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ON THE DUNFORD–PETTIS PROPERTY OF TENSOR PRODUCT SPACES

BY

IOANA GHENCIU (River Falls, WI)

Abstract. We give sufficient conditions on Banach spaces E and F so that their projective tensor product $E \otimes_\pi F$ and the duals of their projective and injective tensor products do not have the Dunford–Pettis property. We prove that if E^* does not have the Schur property, F is infinite-dimensional, and every operator $T : E^* \to F^{**}$ is completely continuous, then $(E \otimes_\epsilon F)^*$ does not have the DPP. We also prove that if E^* does not have the Schur property, F is infinite-dimensional, and every operator $T : F^{**} \to E^*$ is completely continuous, then $(E \otimes_\pi F)^* \simeq L(E, F^*)$ does not have the DPP.

1. Definitions and notation. Our notation and terminology are standard. Throughout this paper, X, Y, E, and F will denote Banach spaces. The unit ball of X will be denoted by B_X, and X^* will denote the continuous linear dual of X. The set of all operators from X to Y will be denoted by $L(X, Y)$, and the compact, resp. completely continuous operators will be denoted by $K(X, Y)$, resp. $CC(X, Y)$. The operator T is *completely continuous* (or Dunford–Pettis) if T maps weakly Cauchy sequences to norm convergent sequences, and T is *unconditionally converging* if T maps weakly unconditionally convergent (wuc) series to unconditionally convergent series.

A Banach space X has the *Dunford–Pettis property* (DPP) if every weakly compact operator T with domain X is completely continuous. Equivalently, X has the DPP if and only if $x_n^*(x_n) \to 0$ for all weakly null sequences (x_n) in X and (x_n^*) in X^* (see [19]). Schur spaces, $C(K)$ spaces, and $L^1(\mu)$ spaces have the DPP ([2], [29], [23]). Reflexive infinite-dimensional spaces do not have the DPP. The Dunford–Pettis property is stable under complemented subspaces. If E^* has the DPP, then E has the DPP [19]. A dual Banach space E^* has the Schur property if and only if E has the DPP and $\ell_1 \not\hookrightarrow E$ ([27], [33]). The reader can check Diestel [18], [19], Diestel and Uhl [21], and Andrews [1] for a guide to the extensive classical literature dealing with the DPP, equivalent formulations of the preceding definitions, and undefined notation and terminology.

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A bounded subset A of X (resp. A of X^*) is called a V^*-*subset* of X (resp. a V-*subset* of X^*) provided that

$$\lim_n \sup\{|x_n^*(x)| : x \in A\} = 0$$
$$(\text{resp. } \lim_n \sup\{|x^*(x_n)| : x^* \in A\} = 0)$$

for each wuc series $\sum x_n^*$ in X^* (resp. wuc series $\sum x_n$ in X). A Banach space X has *property* (V) if every V-subset of X^* is relatively weakly compact, and X has *property* (V^*) if every V^*-subset of X is relatively weakly compact [31]. A Banach space X has property (V) if and only if every unconditionally converging operator with domain X is weakly compact [31]. Pełczyński [31] proved that $C(K)$ spaces have property (V).

A topological space S is called *scattered* (or *dispersed*) if every nonempty closed subset of S has an isolated point [38]. A compact Hausdorff space K is scattered if and only if $\ell_1 \not\hookrightarrow C(K)$ (see [32]).

