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CORRELATION ASYMPTOTICS FROM LARGE DEVIATIONS IN DYNAMICAL SYSTEMS WITH INFINITE MEASURE

BY

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Abstract. We extend a result of Doney [Probab. Theory Related Fields 107 (1997)] on renewal sequences with infinite mean to renewal sequences of operators. As a consequence, we get precise asymptotics for the transfer operator and for correlations in dynamical systems preserving an infinite measure (including intermittent maps with an arbitrarily neutral fixed point).

1. Introduction. Statistical properties of dynamical systems with enough hyperbolicity can often be related to renewal theory. Indeed, if the successive returns to a suitable reference set are sufficiently chaotic, one may expect that they are close to being independent, and then the probability to return exactly at time n behaves like a renewal sequence. This idea, already implicit in Young [You99], has been explicitly developed by Sarig and Gouëzel in [Sar02, Gou04a]. On the technical level, since there is no real independence, one should replace the renewal sequences from probability theory by renewal sequences of operators, but once this is done, many results or arguments from probability theory can be adapted to yield very precise estimates for dynamical systems preserving a probability measure.

Ideas originating in renewal theory have a long history in dynamical systems preserving an infinite measure (see for instance [Sch76, Bow79, Aar86]). Recently, operator renewal theory was extended to this setting by Melbourne and Terhesiu in [MT10]. They were able to adapt (and considerably refine) estimates of Garsia and Lamperti [GL62] on renewal sequences with infinite mean, to obtain precise asymptotics on the iterates of transfer operators for systems having an invariant measure which is "infinite, but not too much".

More specifically, assume that $T : X \to X$ preserves a measure μ, and that there is a set Y of finite measure such that the first return time φ to Y satisfies $\mu(\varphi > n) \sim n^{-\beta}\ell(n)$ for some $\beta > 0$ and some slowly varying function ℓ (i.e., ℓ is a measurable function such that $\ell(\lambda x)/\ell(x)$ tends to 1 as $x \to +\infty$, for all $\lambda > 0$). If $\beta > 1$, then the measure μ

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is finite, while it is infinite for $\beta < 1$. Under suitable assumptions, for $\beta \in (1/2, 1)$, [MT10] obtains precise asymptotics for the transfer operator L, of the form $n^{1-\beta}\ell(n) 1_Y L^n 1_Y \to d_\beta P$, where $Pv = (\int v) 1_Y$ and d_β is a suitable constant (similar asymptotics holds for $\beta = 1$, with an additional logarithm). In particular, this implies estimates on the correlations, of the form $\int u \cdot v \circ T^n \sim n^{\beta-1}\ell(n)^{-1} d_\beta \int u \cdot \int v$ whenever u and v are nice enough functions supported on Y with non-zero integral. Compared to previous results (see for instance [Aar97]), the main novelty of these results is that they hold for each n, while classical arguments give the same asymptotics on average.

The restriction $\beta \in (1/2, 1]$ in their argument is not merely a technical detail: for $\beta \leq 1/2$, even in the probabilistic situation, the result becomes false without additional assumptions, as is explained in [GL62]. More recently, [Don97] developed a different approach to handle also the case $\beta \in (0, 1/2]$ (under stronger assumptions). This approach, in contrast to the analytic one of [GL62], is really probabilistic in nature. Our goal in this article is to adapt it to renewal sequences of operators.

1.1. Doney's result. Let us first explain the result of Doney we will generalize later on. Consider a sequence of independent identically distributed random variables $Z_1, Z_2, \dots$ taking values in $\mathbb{N}^*$, with $P(Z_i > n) \sim n^{-\beta}\ell(n)$ for some $\beta \in (0, 1)$ and some slowly varying function ℓ. We consider the sums $(Z_1 + \dots + Z_k)_{k>0}$, and let T_n be the probability that one of those sums is equal to n.

THEOREM 1.1 ([Don97, Theorem B]). *Assume additionally $P(Z_i = n) \leq Cn^{-\beta-1}\ell(n)$, and that Z_i does not take its values in a smaller lattice $a\mathbb{Z} + b$, $a > 1$. Then*

$$n^{1-\beta}\ell(n) T_n \to d_\beta, \quad \textit{where} \quad d_\beta = \frac{1}{\pi}\sin\beta\pi.$$

The idea of Doney to prove this result is the following. Let $S_k = Z_1 + \dots + Z_k$. Under our assumptions, S_k/a_k converges in distribution to a stable law W of index β, where a_k is such that $k\ell(a_k) \sim a_k^\beta$. A heuristic computation gives

$$\mathbb{P}(S_k = n) \sim \mathbb{P}(W \in [(n-1)/a_k, n/a_k]) \sim \frac{1}{a_k}\psi(n/a_k),$$

where ψ is the density of W. The local limit theorem (based on simple Fourier computations) justifies this statement whenever n/a_k remains bounded. Summing over k, this gives for any $K \geq 1$ good asymptotics on

$$\sum_{k \colon n/a_k \leq K} \mathbb{P}(S_k = n).$$

Since $T_n = \sum_k \mathbb{P}(S_k = n)$, it remains to estimate $\mathbb{P}(S_k = n)$ for $n/a_k \to \infty$ to conclude. This is the main result of Doney:

THEOREM 1.2. *Under the previous assumptions, for $k \in \mathbb{N}$ and $n \geq a_k$,*

$$\mathbb{P}(S_k = n) \leq Ckn^{-\beta-1}\ell(n).$$

To get $S_k = n$, it is possible that one Z_i is larger than $n/2$ while the other Z_j for $j \leq k$ are all smaller than $n/2$ and add up to $n - Z_i$. This has a probability $\asymp n^{-\beta-1}\ell(n)$ if $\mathbb{P}(Z_1 = n) \sim Cn^{-\beta-1}\ell(n)$. Summing over the k possible values of i, we deduce that $kn^{-\beta-1}\ell(n)$ is a lower bound for $\mathbb{P}(S_k = n)$. Hence, Doney's estimate is sharp. The main point of the proof is to show that the configurations we just described give a dominant contribution to $\mathbb{P}(S_k = n)$, i.e., it is very unlikely to get $S_k = n$ unless at least one of the Z_i is already at least $n/2$.

This is in essence a large deviations estimate. The proof of Doney is written in very probabilistic terms (relying in particular on a careful change of probability measure), but it can be reformulated in a more analytic way that is more suitable to an extension to dynamical situations. This reformulation has another advantage: there is a mistake in Doney's computation (in the first displayed equation following (2.31) in [Don97], the first inequality is in the wrong direction). The analytic formulation of the argument (see Section 5) turns out to be significantly simpler than the way it is written in [Don97], and avoids this mistake.

By summing over k the estimates for $\mathbb{P}(S_k = n)$ coming from the local limit theorem and the large deviations estimate of Theorem 1.2, Theorem 1.1 follows.

