{P}}_i \ne \mathbf{S}_{\varphi_r}\mathbf{R}^2\mathbf{S}^{-1}_{\varphi_r}$ for any $r \le 6$ and any $1 \le i \le n$. In this case the assumptions of the lemma imply that $\widehat{\mathbf{P}}_i$ represents a rotation through an angle $\ne k\pi/2$ for $k \in \mathbb{Z}$. Then it is enough to observe by Lemma 3.2 that, for any $n \ge 2$, each of the coefficients in the highest terms of $\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1\dots\widehat{\mathbf{P}}_n\widehat{\mathbf{X}}_n$ is a product of some entries of $\widehat{\mathbf{P}}_1, \dots, \widehat{\mathbf{P}}_n$ and thus it is not zero.

CASE 2: $\widehat{\mathbf{P}}_i = \mathbf{S}_{\varphi_r}\mathbf{R}^2\mathbf{S}^{-1}_{\varphi_r}$ for some $r \le 6$ and some $1 < i < n$. Let j be the minimal number such that $\widehat{\mathbf{P}}_j$ is of the above form. Suppose $\widehat{\mathbf{X}}_{j-1} = \mathbf{A}$ and observe that in this case $\widehat{\mathbf{X}}_j = \mathbf{A}$ and by Lemma 3.2 we can express $\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1\dots\widehat{\mathbf{P}}_{j-1}\widehat{\mathbf{X}}_{j-1}$ as

$$\begin{pmatrix} a^{(1)}_{-(j-1)}\kappa^{-(j-3)} + \dots + a^{(1)}_{j-1}\kappa^{j-1} & a^{(2)}_{-(j-1)}\kappa^{-(j-1)} + \dots + a^{(2)}_{j-1}\kappa^{j-3} \\ a^{(3)}_{-(j-1)}\kappa^{-(j-3)} + \dots + a^{(3)}_{j-1}\kappa^{j-1} & a^{(4)}_{-(j-1)}\kappa^{-(j-1)} + \dots + a^{(4)}_{j-1}\kappa^{j-3} \end{pmatrix}.$$

Since the matrix $\mathbf{S}_{\varphi_r}\mathbf{R}^2\mathbf{S}^{-1}_{\varphi_r}$ corresponds to rotation through π, multiplication of $\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1\dots\widehat{\mathbf{P}}_{j-1}\widehat{\mathbf{X}}_{j-1}$ by it changes the sign of the coefficients in the highest and the lowest terms of all the entries of $\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1\dots\widehat{\mathbf{P}}_{j-1}\widehat{\mathbf{X}}_{j-1}$. Thus finally

$$\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1\dots\widehat{\mathbf{P}}_j\widehat{\mathbf{X}}_j=\begin{pmatrix} a_{-j}^{(1)}\kappa^{-(j-4)}+\dots+a_j^{(1)}\kappa^j & a_{-j}^{(2)}\kappa^{-j}+\dots+a_j^{(2)}\kappa^{j-4}\\ a_{-j}^{(3)}\kappa^{-(j-4)}+\dots+a_j^{(3)}\kappa^j & a_{-j}^{(4)}\kappa^{-j}+\dots+a_j^{(4)}\kappa^{j-4}\end{pmatrix},$$

where the coefficients in the highest terms in the first column and in the lowest terms in the second column are not zero. Clearly we can extend this reasoning to the case $\widehat{\mathbf{P}}_j=\widehat{\mathbf{P}}_{j+1}=\widehat{\mathbf{P}}_{j+p}=\mathbf{S}_{\varphi_r}\mathbf{R}^2\mathbf{S}_{\varphi_r}^{-1}$, where $j+p\le n$. It remains to observe that if $\widehat{\mathbf{P}}_{j+p}\neq\mathbf{S}_{\varphi_r}\mathbf{R}^2\mathbf{S}_{\varphi_r}^{-1}$ for some $j+p\le n$ then by Lemma 3.2 we obtain Case 1 for the matrix $\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1\dots\widehat{\mathbf{P}}_{j+p}\widehat{\mathbf{X}}_{j+p}$. The same reasoning can be applied, mutatis mutandis, to the situation when $\widehat{\mathbf{X}}_{j-1}=\mathbf{A}^{-1}$. ∎

Now we can conclude the proof of Theorem 2.4. Let $\varphi_1,\dots,\varphi_6$, $\widehat{\varphi}_1\widehat{\varphi}_2\widehat{\varphi}_3$ and $\widehat{\varphi}_4\widehat{\varphi}_5\widehat{\varphi}_6$ be the transformations, $\mathbf{A}$, $\mathbf{R}$ be the matrices and D be the set as at the beginning of Section 3. For all positive integers n, the matrices $\mathbf{P}_1,\dots,\mathbf{P}_n$ satisfy the assumptions of Lemma 3.3. Let w be a nontrivial, irreducible composition of elements from Ψ, and $\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1}$ be a matrix corresponding to w, restricted to some $D_i\subset D$, as in Lemma 3.1.