The Banach–Mazur distance $d(E,F)$ between two isomorphic Banach spaces E and F is defined by $\inf(\|T\|\|T^{-1}\|)$, where the infimum is taken over all isomorphisms T from E onto F. A Banach space E is called an $\mathcal{L}_\infty$-*space* (resp. $\mathcal{L}_1$-*space*) [8] if there is a $\lambda \geq 1$ so that every finite-dimensional subspace of E is contained in another subspace N with $d(N, \ell_\infty^n) \leq \lambda$ (resp. $d(N, \ell_1^n) \leq \lambda$) for some integer n. Complemented subspaces of $C(K)$ spaces (resp. $L_1(\mu)$ spaces) are $\mathcal{L}_\infty$-spaces (resp. $\mathcal{L}_1$-spaces) [8, Proposition 1.26]. The dual of an $\mathcal{L}_1$-space (resp. $\mathcal{L}_\infty$-space) is an $\mathcal{L}_\infty$-space (resp. $\mathcal{L}_1$-space) [8, Proposition 1.27]. Complemented subspaces of $\mathcal{L}_\infty$-spaces (resp. $\mathcal{L}_1$-spaces) are $\mathcal{L}_\infty$-spaces (resp. $\mathcal{L}_1$-spaces) [8, Proposition 1.28]. The $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the DPP [8, Corollary 1.30].

Several authors studied whether the DPP of E and F implies the DPP of their projective tensor product $E \otimes_\pi F$ and of their injective tensor product $E \otimes_\epsilon F$ ([1], [5], [9], [14], [13], [22], [24], [26], [28], [36], [40]). Talagrand [40] gave an example of a Banach space E so that E^* has the Schur property, but $C([0,1], E) \simeq C[0,1] \otimes_\epsilon E$ and $L^1([0,1], E^*) \simeq L^1[0,1] \otimes_\pi E^*$ do not have the DPP. Dobrakov [22] showed that if X is a Schur space, then $C(K,X) \simeq C(K) \otimes_\epsilon X$ has the DPP. Andrews [1] proved that if X^* is a Schur space, then $L_1(\mu, X) \simeq L_1(\mu) \otimes_\pi X$ has the DPP if μ is finite. Bourgain [9] showed that for any countable measure μ and any compact Hausdorff space K, the spaces $L_1(\mu, C(K))$, $C(K, L_1(\mu))$, and their duals have the DPP.

Ryan [36] proved that if E and F have the DPP and contain no copy of ℓ_1, then $E \otimes_\pi F$ has the DPP and contains no copy of ℓ_1. Bombal and Villanueva proved that if K_1 and K_2 are infinite compact Hausdorff spaces, then $C(K_1) \otimes_\pi C(K_2)$ has the DPP if and only if both K_1 and K_2 are scattered [5, Theorem 2.2]. González and Gutiérrez proved that if E does not have the Schur property, F contains a copy of ℓ_1, and $L(E,F^*) = CC(E,F^*)$, then $E \otimes_\pi F$ does not have the DPP [28, Theorem 3].

In this paper we give sufficient conditions on Banach spaces E and F so that the projective tensor product $E \otimes_\pi F$, the duals $(E \otimes_\pi F)^*$ and $(E \otimes_\epsilon F)^*$ of their projective and injective tensor products, and the bidual $(E \otimes_\pi F)^{**}$ of their projective tensor product do not have the DPP. We use the sequential characterization [19] of the DPP to show that, in some cases, these spaces fail to have the DPP. Our results generalize those in [5] and [28].

2. The DPP on tensor products. We start by studying the DPP on projective tensor products. We note that there are examples of spaces E and F so that the projective tensor product of E and F has the DPP. Specifically, since $\ell_1 \otimes_\pi \ell_1 \simeq \ell_1$ (see [35, p. 43]), $\ell_1 \otimes_\pi \ell_1$ has the Schur property, and thus the DPP. Ryan [36] proved that if E^* and F^* have the Schur property, then $(E \otimes_\pi F)^* \simeq L(E,F^*)$ has the Schur property. Hence $E \otimes_\pi F$ has the DPP. We will need the following results.

Observation 1. If E has the DPP and property (V) and F has property (V), then $L(E,F^*) = CC(E,F^*)$. Indeed, since F has property (V), F^* is weakly sequentially complete [31], and $c_0 \not\hookrightarrow F^*$. If $T : E \to F^*$ is an operator, then T is unconditionally converging [3]. Since E has property (V) and the DPP, T is weakly compact [31], and thus completely continuous.