1.2. Main result for renewal sequences of operators. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$.

DEFINITION 1.3. Let $(R_n)_{n\geq 1}$ be a sequence of operators on a Banach space $\mathcal{B}$, with $\sum \|R_n\| < +\infty$. They form an *aperiodic renewal sequence of operators* if $R(z) = \sum R_n z^n$, defined for $z \in \overline{\mathbb{D}}$, satisfies:

(1) For $z \in \overline{\mathbb{D}} - \{1\}$, the spectral radius of $R(z)$ is < 1.
(2) The operator $R(1)$ has a simple eigenvalue at 1, and the rest of its spectrum is contained in a disk of radius < 1.
(3) Let P be the eigenprojection of $R(1)$ for the eigenvalue 1, and let r_n be such that $PR_nP = r_nP$. We assume that $r_n \geq 0$.

When $\mathcal{B}$ is $\mathbb{C}$ and R_n is multiplication by a non-negative number r_n, then the second assumption means that $\sum r_n = 1$, while the first one is equivalent to the fact that the corresponding probability measure is not carried by a smaller lattice $a\mathbb{Z} + b$, $a > 1$. This is exactly the setting of Subsection 1.1.

In the scalar case of Subsection 1.1, T_n is the probability that a renewal event takes place at time n. The analogue of this quantity in the non-commutative setting is given by the formula $T_0 = I$ and, for $n \geq 1$,

$$T_n = \sum_{k=1}^{\infty} \sum_{j_1+\cdots+j_k=n} R_{j_1} \cdots R_{j_k}. \tag{1.1}$$

For $|z| < 1$, the operator $T(z) = \sum_{n=0}^{\infty} T_n z^n$ is well defined and satisfies $T(z) = (I - R(z))^{-1}$. This *renewal equation* is of fundamental importance to understand the asymptotics of T_n, since several functional analytic tools can be brought into play.

Our main theorem is the following.

THEOREM 1.4. *Let R_n be an aperiodic renewal sequence of operators. Assume that, for some $\beta \in (0,1)$ and some slowly varying function ℓ, we have $\sum_{j>n} r_j \sim n^{-\beta}\ell(n)$. Assume also that*

$$\|R_n\| \leq C n^{-\beta-1}\ell(n). \tag{1.2}$$

Then

$$n^{1-\beta}\ell(n) T_n \to d_\beta P$$

for $d_\beta = \frac{1}{\pi}\sin\beta\pi$.

For $\beta \in (1/2, 1)$, the theorem is true without the assumption (1.2): this is essentially [GL62, Theorem 1.1] in the scalar case and [MT10, Theorem 2.1] in the operator case. However, the assumption (1.2) becomes necessary for $\beta \in (0, 1/2]$. In the scalar case, this is [Don97, Theorem B].

The strategy of the proof is the same as Doney's, which we described in Subsection 1.1. More specifically, let $T_n(k)$ be the coefficient of z^n in $R(z)^k$. For $k \geq 1$, one has $T_0(k) = 0$ and $T_n(k) = \sum_{j_1+\cdots+j_k=n} R_{j_1} \cdots R_{j_k}$ for $n \geq 1$. By definition, $T_n = \sum_k T_n(k)$. We will get precise asymptotics for $T_n(k)$ in the whole range of n and k, and add them up to get the asymptotics of T_n.

As above, let a_k be a sequence with $k\ell(a_k) \sim a_k^\beta$. The behavior of $T_n(k)$ is different for bounded n/a_k and for n/a_k tending to infinity. The asymptotics in those two regimes are described in the following statements. To describe them, we will use the fully asymmetric stable law of index β, i.e., the real random variable whose characteristic function is given by

$$g_\beta(t) = e^{-\Gamma(1-\beta)\cos(\beta\pi/2)|t|^\beta(1-i\,\mathrm{sgn}(t)\tan(\beta\pi/2))}. \tag{1.3}$$

Its density ψ is continuous, supported on $[0,\infty)$, and decays at infinity like $C/x^{\beta+1}$. In particular, this random variable has a moment of every order $< \beta$, but no moment of order β.

Proposition 1.5. *Let $K > 0$. Uniformly in $k \to \infty$ and $n \in [0, Ka_k]$, one has*

$$T_n(k) = \frac{1}{a_k}(\psi(n/a_k)P + o(1)),$$

where ψ is the density of the fully asymmetric stable law of index β.

This local limit theorem is completely classical (see for instance [AD01] or [Don97]). The main estimate is the following.

Theorem 1.6. *When $n \geq a_k$, we have*

$$\|T_n(k)\| \leq Ckn^{-\beta-1}\ell(n).$$

This is the analogue in the non-commutative setting of Theorem 1.2. Summing the estimates given by these two results, Theorem 1.4 readily follows as in [Don97].

1.3. Applications to dynamical systems. Applications of results such as Theorem 1.4 to different classes of dynamical systems are described in [MT10]. For the sake of simplicity, we will only describe one such example, the Pomeau–Manneville map, and refer the reader to [MT10] for other ones. For $\alpha > 0$, define a map $T = T_\alpha$ on $[0, 1]$ by

$$T(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{for } 0 \leq x \leq 1/2, \\ 2x - 1 & \text{for } 1/2 < x \leq 1. \end{cases}$$

This map has a unique (up to scaling) absolutely continuous invariant measure μ. It is finite for $\alpha < 1$, infinite for $\alpha \geq 1$. Fix some $\alpha > 1$. Let $Y = (1/2, 1)$, and let φ be the first return time from Y to itself. We normalize the invariant measure μ so that $\mu(Y) = 1$. Then $\mu(\varphi = n) \sim cn^{-\beta-1}$ for $\beta = 1/\alpha \leq 1$.

Let L be the transfer operator of T, i.e., the adjoint (with respect to μ) of the composition with T. Denoting by R_n the first return transfer operator to Y at time n, i.e., $R_n u = 1_Y L^n(1_{\{\varphi=n\}}u)$, then

$$1_Y L^n 1_Y = \sum_{k=1}^{\infty} \sum_{j_1+\cdots+j_k=n} R_{j_1} \cdots R_{j_k}.$$

This follows by splitting a trajectory from Y to Y according to its successive returns in Y. This is exactly the same formula as in (1.1). Hence, $T_n = 1_Y L^n 1_Y$.

In order to apply Theorem 1.4, we should check that (R_n) is an aperiodic renewal sequence of operators. When R_n acts on the space $\mathcal{B}$ of Lipschitz functions on Y, it satisfies $\|R_n\| \leq Cn^{-\beta-1}$. Moreover, denoting by L_Y the transfer operator of the induced map on Y, we have $R(z) = \sum R_n z^n = L_Y(z^\varphi \cdot)$. It easily follows that $R(1)$ has a simple eigenvalue at 1 (the corresponding eigenprojector being given by $Pu = (\int_Y u) \cdot 1_Y$), while the spectral radius of $R(z)$ for $z \in \overline{\mathbb{D}} - \{1\}$ is strictly less than 1. In particular, $PR_nP = \mu(\varphi = n)P$, so $r_n = \mu(\varphi = n) \sim cn^{-\beta-1}$.

We have checked all the assumptions of Theorem 1.4. Applying this theorem, we get the following.

PROPOSITION 1.7. *Let u be a Lipschitz function supported on Y. Then $n^{1-\beta}1_Y L^n(u)$ converges to $c\int_Y u$, uniformly on Y, for some constant c independent of u. In particular, if v is an integrable function supported on Y,*

$$n^{1-\beta}\int u \cdot v \circ T^n \to c\int u \cdot \int v.$$

This result is due to Thaler [Tha00] for $\alpha = 1$, to Melbourne and Terhesiu [MT10] for $\alpha \in (1, 2)$, and is new for $\alpha \geq 2$. As in [MT10], it can be extended to functions that are not supported in Y, and to other classes of maps (for instance, non-markovian ones, thanks to [Zwe98]).