First we show that for any positive integer n,

$$(*)\qquad \operatorname{tr}(\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1})=\operatorname{tr}(\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1\dots\widehat{\mathbf{P}}_j\widehat{\mathbf{X}}_j),$$

for some integer j such that $1\le j\le n$, where $\widehat{\mathbf{P}}_i$ is described by one of the conditions (i)–(iii), and $\widehat{\mathbf{X}_i}\in\{\mathbf{A},\mathbf{A}^{-1}\}$ for $1\le i\le j$.

We have two cases:

Case 1: $\mathbf{P}_1\neq\mathbf{P}_{n+1}^{-1}$. By Lemma 3.1 we can write $\mathbf{P}_{n+1}\mathbf{P}_1=\mathbf{S}_{\varphi_r}\mathbf{R}^{k_1}\mathbf{S}_{\varphi_s}^{-1}$ for some $r,s\le 6$ and some $k_1\in\{0,1,2,3\}$. Since similar matrices have equal traces,

$$\operatorname{tr}(\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1})=\operatorname{tr}(\widehat{\mathbf{P}}\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n),\quad\text{where}\quad \widehat{\mathbf{P}}=\mathbf{S}_{\varphi_r}\mathbf{R}^{k_1}\mathbf{S}_{\varphi_s}^{-1}.$$

Case 2: $\mathbf{P}_1=\mathbf{P}_{n+1}^{-1}$. By Lemma 3.1 we obtain $\mathbf{P}_1=\mathbf{R}^{k_1}\mathbf{S}_{\varphi_r}^{-1}$ for some $r\le 6$ and some $k_1\in\{0,1,2,3\}$, and then $\mathbf{P}_{n+1}^{-1}=\mathbf{S}_{\varphi_r}\mathbf{R}^{-k_1}$. Since the bases $\mathcal{B}_i$ are pairwise different, we apply the property (P) to deduce that $\mathbf{X}_1=\mathbf{X}_n^{-1}$. Thus

$$\begin{aligned}\operatorname{tr}(\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1})&=\operatorname{tr}(\mathbf{P}_{n+1}^{-1}\mathbf{X}_n^{-1}\mathbf{P}_2\mathbf{X}_2\dots\mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1})\\&=\operatorname{tr}(\mathbf{X}_n\mathbf{P}_{n+1}\mathbf{P}_{n+1}^{-1}\mathbf{X}_n^{-1}\mathbf{P}_2\mathbf{X}_2\dots\mathbf{P}_{n-1}\mathbf{X}_{n-1}\mathbf{P}_n)\\&=\operatorname{tr}(\mathbf{P}_2\mathbf{X}_2\dots\mathbf{P}_{n-1}X_{n-1}\mathbf{P}_n).\end{aligned}$$

We can repeat this operation if necessary, to obtain finally

$$\operatorname{tr}(\mathbf{P}_1\mathbf{X}_1\dots\mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1})=\operatorname{tr}(\mathbf{P}_{1+j}\mathbf{X}_{1+j}\dots\mathbf{P}_{n-j}\mathbf{X}_{n-j}\mathbf{P}_{n+1-j}),$$

where j is the number of the above cancelations and $\mathbf{P}_{1+j} \neq \mathbf{P}_{n+1-j}^{-1}$ (such a j exists since w is nontrivial and irreducible).

Thus Case 2 is reduced to Case 1.

Further, we observe that, by Lemmas 3.2 and 3.3, the condition $\operatorname{tr}(\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1 \ldots \widehat{\mathbf{P}}_n\widehat{\mathbf{X}}_n) = 2$ leads to a nonconstant polynomial in κ. Since each of the above polynomials has only finitely many roots and there are only countably many expressions of the form $\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1 \ldots \widehat{\mathbf{P}}_n\widehat{\mathbf{X}}_n$, where n is any positive integer, we conclude that the set $K = \{\kappa \in \mathbb{R} : \operatorname{tr}(\widehat{\mathbf{P}}_1\widehat{\mathbf{X}}_1 \ldots \widehat{\mathbf{P}}_n\widehat{\mathbf{X}}_n) = 2$, $n \in \mathbb{N}$, $\widehat{\mathbf{P}}_i$ are of the form (i)–(iii), $\widehat{\mathbf{X}}_i \in \{\mathbf{A}, \mathbf{A}^{-1}\}$ for each $1 \leq i \leq n\}$ is countable. Thus we can choose some $\kappa_1 \in \mathbb{R}$ such that $\kappa_1 \notin K$ and $1 < \kappa_1 < \kappa_0$.

By the properties (P) and $(*)$ we have $\operatorname{tr}(\mathbf{P}_1\mathbf{X}_1 \ldots \mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1}) \neq 2$ for this κ_1, any positive integer n and any $\mathbf{P}_i, \mathbf{X}_j$, $j < n+1$, $i \leq n+1$. To conclude the proof it is enough to use Lemmas 2.1 and 3.1. ∎

COROLLARY 3.4. *If the entries of the matrices* $\mathbf{P}_i$ *from* $\mathbf{P}_1\mathbf{X}_1 \ldots \ldots \mathbf{P}_n\mathbf{X}_n\mathbf{P}_{n+1}$ *are nonzero algebraic numbers and* κ_0 *is a real number, occurring in the property* (P), *then one can use as* κ_1, *in the above proof, all transcendental numbers* κ *such that* $1 < \kappa < \kappa_0$.

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