Observation 2.

(i) If $\ell_1 \hookrightarrow X$, then X^* does not have the Schur property (since $\ell_1 \hookrightarrow X$ implies $L_1 \hookrightarrow X^*$; see [18, p. 212]).
(ii) If X is infinite-dimensional and has the Schur property, then X^* does not have the Schur property. Indeed, let (x_n) be a sequence of norm one elements of X such that $\|x_n - x_m\| \geq 1/2$ for all $n \neq m$. Then (x_n) has no norm Cauchy subsequence, hence no weakly Cauchy subsequence. By Rosenthal's ℓ_1-theorem, $\ell_1 \hookrightarrow X$, and (i) applies.
(iii) If X is infinite-dimensional, then X^{**} does not have the Schur property. Indeed, if X^{**} has the Schur property, so does X. Then $\ell_1 \hookrightarrow X$, hence $\ell_1 \hookrightarrow X^*$ (see [18, p. 211]), and (i) applies.

Observation 3. If X^* does not have the Schur property and X has the DPP, then $\ell_1 \hookrightarrow X$ (see [27], [33]).

Observation 4. If X is infinite-dimensional and has property (V), then X does not have the Schur property. To see this, suppose that X has the Schur property. Then $c_0 \not\hookrightarrow X$, and the identity map i on X is unconditionally converging [3]. Since X has property (V), i is weakly compact. Hence B_X is relatively (weakly) compact, a contradiction.

OBSERVATION 5. If E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces, then $L(E, F^*) = CC(E, F^*)$ (see [20, Theorems 3.7 and 2.17]).

LEMMA 1 ([5]). *Suppose that* $L(E, F^*) = CC(E, F^*)$, (x_n) *is a weakly null sequence in* E *and* (y_n) *is a bounded sequence in* F. *Then the sequence* $(x_n \otimes y_n)$ *is weakly null in* $E \otimes_\pi F$.

The following lemma is essentially contained in [28] (see the proof of Theorem 12 there); see also [21, p. 256]. We include a different proof for the convenience of the reader.

LEMMA 2 ([28]). *If* (x_n) *is weakly null in* E *and* (y_n) *is bounded in* F, *then* $(x_n \otimes y_n)$ *is weakly null in* $E \otimes_\epsilon F$.

Proof. Let (x_n) be weakly null in E and (y_n) be bounded in F. Then $\langle x_n \otimes y_n, x^* \otimes y^* \rangle = x^*(x_n)y^*(y_n) \to 0$ for all $x^* \in E^*$, $y^* \in F^*$. Consider the compact product space $B_{E^*} \times B_{F^*}$, where B_{E^*} and B_{F^*} are equipped with their compact w^*-topologies. Since E and F are closed subspaces of $C(B_{E^*})$ and $C(B_{F^*})$, $E \otimes_\epsilon F$ is a closed subspace of $C(B_{E^*} \times B_{F^*})$. By the Lebesgue dominated convergence theorem, $(x_n \otimes y_n)$ is weakly null in $C(B_{E^*} \times B_{F^*})$, hence in $E \otimes_\epsilon F$. ∎

THEOREM 3. *Suppose that* E *and* F^* *do not have the Schur property, and* $L(E, F^*) = CC(E, F^*)$. *Then* $E \otimes_\pi F$ *does not have the DPP.*

Proof. Let (x_n) be a weakly null sequence in E and (x_n^*) be a bounded sequence in E^* such that $x_n^*(x_n) = 1$ for all n. Let (y_n^*) be a weakly null sequence in F^* and (y_n) be a bounded sequence in F such that $y_n^*(y_n) = 1$ for all n. Then $\langle x_n^* \otimes y_n^*, x_n \otimes y_n \rangle = 1$. By Lemma 1, $(x_n \otimes y_n)$ is weakly null in $E \otimes_\pi F$. By Lemma 2, $(x_n^* \otimes y_n^*)$ is weakly null in $E^* \otimes_\epsilon F^*$, hence in $(E \otimes_\pi F)^* \simeq L(E, F^*)$. Then $E \otimes_\pi F$ does not have the DPP. ∎