The paper is organized as follows. In Section 2, we prove Proposition 1.5, and derive Theorem 1.4 from this proposition and the large deviations estimate, Theorem 1.6. The rest of the paper is devoted to the proof of Theorem 1.6. In Section 3, we describe the overall strategy, state two crucial estimates (in Lemmas 3.1 and 3.2), and deduce the theorem from those estimates. Finally, the last two sections are devoted to the proofs, respectively, of Lemmas 3.1 and 3.2. It is only in those two sections that our arguments deviate significantly from Doney's.

2. Deriving the main theorem from the large deviations estimate. In this section, we assume the crucial large deviations estimate, Theorem 1.6, and show how to derive our main theorem from it. The proofs are classical (they are the same as Doney's); we give some details for the convenience of the reader.

For z close to 1, the operator $R(z)$ is close to $R(1)$. Since $R(1)$ has an isolated eigenvalue at 1, it follows from standard perturbation theory that $R(z)$ has a unique eigenvalue $\lambda(z)$ close to 1, while the rest of its spectrum is contained in a disk of radius uniformly less than 1. More specifically, we may write

$$R(z) = \lambda(z)P(z) + Q(z),$$

where $P(z)$ is a one-dimensional projection, $P(z)Q(z) = Q(z)P(z) = 0$ and $\|Q(z)^n\| \leq C\rho^n$ for some $\rho < 1$.

We will need the asymptotics of $\lambda(e^{it})$ for t close to 0, due to Aaronson and Denker [AD01] (see also [MT10, Lemma 3.1]): if $t > 0$, then

$$\lambda(e^{it}) = e^{-c_\beta t^\beta \ell(1/t)(1+o(1))},$$

where

$$c_\beta = -i \int_0^\infty e^{i\sigma}\sigma^{-\beta}\, d\sigma = \Gamma(1-\beta)(\cos(\beta\pi/2) - i\sin(\beta\pi/2)).$$

A similar formula holds for $t < 0$, but with c_β replaced by its complex conjugate. In particular, for all $t \in \mathbb{R}$,

$$\lambda(e^{it/a_k})^k \to g_\beta(t), \tag{2.1}$$

where g_β is the characteristic function of the totally asymmetric stable law of parameter β (defined in (1.3)). Moreover, it follows from Potter bounds [BGT87, Theorem 1.5.6] that there exist $C, c > 0$ such that, for any t close enough to 0 and any $k \in \mathbb{N}$,

$$|\lambda(e^{it/a_k})^k| \le Ce^{-c|t|^{\beta/2}}. \tag{2.2}$$

Proof of Proposition 1.5. We want to estimate $T_n(k)$. The integral formula for Fourier coefficients gives

$$T_n(k) = \int_{-\pi}^{\pi} R(e^{it})^k e^{-int}\, dt/2\pi.$$

Outside a small neighborhood of $z = 1$, $\|R(z)^k\|$ decays exponentially fast, giving a negligible contribution to $T_n(k)$. Therefore, we may restrict the integral to an interval $[-\delta, \delta]$ (in which λ, P and Q are well defined). Writing $u = t/a_k$, we get

$$a_k T_n(k) = \frac{1}{2\pi}\int_{-\delta a_k}^{\delta a_k} R(e^{iu/a_k})^k e^{-i(n/a_k)u}\, du + O(\rho^k). \tag{2.3}$$

We have

$$\|R(e^{iu/a_k})^k\| \le C|\lambda(e^{iu/a_k})|^k \le Ce^{-c|u|^{\beta/2}},$$

by (2.2). This function is independent of k and integrable.

This shows that, uniformly in k and n, the integral in (2.3) satisfies the assumptions of the Lebesgue dominated convergence theorem. Together with pointwise convergence, this easily implies the proposition. To write it formally, it is more convenient to argue by contradiction. Assume therefore that, for some sequences $k_j \to \infty$ and $n_j \in [0, Ka_{k_j}]$,

$$a_{k_j} T_{n_j}(k_j) - \psi(n_j/a_{k_j})P \not\to 0. \tag{2.4}$$

We can assume that n_j/a_{k_j} converges to a number $x \in [0, K]$.

When k tends to infinity, $R(e^{iu/a_k})^k$ converges simply to $g_\beta(u)P$ by (2.1). Since n_j/a_{k_j} converges to x, it follows from the dominated convergence theorem that the integral (2.3) converges to P multiplied by the inverse Fourier transform of g_β at x, i.e., $\psi(x)$. Since ψ is continuous, for large enough j, $a_{k_j}T_{n_j}(k_j)-\psi(n_j/a_{k_j})P$ is arbitrarily small. This contradicts (2.4), and concludes the proof of the proposition. ∎

Proof of Theorem 1.4 using Proposition 1.5 and Theorem 1.6. We want to estimate $T_n = \sum_k T_n(k)$. Fix some $K \geq 1$, and decompose T_n as

$$T_n = \sum_{k\colon n<Ka_k} T_n(k) + \sum_{k\colon n\geq Ka_k} T_n(k) =: T_n^{(1)} + T_n^{(2)}.$$

If n is large, the ks appearing in the first sum are large enough so that Proposition 1.5 applies and gives

$$T_n(k) = \frac{1}{a_k}\psi(n/a_k)P \pm \varepsilon/a_k \tag{2.5}$$

for any fixed $\varepsilon > 0$ (where $\pm c$ means a term in the interval $[-c, c]$).

Since a_k is regularly varying of index $1/\beta > 1$, Karamata's Theorem [BGT87, Proposition 1.5.10] gives $\sum_{k=N}^{\infty} 1/a_k \sim cN/a_N$. Hence, the error terms ε/a_k in (2.5) add up to at most $\varepsilon N/a_N$, where N is such that $n = Ka_N$. Since $N\ell(a_N) \sim a_N^{\beta}$, this is at most

$$C\varepsilon a_N^{\beta-1}\ell(a_N)^{-1} \leq C(K)\varepsilon n^{\beta-1}\ell(n)^{-1}.$$

Hence, the error term in $T_n^{(1)}$ is $o_K(n^{\beta-1}\ell(n)^{-1})$.