COROLLARY 4 ([28, Theorem 3]). *Suppose that* E *does not have the Schur property,* $\ell_1 \hookrightarrow F$, *and* $L(E, F^*) = CC(E, F^*)$. *Then* $E \otimes_\pi F$ *does not have the DPP.*

Proof. By Observation 2(i), F^* does not have the Schur property. Apply Theorem 3. ∎

Actually Corollary 4 implies Theorem 3, so they are equivalent. Indeed, assume that Theorem 3 does not hold. In particular, F^* does not have the Schur property and $E \otimes_\pi F$ has the DPP. Hence F, which is isomorphic to a complemented subspace of $E \otimes_\pi F$, has the DPP. By Observation 3, $\ell_1 \hookrightarrow F$. Hence Corollary 4 does not hold.

The next result is similar to [28, Corollary 5]. We include the proof for the convenience of the reader.

Corollary 5. *Suppose that* E *and* F *are infinite-dimensional,* $L(E,F^*) = CC(E,F^*)$, $L(F,E^*) = CC(F,E^*)$, *and* $E\otimes_\pi F$ *has the DPP. Then either* E *and* F *have the Schur property, or* E^* *and* F^* *have the Schur property.*

Proof. If E has the Schur property, then E^* does not have the Schur property (by Observation 2(ii)). Since $F\otimes_\pi E \simeq E\otimes_\pi F$ has the DPP, Theorem 3 implies that F has the Schur property.

If E does not have the Schur property, Theorem 3 implies that F^* has the Schur property. Then F does not have the Schur property. Since $F\otimes_\pi E$ has the DPP, Theorem 3 implies that E^* has the Schur property. ∎

Corollary 6.

(i) *If* E *and* F *are infinite-dimensional,* $L(E^{**},F^*) = CC(E^{**},F^*)$, *and* $L(F,E^{***}) = CC(F,E^{***})$, *then* $E^{**}\otimes_\pi F$ *does not have the DPP.*
(ii) *If* E *and* F *are infinite-dimensional,* $L(E^{**},F^{***}) = CC(E^{**},F^{***})$, *and* $L(F^{**},E^{***}) = CC(F^{**},E^{***})$, *then* $E^{**}\otimes_\pi F^{**}$ *does not have the DPP.*

Proof. Suppose that $E^{**}\otimes_\pi F$ has the DPP. By Corollary 5, either E^{**} and F have the Schur property, or E^{***} and F^* have the Schur property. By Observation 2(iii), neither E^{**} nor E^{***} can have the Schur property.

(ii) Apply (i). ∎

If E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces, then they satisfy the hypotheses of Corollary 6 (by Observation 5), and so $E^{**}\otimes_\pi F$ and $E^{**}\otimes_\pi F^{**}$ do not have the DPP. The hypotheses of Corollary 6(i) (resp. (ii)) are also satisfied if E and F are infinite-dimensional spaces such that E^{**} and F (resp. F^{**}) have the DPP and property (V) (by Observation 1).

Examples of spaces with the DPP and property (V) are the disk algebra A, H^∞, $(H^\infty)^{**}$ [6], [15], and biduals of $\mathcal{L}_\infty$-spaces (since they are complemented in $C(K)$ spaces [8]). Thus, e.g., $(H^\infty)^{**}\otimes_\pi A$, $(H^\infty)^{**}\otimes_\pi H^\infty$, and $(H^\infty)^{**}\otimes_\pi (H^\infty)^{**}$ do not have the DPP.