Let us now study the dominating term in $T_n^{(1)}$. Define a measure μ_n on $[0, K]$ as the sum of Dirac masses at n/a_k for $n/a_k < K$, so that

$$\sum_{k\colon n<Ka_k} \frac{1}{a_k}\psi(n/a_k) = n^{-1}\int x\psi(x)\,d\mu_n.$$

Define $A(x) = x^{\beta}/\ell(x)$, so that $k \sim A(a_k)$. For any fixed $0 < x < y < K$ we have

$$\begin{aligned}\mu_n([x,y]) &= \sum_{x\leq n/a_k\leq y} 1 \sim \sum_{k\in[A(n/y),A(n/x)]} 1 \sim A(n/x) - A(n/y)\\ &= \frac{(n/x)^{\beta}}{\ell(n/x)} - \frac{(n/y)^{\beta}}{\ell(n/y)} \sim n^{\beta}\ell(n)^{-1}\cdot(1/x^{\beta} - 1/y^{\beta})\\ &= n^{\beta}\ell(n)^{-1}\cdot\nu([x,y]),\end{aligned}$$

where ν is the measure with density $\beta x^{-\beta-1}$ on $(0, K]$. This shows that $n^{-\beta}\ell(n)\mu_n$ converges weakly to ν on $(0, K]$. There is no problem at 0 since everything can be controlled uniformly as in the estimate of the error term. We obtain

$$\sum_{k\colon n<Ka_k} \frac{1}{a_k}\psi(n/a_k) \sim n^{\beta-1}\ell(n)^{-1}\int x\psi(x)\,d\nu.$$

Therefore,

$$n^{1-\beta}\ell(n)T_n^{(1)} = \beta P \int_0^K \psi(x)x^{-\beta}\,dx + o_K(1). \tag{2.6}$$

We now turn to $T_n^{(2)}$. We bound it directly using Theorem 1.6, by

$$\sum_{k:n\geq Ka_k} Ckn^{-1-\beta}\ell(n) \sim Cn^{-1-\beta}\ell(n)\sum_{k\leq A(n/K)} k \leq Cn^{-1-\beta}\ell(n)A(n/K)^2$$
$$\leq Cn^{-1+\beta}K^{-2\beta}\ell(n)^{-1}\cdot\frac{\ell(n)^2}{\ell(n/K)^2}.$$

By Potter bounds, $\ell(n)/\ell(n/K) \leq CK^{\beta/2}$, yielding

$$\|T_n^{(2)}\| \leq Cn^{-1+\beta}\ell(n)^{-1}\cdot K^{-\beta}.$$

Together with (2.6), this yields

$$n^{1-\beta}\ell(n)T_n = \beta P\int_0^K \psi(x)x^{-\beta}\,dx + o_K(1) + O(K^{-\beta}).$$

Choosing first K large enough so that $K^{-\beta}$ and $\int_K^\infty \psi(x)x^{-\beta}\,dx$ are small, and then n large enough so that the term $o_K(1)$ is small, we obtain the convergence of $n^{1-\beta}\ell(n)T_n$ to cP, for $c = \beta\int_0^\infty x^{-\beta}\psi(x)\,dx$, which is equal to d_β (see for instance [Zol86, Theorem 2.6.3]). ∎

REMARK 2.1. In the proof of Theorem 1.4, we have relied on the local limit theorem to estimate each individual term $T_n(k)$ in $T_n^{(1)}$, as in [Don97]. However, it is also possible to give a direct proof for this term, bypassing the local limit theorem, more in the spirit of [GL62] and [MT10]. Indeed, using the integral formula for Fourier coefficients, one gets

$$T_n^{(1)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}\sum_{k>A(n/K)} R(e^{it})^k\,dt$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}R(e^{it})^{A(n/K)}(I-R(e^{it}))^{-1}\,dt.$$

Using as in the proof of Proposition 1.5 a reduction to a neighborhood of $t=0$, the change of variables $u = nt$ and the Lebesgue dominated convergence theorem, one obtains good asymptotics for this term. The main point of the computation is that $R(e^{it})^{A(n/K)}$ is small if t is not very close to 0, making everything uniformly integrable. If one tries to use the analogous formula for T_n, i.e.,

$$T_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}(I-R(e^{it}))^{-1}\,dt,$$

as in [GL62] and [MT10], one loses the uniform integrability, and the contribution far away from 0 becomes more complicated to control. This is why we need to resort to a different technique to handle $T_n^{(2)}$.

3. Large deviations: the strategy. In this section, we describe Doney's strategy to prove Theorem 1.6. Consider some $k \geq 3$ (since the theorem is trivial for any fixed k, we can assume without loss of generality that k is as large as we like), and $n \geq a_k$. We write $n = wa_k$ for some $w \geq 1$. Introduce a truncation level $\zeta = w^\gamma a_k/2 \in [a_k/2, n/2]$, where $\gamma \in (0,1)$ is close enough to 1 (how close will be specified below). We expand $T_n(k) = \sum_{j_1+\cdots+j_k=n} R_{j_1}\cdots R_{j_k}$, and we will estimate separately the contributions of different $(j_1,\dots,j_k)$, depending on the size of the different indices.

More precisely, the set $J = \{(j_1,\dots,j_k) : j_1+\cdots+j_k = n\}$ is partitioned into four disjoint subsets as follows: $J = J_3 \cup J_2 \cup J_1 \cup J_0$ where

$$\begin{aligned}
J_3 &= \{(j) \in J : \exists p,\ j_p \geq n/2\},\\
J_2 &= \{(j) \in J : \forall p,\ j_p < n/2 \text{ and } \exists u < v \text{ with } j_u, j_v \geq \zeta\},\\
J_1 &= \{(j) \in J : \forall p,\ j_p < n/2 \text{ and } \exists! u \text{ with } j_u \geq \zeta\},\\
J_0 &= \{(j) \in J : \forall p,\ j_p < \zeta\}.
\end{aligned}$$

We will show that the sums

$$\Sigma_i = \sum_{(j)\in J_i} R_{j_1}\cdots R_{j_k}$$

satisfy an inequality $\|\Sigma_i\| \leq Ckn^{-\beta-1}\ell(n)$ for $i = 0,\dots,3$. This will conclude the proof.

To compute these norms, we will use specific estimates for the indices which are relatively large (i.e., $\geq \zeta$), but we will also need to control the terms with small indices. For them, in the scalar case, it suffices to use the trivial fact that a probability measure has mass 1, implying that $\sum_j \mathbb{P}(S_k = j) = 1$ for all k. The analogue of this fact in the non-commutative setting reads $\|R(z)^k\|_A \leq C$, where C is a constant independent of k and we write $\|\sum F_p z^p\|_A = \sum \|F_p\|$. It turns out that this crucial a priori estimate is highly non-trivial in our situation. The next lemma gives a slightly strengthened version of this estimate.

For any $s \in \mathbb{N} \cup \{\infty\}$, we define a truncated series $R^{(s)}(z) = \sum_{j<s} R_j z^j$ (for $s = \infty$, this is simply $R(z)$).

Lemma 3.1. *There exists a constant C such that, for all $k \in \mathbb{N}$ and all $s \in [a_k/2, \infty]$,*

$$\|R^{(s)}(z)^k\|_A \leq C.$$

Let us stress that we do not know if this lemma holds for every renewal sequence of operators in the sense of Definition 1.3 (even for $s = \infty$, i.e., the non-truncated series): we really need the full strength of the assumption $\|R_n\| \leq Cn^{-\beta-1}\ell(n)$ to prove this estimate. It is the most complicated step of the proof in the operator situation (while it is completely trivial in the scalar case). The introduction of the truncation level s is merely for technical reasons in the argument below. The proof of this lemma is given in Section 4.

Let us introduce a convenient notation for the coefficient of z^j in a power series: we write $c_j(\sum F_p z^p) = F_j$. By the very definition of the set J_0, the sum over J_0 is simply $c_n(R^{(\zeta)}(z)^k)$. The hardest part of Doney's argument is an estimate of this term. Let us formulate the main estimate we will need in this direction.

LEMMA 3.2. *There exists a constant C such that, for all $k \in \mathbb{N}$ and all $s \in [a_k/2, \infty]$, for all $n \in \mathbb{N}$,*

$$\|c_n(R^{(s)}(z)^k)\| \leq Ce^{-n/s}/a_k.$$

This lemma is proved in Section 5.

Assuming those two results, we will now prove Theorem 1.6. The arguments here are similar to those of Doney, with an important difference: since we are dealing with (non-commutative) products of operators, we have to keep track of the indices where summands are large.

Bounding Σ_3. When $(j_1, \ldots, j_k) \in J_3$ and $k \geq 3$, there is at most one index u where $j_u \geq n/2$. We can thus decompose J_3 according to the value of u, to obtain

$$\Sigma_3 = \sum_{u=1}^{k} c_n\Big(R(z)^{u-1}\Big(\sum_{j=n/2}^{\infty} R_j z^j\Big)R(z)^{k-u}\Big).$$

Therefore,

$$\begin{aligned}
\|\Sigma_3\| &\leq \sum_{u=1}^{k} \sum_{p+q\leq n/2} \|c_p(R(z)^{u-1})\| \, \|c_q(R(z)^{k-u})\| \sup_{j\geq n/2} \|R_j\| \\
&\leq \sum_{u=1}^{k} \|R(z)^{u-1}\|_A \|R(z)^{k-u}\|_A Cn^{-1-\beta}\ell(n).
\end{aligned}$$

By Lemma 3.1, this is bounded by $Ckn^{-1-\beta}\ell(n)$, as desired.

Bounding Σ_2. For $(j_1, \ldots, j_k) \in \Sigma_2$, we can consider the first index u such that $j_u \geq \zeta$, and the first index $v > u$ such that $j_v \geq \zeta$. This gives a partition of Σ_2. Hence, Σ_2 is equal to the sum over $u<v$ of

$$c_n\Big(R^{(\zeta)}(z)^{u-1}\Big(\sum_{i=\zeta}^{n/2-1} R_i z^i\Big)R^{(\zeta)}(z)^{v-u-1}\Big(\sum_{j=\zeta}^{n/2-1} R_j z^j\Big)R^{(n/2)}(z)^{k-v}\Big).$$

We expand the product as for Σ_3, to get a bound

$$\|R^{(\zeta)}(z)^{u-1}\|_A \cdot \sup_{i\geq\zeta}\|R_i\|\cdot\|R^{(\zeta)}(z)^{v-u-1}\|_A\cdot\Big\|\sum_{j=\zeta}^{n/2-1} R_j z^j\Big\|_A\cdot\|R^{(n/2)}(z)^{k-v}\|_A.$$

Thanks to the a priori bounds given by Lemma 3.1, this is bounded by $C\zeta^{-\beta-1}\ell(\zeta)\cdot\zeta^{-\beta}\ell(\zeta)$. After a summation over u and v, we get

$$\|\Sigma_2\|\leq Ck^2\zeta^{-2\beta-1}\ell(\zeta)^2.$$

Since $k\sim a_k^\beta/\ell(a_k)$ and $\zeta/a_k=w^\gamma/2$, $n/\zeta=2w^{1-\gamma}$, this can be written as

$$\begin{aligned}\|\Sigma_2\|&\leq Ck\cdot\frac{a_k^\beta}{\ell(a_k)}\frac{\ell(\zeta)}{\zeta^\beta}\cdot\frac{\ell(\zeta)}{\zeta^{\beta+1}}\frac{n^{\beta+1}}{\ell(n)}\cdot n^{-\beta-1}\ell(n)\\&\leq Ck\cdot w^{-\beta\gamma\pm\varepsilon}\cdot w^{(1-\gamma)(\beta+1)\pm\varepsilon}\cdot n^{-\beta-1}\ell(n),\end{aligned}$$

thanks to Potter bounds. If $\beta\gamma>(1-\gamma)(\beta+1)$ (i.e., $\gamma>(1+\beta)/(1+2\beta)$), the exponent of w can be made negative by choosing ε small enough. This gives a bound $Ckn^{-\beta-1}\ell(n)$ as desired.

Bounding Σ_1. In Σ_1, there is a unique index u such that $j=j_u>\zeta$. We can therefore decompose Σ_1 according to the value of u, to obtain

$$\Sigma_1=\sum_{u=1}^{k}c_n\Big(R^{(\zeta)}(z)^{u-1}\Big(\sum_{j=\zeta}^{n/2-1}R_j z^j\Big)R^{(\zeta)}(z)^{k-u}\Big).$$

Expanding the product, we obtain

$$\|\Sigma_1\|\leq\sum_{u=1}^{k}\sum_{p+j+q=n}\|c_p(R^{(\zeta)}(z)^{u-1})\|\,\|R_j\|\,\|c_q(R^{(\zeta)}(z)^{k-u})\|, \tag{3.1}$$

where the index j has to belong to $[\zeta,n/2)$.

If $w\leq k$, we need to treat separately the case where u is too close to 1 or k, i.e., $u\leq k/w$ or $k+1-u\leq k/w$. For such a u, the corresponding term in the sum is at most

$$\begin{aligned}&\sum_{p,q}\|c_p(R^{(\zeta)}(z)^{u-1})\|\,\|c_q(R^{(\zeta)}(z)^{k-u})\|\sup_{j\geq\zeta}\|R_j\|\\&\qquad\leq\|R^{(\zeta)}(z)^{u-1}\|_A\|R^{(\zeta)}(z)^{k-u}\|_A\zeta^{-\beta-1}\ell(\zeta).\end{aligned}$$

By Lemma 3.1, the A-norms are bounded. Since there are at most $2k/w$

such terms, their overall contribution is bounded by

$$\begin{aligned}Ckw^{-1}\zeta^{-\beta-1}\ell(\zeta) &= Ckw^{-1}\cdot\frac{\ell(\zeta)}{\zeta^{\beta+1}}\,\frac{n^{\beta+1}}{\ell(n)}\cdot n^{-\beta-1}\ell(n)\\ &\le Ckw^{-1}w^{(1-\gamma)(\beta+1)\pm\varepsilon}\cdot n^{-\beta-1}\ell(n),\end{aligned}$$

since $n/\zeta = w^{1-\gamma}/2$. If γ is close enough to 1 (i.e., $\gamma > \beta/(1+\beta)$), the exponent of w in this term is negative, hence this contribution is bounded by $Ckn^{-\beta-1}\ell(n)$ as desired.