Corollary 7. *Suppose that* E *and* F *have the DPP and do not have the Schur property,* $L(E,F^*) = CC(E,F^*)$, *and* $L(F,E^*) = CC(F,E^*)$. *Then the following are equivalent:*

(i) $\ell_1 \not\hookrightarrow E$ *and* $\ell_1 \not\hookrightarrow F$.
(ii) $(E\otimes_\pi F)^*$ *has the Schur property.*
(iii) $(E\otimes_\pi F)^*$ *has the DPP.*

(iv) *$E \otimes_\pi F$ has the DPP.*
(v) *E^* and F^* have the Schur property.*

Proof. (i)⇒(ii). By Observation 3, E^* and F^* have the Schur property. Then $(E \otimes_\pi F)^* \simeq L(E, F^*)$ has the Schur property [36].

(ii)⇒(iii) and (iii)⇒(iv) are clear. (iv)⇒(v) follows by Corollary 5. (v)⇒(i) is justified by Observation 2(i). ∎

The following result generalizes Theorem 2.2 of [5].

Corollary 8. *Suppose that E and F are infinite-dimensional spaces with the DPP and property (V). Then the conclusion of Corollary 7 is true.*

Proof. By Observation 1, we have $L(E, F^*) = CC(E, F^*)$ and $L(F, E^*) = CC(F, E^*)$. By Observation 4, E and F do not have the Schur property. Apply Corollary 7. ∎

Example. The spaces A and H^∞ have the DPP and property (V), and contain copies of ℓ_1 (see [6], [7], [17], [37]). Let E, F be A or H^∞. Then, by Corollary 8, $E \otimes_\pi F$ does not have the DPP.

Next we give some results about the DPP of duals of injective tensor products. It is known that $(E \otimes_\epsilon F)^* = I(E, F^*)$, the space of integral operators from E to F^* (see [21, Corollary VIII.2.12]).

Lemma 9 ([26]). *$E^{**} \otimes_\epsilon F$ is a closed subspace of $(E \otimes_\epsilon F)^{**}$.*

Theorem 10. *If E^* does not have the Schur property, F is infinite-dimensional, and $L(E^*, F^{**}) = CC(E^*, F^{**})$, then*

(i) *$(E \otimes_\epsilon F)^*$ does not have the DPP.*
(ii) *$E \otimes_\epsilon F^{**}$ does not have the DPP.*

Proof. By Observation 2(iii), F^{**} does not have the Schur property. Let (x_n^*) be a weakly null sequence in E^* and (x_n) be a bounded sequence in E so that $x_n^*(x_n) = 1$ for all n. Let (y_n^{**}) be a weakly null sequence in F^{**} and (y_n^*) be a bounded sequence in F^* so that $y_n^{**}(y_n^*) = 1$ for all n. Then we have (1) $\langle x_n \otimes y_n^{**}, x_n^* \otimes y_n^* \rangle = 1$.

By Lemma 1, $(x_n^* \otimes y_n^*)$ is weakly null in $E^* \otimes_\pi F^*$, hence (2) $(x_n^* \otimes y_n^*)$ is weakly null in $(E \otimes_\epsilon F)^*$. By Lemma 2, we deduce (3) $(x_n \otimes y_n^{**})$ is weakly null in $E \otimes_\epsilon F^{**}$.

(i) By Lemma 9, $(x_n \otimes y_n^{**})$ is weakly null in $(E \otimes_\epsilon F)^{**}$. By (1) and (2), $(E \otimes_\epsilon F)^*$ does not have the DPP.

(ii) $(E \otimes_\epsilon F)^*$ is a closed linear subspace of $(E \otimes_\epsilon F^{**})^*$ ([21, Corollary VIII.2.13]), hence by (2), $(x_n^* \otimes y_n^*)$ is weakly null in $(E \otimes_\epsilon F^{**})^*$. By (1) and (3), $E \otimes_\epsilon F^{**}$ does not have the DPP. ∎

Theorem 10(i) simplifies the proof of [28, Theorem 10]; the assumption $\ell_1 \hookrightarrow F^*$ there is superfluous.

Corollary 11.