Assume now that $u \in [k/w, k+1-k/w]$. In (3.1), $p+j+q=n$ and $j < n/2$, hence $p \ge n/4$ or $q \ge n/4$. We will handle the part of the sum where $p \ge n/4$, the other one is similar. We have

$$\begin{aligned}&\sum_{\substack{p+j+q=n\\ p\ge n/4}} \|c_p(R^{(\zeta)}(z)^{u-1})\|\,\|R_j\|\,\|c_q(R^{(\zeta)}(z)^{k-u})\|\\ &\qquad\le \Big(\sup_{p\ge n/4}\|c_p(R^{(\zeta)}(z)^{u-1})\|\Big)\cdot\Big\|\sum_{j=\zeta}^{n/2-1}R_j z^j\Big\|_A\cdot\|R^{(\zeta)}(z)^{k-u}\|_A.\end{aligned}$$

By Lemma 3.2, $\sup_{p\ge n/4}\|c_p(R^{(\zeta)}(z)^{u-1})\| \le Ce^{-n/(4\zeta)}/a_{u-1}$. Moreover, we have $\|\sum_{j=\zeta}^{n/2-1}R_j z^j\|_A \le C\zeta^{-\beta}\ell(\zeta)$, and the last A-norm is bounded by Lemma 3.1. Therefore, since $1/a_{u-1} \le C/a_u$, this term is bounded by

$$Ce^{-n/(4\zeta)}\zeta^{-\beta}\ell(\zeta)/a_u.$$

Since $u \ge k/w$ and a_i is a regularly varying function of i of index $1/\beta$, we have $a_u \ge Ca_k/w^{1/\beta+\varepsilon}$. Since $a_k = n/w$ and $n/\zeta = 2w^{1-\gamma}$, we finally get a bound of the form

$$Ce^{-w^{1-\gamma}/2}w^{C'}n^{-\beta-1}\ell(n)$$

for some constant C'. Since $\gamma < 1$, this is bounded by $Cn^{-\beta-1}\ell(n)$. Summing over the at most k possible values of u, we get the result.

Bounding Σ_0. This is easier, since $\Sigma_0 = c_n(R^{(\zeta)}(z)^k)$. By Lemma 3.2, it is bounded by $Ce^{-n/\zeta}/a_k = Ce^{-2w^{1-\gamma}}/a_k$, to be compared with the desired upper bound

$$kn^{-\beta-1}\ell(n) \sim \frac{a_k^\beta}{\ell(a_k)}\cdot(wa_k)^{-\beta-1}\ell(wa_k) = w^{-\beta-1\pm\varepsilon}/a_k.$$

The result follows. ∎

4. Proof of the a priori estimates. The statement of Lemma 3.1 deals with functions $\sum_{j\ge0}F_j z^j$ defined on the whole unit disk. However, in the proof, it will be important to work with series $\sum_{j\in\mathbb{Z}}F_j z^j$ defined only on S^1 (to be able to introduce partitions of unity). We will write

$\|\sum F_j z^j\|_A = \sum \|F_j\|$ both in the unilateral and bilateral contexts. When convenient, we will use the variable $t \in \mathbb{R}/2\pi\mathbb{Z}$ instead of z.

For $\varepsilon > 0$, let $\Delta_\varepsilon(t) = \min(0, 1 - |t|/\varepsilon)$ be the piecewise affine function that vanishes outside of $[-\varepsilon, \varepsilon]$ and takes the value 1 at 1. We will use such functions to construct partitions of unity on S^1.

LEMMA 4.1. *If $\|F_n\| \leq |n|^{-1-\beta}\ell(|n|)$ and $\sum_{n\in\mathbb{Z}} F_n = 0$, then*

$$\left\|\Delta_\varepsilon \cdot \sum F_n e^{int}\right\|_A \leq C\varepsilon^\beta \ell(\varepsilon^{-1}),$$

where C only depends on β and ℓ.

Proof. Define a real function G by $G(x) = \sin^2(x/2)/x^2$. An easy computation (see, e.g., [Kah70, p. 9]) shows that the Fourier coefficients of Δ_ε are given by

$$c_n(\Delta_\varepsilon) = \frac{2}{\pi}\varepsilon G(\varepsilon n).$$

It is easy to check that G is C^∞, with $|G'(x)| \leq C\min(|x|, 1/x^2)$. Therefore,

$$|G(x) - G(y)| \leq C|x - y|\min(|x| + |y|, 1/|x|^2 + 1/|y|^2).$$

We have

$$\begin{aligned}\Delta_\varepsilon \cdot \sum F_n e^{int} &= \Big(\sum c_k(\Delta_\varepsilon)e^{ikt}\Big) \cdot \Big(\sum F_n(e^{int} - 1)\Big)\\ &= \sum_{n,k} F_n(c_{k-n}(\Delta_\varepsilon) - c_k(\Delta_\varepsilon))e^{ikt}.\end{aligned}$$

To conclude, it is therefore sufficient to bound

$$\begin{aligned}\sum_{n,k}\frac{\ell(|n|)}{|n|^{\beta+1}}\varepsilon|G(\varepsilon(k-n)) - G(\varepsilon k)| &= \sum_{|n|\leq 1/\varepsilon,\, |k|\leq 2/\varepsilon} + \sum_{|n|\leq 1/\varepsilon,\, |k|>2/\varepsilon} + \sum_{|n|>1/\varepsilon,\, k}\\ &= \Sigma_1 + \Sigma_2 + \Sigma_3.\end{aligned}$$

For Σ_1, we bound $|G(\varepsilon(k-n)) - G(\varepsilon k)|$ by $C\varepsilon^2|n|(|k| + |n|)$. This yields

$$\begin{aligned}\Sigma_1 &\leq C\sum_{|n|\leq 1/\varepsilon}\frac{\ell(|n|)}{|n|^{\beta+1}}\varepsilon^3|n|\Big(\sum_{|k|\leq 2/\varepsilon}|k| + |n|\Big)\\ &\leq C\sum_{|n|\leq 1/\varepsilon}\frac{\ell(|n|)}{|n|^{\beta+1}}\varepsilon^3|n|/\varepsilon^2 \leq C\varepsilon^\beta\ell(\varepsilon^{-1}).\end{aligned}$$

For Σ_2, we bound $|G(\varepsilon(k-n)) - G(\varepsilon k)|$ by $C\varepsilon^{-1}|n|(1/|k-n|^2 + 1/|k|^2) \leq C\varepsilon^{-1}|n|/k^2$ (since $|k - n| \geq |k|/2$). We obtain

$$\Sigma_2 \leq C\sum_{|n|\leq 1/\varepsilon}\frac{\ell(|n|)}{|n|^{\beta+1}}\Big(\sum_{|k|>2/\varepsilon}|n|/k^2\Big) \leq C\varepsilon\sum_{|n|\leq 1/\varepsilon}\frac{\ell(|n|)}{|n|^\beta} \leq C\varepsilon^\beta\ell(\varepsilon^{-1}).$$

Finally,

$$\Sigma_3 \leq C \sum_{|n|>1/\varepsilon} \frac{\ell(|n|)}{|n|^{\beta+1}} \cdot 2\varepsilon \sum_k |G(\varepsilon k)| \leq C \sum_{|n|>1/\varepsilon} \frac{\ell(|n|)}{|n|^{\beta+1}} \|\Delta_\varepsilon\|_A \leq C\varepsilon^\beta \ell(\varepsilon^{-1}),$$

since $\|\Delta_\varepsilon\|_A$ is bounded independently of ε: as the Fourier coefficients of Δ_ε are non-negative, this norm is simply equal to $\Delta_\varepsilon(0) = 1$. ∎

We can now prove Lemma 3.1 for non-truncated series (i.e., $s = \infty$):

Lemma 4.2. *There exists a constant C such that, for all $k \in \mathbb{N}$,*

$$\|R(z)^k\|_A \leq C.$$

Proof. In this proof, we will denote by $O_{L,\beta}(\mathcal{B})$ the set of power series $\sum_{n\in\mathbb{Z}} F_n z^n$ such that F_n is an operator on the Banach space $\mathcal{B}$ with $\|F_n\| \leq C|n|^{-\beta-1}\ell(|n|)$. The norm in $O_{L,\beta}(\mathcal{B})$ is then the best such C. This is a Banach algebra, i.e., it is stable under multiplication. Moreover, a crucial fact about this space is the following Wiener lemma: the spectrum of $\sum F_n z^n \in O_{L,\beta}(\mathcal{B})$ is the union of the spectra of all the operators $\sum F_n z^n$ for $z \in S^1$ (see for instance [Gou04b, paragraphe 2.2.4]). In particular, if $\sum F_n z^n$ is an invertible operator for each $z \in S^1$, then its pointwise inverse still belongs to $O_{L,\beta}(\mathcal{B})$. Similarly, we will write $O_{L,\beta}(\mathbb{C})$ for $\mathbb{C}$-valued power series with the same condition on the modulus of the Taylor coefficients (this is a special instance of the previous space, with $\mathcal{B} = \mathbb{C}$).

Given $\delta > 0$ small enough, we define for z close to 1 the spectral projector

$$P(z) = \frac{1}{2i\pi} \int_{|u-1|=\delta} (uI - R(z))^{-1}\, du$$

associated to the eigenvalue $\lambda(z)$ close to 1 of $R(z)$. Therefore, we can write $R(z) = \lambda(z)P(z) + Q(z)$, where the spectrum of $Q(z)$ is included in a disk of radius < 1.

We will first give the proof assuming that $P(z)$ is well defined for every $z \in S^1$ and that it is close to $P(1)$. In this case, since R belongs to the Banach algebra $O_{L,\beta}(\mathcal{B})$, this is also the case of P (this follows from the integral formula for P and the Wiener property of $O_{L,\beta}(\mathcal{B})$). Consider $\xi \in \mathcal{B}^*$ and $\eta \in \mathcal{B}$ such that $\langle \xi, P(1)\eta\rangle \neq 0$. We obtain $\lambda(z) = \langle \xi, R(z)P(z)\eta\rangle / \langle \xi, P(z)\eta\rangle$ for $z \in S^1$. Therefore, $\lambda \in O_{L,\beta}(\mathbb{C})$ again by the Wiener lemma.

For every z, the spectrum of $Q(z)$ (as an operator acting on $\mathcal{B}$) is contained in a disk of radius < 1. The Banach algebra A also has the Wiener property. Therefore, the spectrum of Q (as an element of the Banach algebra A) is also contained in a disk of radius < 1. Hence, $\|Q^k\|_A$ decays exponentially fast in k. Since $R^k = \lambda^k P + Q^k$, it suffices to prove that $\|\lambda^k\|_A$ is bounded independently of k to conclude.

The idea of the proof is to use a clever partition of unity, and estimate the A-norm of λ^k on each piece of the partition of unity. Let us write $V_\varepsilon = 2\Delta_{2\varepsilon} - \Delta_\varepsilon$: this function is equal to 1 on $[-\varepsilon, \varepsilon]$, vanishes outside of $[-2\varepsilon, 2\varepsilon]$ and is affine in between. Let also $\Delta_{\varepsilon,x}(y) = \Delta_\varepsilon(x-y)$ and $V_{\varepsilon,x}(y) = V_\varepsilon(y-x)$.

Consider a fixed integer k. Let b_k be an even integer such that $b_k \sim a_k$, i.e., $k\ell(b_k) \sim b_k^\beta$, and let $\varepsilon_k = 2\pi/b_k$. Let $x_j = j\varepsilon_k$ for $-b_k/2 \leq j < b_k/2$, so that $\sum_j \Delta_{\varepsilon_k,x_j} = 1$ on $\mathbb{R}/2\pi\mathbb{Z}$. Moreover, V_{ε_k,x_j} is equal to 1 on the support of $\Delta_{\varepsilon_k,x_j}$. Therefore, we have

$$\lambda(z)^k = \sum_j \lambda(z)^k \Delta_{\varepsilon_k,x_j} = \sum_j \big(\lambda(x_j) + (\lambda(z) - \lambda(x_j))V_{\varepsilon_k,x_j}(z)\big)^k \Delta_{\varepsilon_k,x_j}.$$

As $\|\Delta_{\varepsilon_k,x_j}\|_A \leq 1$ and A is a Banach algebra, we obtain

$$\|\lambda(z)^k\|_A \leq \sum_j \big(|\lambda(x_j)| + \|(\lambda(z) - \lambda(x_j))V_{\varepsilon_k,x_j}(z)\|_A\big)^k.$$

By Lemma 4.1, since $\lambda \in O_{L,\beta}(\mathbb{C})$, we have $\|(\lambda(z) - \lambda(x_j))V_{\varepsilon_k,x_j}(z)\|_A \leq C\varepsilon_k^\beta \ell(\varepsilon_k^{-1}) \leq C/k$. Hence,

$$\|\lambda(z)^k\|_A \leq \sum_j (|\lambda(x_j)| + C/k)^k.$$

Since $|\lambda(x_j)|$ is bounded from below,

$$|\lambda(x_j)| + C/k \leq |\lambda(x_j)|(1+C'/k) = |\lambda(2\pi j/b_k)|(1+C'/k) \leq e^{-C|j|^{\beta/2}/k} \cdot e^{C'/k}$$

by (2.2). When j is large enough, say $|j| > M$, we get $|\lambda(x_j)| + C/k \leq e^{-C''|j|^{\beta/2}/k}$, while for $|j| \leq M$ we simply have the trivial bound $1 + C/k$. This gives

$$\|\lambda(z)^k\|_A \leq \sum_{|j|\leq M} (1 + C/k)^k + \sum_{|j|>M} (e^{-C''|j|^{\beta/2}/k})^k.$$

Since this quantity is bounded independently of k, this proves the lemma under the assumption that the eigenvalue $\lambda(z)$ is everywhere well defined.

The general case reduces to the previous one using partitions of unity, as follows. We first define a function $\tilde{R}(z)$ which coincides with $R(z)$ for z close to 1, such that $\tilde{R}(z)$ is everywhere close to $R(1)$ (therefore, it has a unique eigenvalue $\tilde{\lambda}(z)$ close to 1). We can also make sure that $|\tilde{\lambda}(z)| < 1$ for $z \neq 1$. This construction is easily made using smooth partitions of unity (see for instance Step 3 of the proof of Theorem 2.2.5 in [Gou04b]). In particular, $\tilde{R}$ still belongs to the Banach algebra $O_{L,\beta}(\mathcal{B})$. The previous argument applies to $\tilde{R}$ to show that $\|\tilde{R}(z)^k\|_A$ is uniformly bounded. Let us also define another function $\bar{R}(z)$, which coincides with $R(z)$ outside of a small neighborhood of 1, and such that the spectrum of $\bar{R}(z)$ is contained for every z in a disk of radius < 1. By the Wiener property, $\|\bar{R}(z)^k\|_A$ decays exponentially fast.