(i) *If E and F are infinite-dimensional, E^* has the DPP and property (V), and F^* has property (V), then $E \otimes_\epsilon F^{**}$ does not have the DPP.*

(ii) *If $\ell_1 \overset{c}{\hookrightarrow} E$ and $\ell_1 \overset{c}{\not\hookrightarrow} F^*$, then $E \otimes_\epsilon F^{**}$ does not have the DPP.*

Proof. (i) By Observation 1, $L(E^*, F^{**}) = CC(E^*, F^{**})$. By Observation 4, E^* does not have the Schur property. Apply Theorem 10(ii).

(ii) Suppose first that $E = \ell_1$. Since $\ell_1 \overset{c}{\not\hookrightarrow} F^*$, we have $c_0 \not\hookrightarrow F^{**}$ (see [3]). Then every operator $T : \ell_\infty \to F^{**}$ is unconditionally converging, and thus completely continuous (since ℓ_∞ has property (V) and the DPP). By Theorem 10(ii), $\ell_1 \otimes_\epsilon F^{**}$ does not have the DPP.

Now suppose that $\ell_1 \overset{c}{\hookrightarrow} E$. If I is the identity on F^{**} and $P : E \to \ell_1$ is a projection, then $P \otimes_\epsilon I : E \otimes_\epsilon F^{**} \to \ell_1 \otimes_\epsilon F^{**}$ is a projection. Since $\ell_1 \otimes_\epsilon F^{**}$ does not have the DPP, $E \otimes_\epsilon F^{**}$ does not have the DPP. ∎

Corollary 12 ([28]). *If E and F are infinite-dimensional $\mathcal{L}_1$-spaces, then*

(i) *$(E \otimes_\epsilon F)^*$ does not have the DPP.*

(ii) *$E \otimes_\epsilon F^{**}$ does not have the DPP.*

Proof. Since E and F are infinite-dimensional $\mathcal{L}_1$-spaces, $\ell_1 \overset{c}{\hookrightarrow} E$ and $\ell_1 \overset{c}{\hookrightarrow} F$ (see [8, Proposition 1.24]). Then $c_0 \hookrightarrow E^*$ (see [3]), and thus E^* does not have the Schur property. Further, E^* and F^* are infinite-dimensional $\mathcal{L}_\infty$-spaces, and thus $L(E^*, F^{**}) = CC(E^*, F^{**})$ (by Observation 5).

(i) Apply Theorem 10(i).

(ii) Apply Theorem 10(ii). ∎

In the next results we consider the DPP of duals of projective tensor products.

Theorem 13. *Suppose that E^* does not have the Schur property, F is infinite-dimensional, and $L(F^{**}, E^*) = CC(F^{**}, E^*)$. Then $(E \otimes_\pi F)^* \simeq L(E, F^*)$ does not have the DPP.*

Proof. By Observation 2(iii), F^{**} does not have the Schur property. Replacing E and F in Theorem 3 by F^{**} and E respectively, one obtains from its proof a weakly null sequence $(x_n \otimes y_n^{**})$ in $E \otimes_\pi F^{**}$, and a weakly null sequence $(x_n^* \otimes y_n^*)$ in $E^* \otimes_\epsilon F^*$, hence in $L(E, F^*)$, such that $x_n^*(x_n) y_n^{**}(y_n^*) = 1$ for all n.

Define the bounded operator $S : E \otimes_\pi F^{**} \to L(E, F^*)^*$ by

$$\langle S(x \otimes y^{**}), T \rangle = \langle T^*(y^{**}), x \rangle \quad \text{for } T \in L(E, F^*),\ x \in E,\ y^{**} \in F^{**}.$$

Note that $(S(x_n \otimes y_n^{**}))$ is weakly null in $L(E, F^*)^*$ and $\langle S(x_n \otimes y_n^{**}), x_n^* \otimes y_n^* \rangle = x_n^*(x_n) y_n^{**}(y_n^*) = 1$. Then $L(E, F^*) \simeq (E \otimes_\pi F)^*$ does not have the DPP. ∎

Corollary 14.

(i) *Suppose that E^* does not have the Schur property, $\ell_1 \overset{c}{\not\hookrightarrow} E$, and $\ell_1 \overset{c}{\hookrightarrow} F^*$. Then $(E \otimes_\pi F)^*$ does not have the DPP.*

(ii) ([28, Theorem 15]) *Suppose that E^* does not have the Schur property, $\ell_1 \overset{c}{\not\hookrightarrow} E^{**}$, and $\ell_1 \overset{c}{\hookrightarrow} F^*$. Then $(E \otimes_\pi F)^*$ does not have the DPP.*

(iii) *Suppose that E has property (V), $\ell_1 \hookrightarrow E$, and $\ell_1 \overset{c}{\hookrightarrow} F^*$. Then $(E \otimes_\pi F)^*$ does not have the DPP.*

(iv) *Suppose that $\ell_1 \hookrightarrow E$, $\ell_1 \overset{c}{\not\hookrightarrow} E$, F^* has property (V^*), and $\ell_1 \hookrightarrow F^*$. Then $(E \otimes_\pi F)^*$ does not have the DPP.*

Proof. (i) Suppose first that $F^* = \ell_1$. Since $\ell_1 \overset{c}{\not\hookrightarrow} E$, we have $c_0 \not\hookrightarrow E^*$. Then every operator $T : \ell_\infty \to E^*$ is unconditionally converging, and thus completely continuous. By Theorem 13, $L(E, \ell_1)$ does not have the DPP.

Now suppose that $\ell_1 \overset{c}{\hookrightarrow} F^*$. If $P : F^* \to \ell_1$ is a projection, then the operator $Q : L(E, F^*) \to L(E, \ell_1)$ defined by $Q(T) = PT$ is a projection of $L(E, F^*)$ onto $L(E, \ell_1)$. Since $L(E, \ell_1)$ does not have the DPP, $L(E, F^*)$ does not have the DPP.

(ii) If $\ell_1 \overset{c}{\hookrightarrow} E$, then $c_0 \hookrightarrow E^*$. So ℓ_1 is a quotient of E^{**}, hence $\ell_1 \overset{c}{\hookrightarrow} E^{**}$ (see [18, p. 72]). Apply (i).

(iii) Since E has property (V), E^* is weakly sequentially complete [31]. Then $c_0 \not\hookrightarrow E^*$, and thus $\ell_1 \overset{c}{\not\hookrightarrow} E$. By Observation 2(i), E^* does not have the Schur property. Apply (i).

(iv) By Observation 2(i) again, E^* does not have the Schur property. If F^* has property (V^*) and $\ell_1 \hookrightarrow F^*$, then $\ell_1 \overset{c}{\hookrightarrow} F^*$, by results of [4], [25]. Apply (i). ∎

Corollary 15. *Suppose that E and F have the DPP, $\ell_1 \overset{c}{\not\hookrightarrow} E$, $\ell_1 \overset{c}{\not\hookrightarrow} F$, and $\ell_1 \overset{c}{\hookrightarrow} E^*$, $\ell_1 \overset{c}{\hookrightarrow} F^*$. Then the properties* (i), (ii), (iii), *and* (v) *in Corollary 7 are equivalent.*

Proof. The proof is the same as for Corollary 7, except for (iii)⇒(v), where one uses Corollary 14(i). ∎

A Banach space X has the *approximation property* if for every compact subset K of X and every $\epsilon > 0$ there exists a finite rank operator $S : X \to X$ such that $\|x - Sx\| \leq \epsilon$ for every $x \in K$. Examples of spaces with the approximation property include $C(K)$ spaces, c_0, ℓ_p, $1 \leq p < \infty$, $L_p(\mu)$ (μ any measure), $1 \leq p < \infty$, and $C(K)^*$ (see [21], [35]). We recall that if X^* or Y has the approximation property, then $K(X, Y) = X^* \otimes_\epsilon Y$ (see [35, Corollary 4.13]).