Finally, let us consider a smooth partition of unity $\varphi + \psi = 1$ such that $R = \tilde{R}$ on the support of φ and $R = \bar{R}$ on the support of ψ. This gives $R^k = \varphi \cdot \tilde{R}^k + \psi \cdot \bar{R}^k$. Since the A-norm of all those terms is uniformly bounded, we get the same estimate for R^k. ∎

Proof of Lemma 3.1. We will use the following general fact in Banach algebras: if f and g are two elements of a Banach algebra and $C_0 \geq 1$ is such that $\|f^k\| \leq C_0$ for all k, then

$$\|(f+g)^k\| \leq C_0(1 + C_0\|g\|)^k. \tag{4.1}$$

To prove this estimate, let us develop the product in $(f+g)^k$ and consider one of the resulting terms $h_1 \cdots h_k$ with $h_i \in \{f, g\}$. The number of blocks of consecutive fs is bounded by $j+1$ where j is the number of g factors. Therefore, the norm of $h_1 \cdots h_k$ is at most $C_0^{j+1}\|g\|^j$ (since each block of consecutive fs gives a contribution at most C_0 by assumption). Summing over all possible terms, we obtain

$$\|(f+g)^k\| \leq \sum_{j=0}^{k} \binom{k}{j} C_0 \cdot (C_0\|g\|)^j = C_0(1 + C_0\|g\|)^k.$$

This proves (4.1).

In our case, we apply this estimate in the Banach algebra A to $f = R$ (by Lemma 4.2, its powers indeed have uniformly bounded norm) and $g = -\sum_{n=s}^{\infty} R_n z^n$, whose norm is bounded by $C\sum_{n=s}^{\infty} n^{-\beta-1}\ell(n) \leq Cs^{-\beta}\ell(s)$, which is bounded by C/k when $s \geq a_k/2$. Therefore, we get, from (4.1),

$$\|R^{(s)}(z)^k\|_A \leq C(1 + C/k)^k \leq C'. \ ∎$$

5. Proof of truncated series bounds. In this section, we prove Lemma 3.2. Let us first note that the result is trivial for fixed k, since $c_n(R^{(s)}(z)^k)$ vanishes for $n \geq ks$, while $e^{-n/s}/a_k$ is bounded from below by e^{-k}/a_k for $n \leq ks$. Therefore, it is sufficient to prove the result for large k.

We should estimate the coefficient of z^n in $R^{(s)}(z)^k$, which can be written as

$$\frac{1}{2\pi} \int_{|z|=1} R^{(s)}(z)^k z^{-n-1}\, dz.$$

The truncated series $R^{(s)}(z)$ is a polynomial, and is therefore holomorphic in the whole complex plane. By holomorphy, for any $\rho > 0$, the last equation is equal to

$$\frac{1}{2\pi} \int_{|z|=e^{\rho}} R^{(s)}(z)^k z^{-n-1}\, dz = \frac{e^{-\rho n}}{2\pi} \int_{|z|=1} R^{(s)}(ze^{\rho})^k z^{-n-1}\, dz.$$

We should choose ρ so that the integral can be well estimated. Doney chooses

ρ in an implicit way and then has to study its asymptotics. It seems much simpler to directly choose $\rho = 1/s$. Then the term $e^{-\rho n}$ gives the desired asymptotics $e^{-n/s}$. To conclude, it is sufficient to prove that

$$\int_{|z|=1} \|R^{(s)}(ze^{\rho})^k\| |\mathrm{d}z| \leq C/a_k. \tag{5.1}$$

We first estimate $R^{(s)}(ze^{\rho}) - R(z)$ for $|z| = 1$:

$$\begin{aligned}
\|R^{(s)}(ze^{\rho}) - R(z)\| &= \Big\| \sum_{j=1}^{s-1} z^j R_j (e^{\rho j} - 1) - \sum_{j \geq s} z^j R_j \Big\| \\
&\leq C \sum_{j=1}^{s-1} j^{-\beta-1} \ell(j) \rho j e^{\rho j} + C \sum_{j=s}^{\infty} j^{-\beta-1} \ell(j) \\
&\leq C s^{1-\beta} \ell(s) \rho e^{\rho s} + C s^{-\beta} \ell(s).
\end{aligned}$$

Since $\rho = 1/s$ with $s \geq a_k/2$, this is bounded by

$$C s^{-\beta} \ell(s) \leq C a_k^{-\beta} \ell(a_k) \leq C/k.$$

This tends to 0 as $k \to \infty$. For t in a small neighborhood of 0 (say $|t| \leq \delta$) and large enough k, we deduce that $R^{(s)}(e^{it}e^{\rho})$ has a unique eigenvalue $\mu(t, \rho)$ close to 1, and moreover

$$\|R^{(s)}(e^{it}e^{\rho})^k\| \leq C|\mu(t, \rho)|^k, \quad |\mu(t, \rho) - \lambda(t)| \leq C/k.$$

For t outside of this neighborhood, the spectral radius of the operators $R(e^{it})$ is uniformly less than 1. By continuity, we deduce that $\|R^{(s)}(e^{it}e^{\rho})^k\|$ decays exponentially in k, uniformly for $|t| \geq \delta$.

We use these estimates to bound the integral (5.1), which can also be written as

$$\int_{-\pi}^{\pi} \|R^{(s)}(e^{it}e^{\rho})^k\| \, dt.$$

The contribution of the set $|t| \geq \delta$ is exponentially small in k, and therefore smaller than C/a_k.

Since $|\lambda(t)| \leq 1$, we have $|\mu(t, \rho)| \leq 1 + C/k$, hence $\|R^{(s)}(e^{it}e^{\rho})^k\| \leq C$. As a consequence, the contribution of the set $\{t : |t| \leq C_1/a_k\}$ is bounded by CC_1/a_k for any $C_1 > 0$.

Finally, for $|t| \in [C_1/a_k, \delta]$, we have

$$|\mu(t, \rho)| \leq |\lambda(t)| + C/k \leq 1 - c|t|^{\beta} \ell(1/|t|) + C/k.$$

Moreover, if C_1 is large, $|t|^{\beta} \ell(1/|t|) \geq (C_1/a_k)^{\beta} \ell(a_k/C_1)$ is much larger than $(1/a_k)^{\beta} \ell(a_k) \sim 1/k$. Therefore, in the last equation, the second term is dominated by the first one if C_1 is large enough, and we get

$$|\mu(t, \rho)| \leq 1 - c'|t|^{\beta} \ell(1/|t|).$$

Hence,

$$\int_{|t|\in[C_1/a_k,\delta]} \|R^{(s)}(e^{it}e^{\rho})^k\|\,dt \le C\int_{C_1/a_k}^{\delta}(1-c't^{\beta}\ell(1/t))^k\,dt$$
$$\le C\int_{C_1/a_k}^{\delta} e^{-c'kt^{\beta}\ell(1/t)}\,dt.$$

Writing $u = ta_k$ and using the asymptotics $k \sim a_k^{\beta}/\ell(a_k)$ and Potter bounds, we find that this is at most

$$C\int_{C_1}^{\delta/a_k} e^{-c'u^{\beta/2}}\,\frac{du}{a_k}.$$

This is bounded by C/a_k as desired. ■

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