Corollary 16.

(i) ([11]) *If* E *and* F *are infinite-dimensional* $\mathcal{L}_\infty$*-spaces, then* $(E \otimes_\pi F)^{**}$ *does not have the DPP.*

(ii) ([28, Corollary 11]) *The space* $(C(K_1) \otimes_\pi C(K_2))^{**}$ *does not have the DPP, for all infinite compact spaces* K_1 *and* K_2.

Proof. (i) First assume that $\ell_1 \not\hookrightarrow E$. Then $CC(E, F^*) = K(E, F^*)$ (see [34, p. 377]). Further, since E^* (or F^*) has the approximation property [16, p. 306], we have $K(E, F^*) = E^* \otimes_\epsilon F^*$. By Observation 5, $CC(E, F^*) = L(E, F^*) \simeq (E \otimes_\pi F)^*$. Therefore

$$(E \otimes_\pi F)^* \simeq E^* \otimes_\epsilon F^*.$$

Since E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces, E^* and F^* are infinite-dimensional $\mathcal{L}_1$-spaces. By Corollary 12(i), $(E^* \otimes_\epsilon F^*)^*$ does not have the DPP. Therefore $(E \otimes_\pi F)^{**} \simeq (E^* \otimes_\epsilon F^*)^*$ does not have the DPP.

Now assume that $\ell_1 \hookrightarrow E$. If E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces, then they satisfy the hypotheses of Corollary 15. Indeed, since E^* is an $\mathcal{L}_1$-space, E^* is weakly sequentially complete [8, Corollary 1.29]. Then $c_0 \not\hookrightarrow E^*$, and thus $\ell_1 \overset{c}{\not\hookrightarrow} E$ (see [3]). Further, $\ell_1 \overset{c}{\hookrightarrow} E^*$ (see [8, Proposition 1.24]). Similarly, $\ell_1 \overset{c}{\not\hookrightarrow} F$ and $\ell_1 \overset{c}{\hookrightarrow} F^*$. Corollary 15 implies that $(E \otimes_\pi F)^*$, hence $(E \otimes_\pi F)^{**}$, does not have the DPP.

(ii) For infinite compact Hausdorff spaces K, $C(K)$ spaces are infinite-dimensional $\mathcal{L}_\infty$-spaces (by [8, Proposition 1.26]). Apply (i). ■

In [12, Corollary 1.5], the authors proved (using different techniques) that the space $(C(K_1) \otimes_\pi C(K_2))^{**}$ even contains a complemented copy of ℓ_2.

We can now give families of spaces with the Schur property whose duals fail to have the DPP. See also [19], [39].

Remark 1. Suppose that E and F are infinite-dimensional $\mathcal{L}_\infty$-spaces not containing copies of ℓ_1. Then $(E \otimes_\pi F)^*$ has the Schur property by Observation 5 and Corollary 7, while $(E \otimes_\pi F)^{**}$ does not have the DPP by Corollary 16. Examples are $C(K)$ spaces, where K is scattered, or a separable $\mathcal{L}_\infty$-space Y so that Y is somewhat reflexive, $\ell_1 \not\hookrightarrow Y$, and $Y^* \simeq \ell_1$ (see [10]).

Remark 2. Suppose that E and F are infinite-dimensional spaces with the Schur property. By [30], [36], $E \otimes_\epsilon F$ has the Schur property.

(i) If moreover E^* has the DPP and property (V) and F^* has property (V), then $(E \otimes_\epsilon F)^*$ does not have the DPP by Observation 1, Observation 2(i), and Theorem 10.

(ii) If moreover E and F are infinite-dimensional $\mathcal{L}_1$-spaces, then $(E \otimes_\epsilon F)^*$ does not have the DPP by Corollary 12(i).

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Ioana Ghenciu
Mathematics Department
University of Wisconsin-River Falls
River Falls, WI 54022, U.S.A.
E-mail: ioana.ghenciu@uwrf.edu

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