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EQUATIONS RELATING FACTORS IN DECOMPOSITIONS INTO FACTORS OF SOME FAMILY OF PLANE TRIANGULATIONS, AND APPLICATIONS

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(with an appendix by Andrzej Schinzel)

Abstract. Let $\mathcal{P}$ be the family of all 2-connected plane triangulations with vertices of degree three or six. Grünbaum and Motzkin proved (in dual terms) that every graph $P \in \mathcal{P}$ has a decomposition into factors P_0, P_1, P_2 (indexed by elements of the cyclic group $Q = \{0, 1, 2\}$) such that every factor P_q consists of two induced paths of the same length $M(q)$, and $K(q)-1$ induced cycles of the same length $2M(q)$. For $q \in Q$, we define an integer $S^+(q)$ such that the vector $(K(q), M(q), S^+(q))$ determines the graph P (if P is simple) uniquely up to orientation-preserving isomorphism. We establish arithmetic equations that will allow calculating $(K(q+1), M(q+1), S^+(q+1))$ from $(K(q), M(q), S^+(q))$, $q \in Q$. We present some applications of these equations. The set $\{(K(q), M(q), S^+(q)) : q \in Q\}$ is called the orbit of P. If P has a one-point orbit, then there is an orientation-preserving automorphism σ such that $\sigma(P_i) = P_{i+1}$ for every $i \in Q$ (where $P_3 = P_0$). We characterize one-point orbits of graphs in $\mathcal{P}$. It is known that every graph in $\mathcal{P}$ has an even order. We prove that if P is of order $4n + 2$, $n \in \mathbb{N}$, then it has two disjoint induced trees of the same order, which are equitable 2-colorable and together cover all vertices of P.

1. Introduction. Let G_i, $i = 1, 2$, be a plane graph with vertex set $V(G_i)$, edge set $E(G_i)$, and face set $F(G_i)$. An isomorphism σ between G_1 and G_2 is called *combinatorial* if it can be extended to a bijection

$$\sigma : V(G_1) \cup E(G_1) \cup F(G_1) \to V(G_2) \cup E(G_2) \cup F(G_2)$$

that preserves incidence not only of vertices with edges but also of vertices and edges with faces (Diestel [3, p. 93]). Furthermore, we say that G_1 and G_2 are *op-equivalent* (equivalent up to orientation-preserving isomorphism) if σ is a combinatorial isomorphism which preserves the counterclockwise orientation. (Formally: we require that g_1, g_2, g_3 are counterclockwise successive edges incident with a vertex v if and only if $\sigma(g_1), \sigma(g_2), \sigma(g_3)$ are counterclockwise successive edges incident with $\sigma(v)$.)

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A *factor* of a graph is a subgraph whose vertex set is that of the whole graph. A graph H is said to be *factorable* into factors $H_1, \ldots, H_t$ if these factors are pairwise edge-disjoint and $E(H) = E(H_1) \cup \cdots \cup E(H_t)$ (see Chartrand–Lesniak [1, p. 246]). An edge (respectively a subgraph) of H is said to be of *class* q if this edge (respectively any edge of this subgraph) belongs to the factor H_q.

Let $\mathcal{P}$ be the family of all 2-connected plane triangulations all of whose vertices are of degree 3 or 6, and suppose that $P \in \mathcal{P}$. Grünbaum and Motzkin [8, Lemma 2] proved (in dual terms) that the graph P is factorable into factors P_0, P_1, P_2 (indexed by elements of the cyclic group $Q = \{0, 1, 2\}$) satisfying the following condition:

(GM1) if three edges in P are counterclockwise successive edges incident with a common vertex, then these edges belong to successive factors of P.

Notice that every edge of class $q \in Q$ belongs to a maximal path with ends of degree 3 in P, or it belongs to a cycle of class q. Since P has four vertices of degree 3, there are two maximal paths of class q. More precisely,

(GM2) for $q \in Q$, there is a drawing of P (called the *q-drawing*) which is op-equivalent to P. The q-drawing of P consists of a maximal path of class q and length $M(q)$, and this path is surrounded by $K(q) - 1$ disjoint cycles of class q and the same length $2M(q)$. Finally, there is another maximal path of class q and length $M(q)$ (called the *outer path*) around the outside of the last cycle (see Example 1.1, Figs. 1 and 2).

By (GM2) we have the following result of Grünbaum and Motzkin [8, Theorem 2]:

(1) $$2K(q)M(q) + 2 \quad \text{is the order of } P.$$

Notice that the outer path may be added at different positions. We define (Definition 2.2) an integer $0 \le S^+(q) < M(q)$ (and also $0 < S^-(q) \le M(q)$) that determines that position. In Theorem 2.1 we show the following relation between $S^+(q)$ and $S^-(q)$:

(2) $$S^-(q) - S^+(q) \equiv K(q) \pmod{M(q)}.$$

The vector $(K(q), M(q), S^+(q))$, for $q \in Q$, is called the *q-index-vector* of P, and the set $\{(K(q), M(q), S^+(q)) : q \in Q\}$ is called the *orbit* of P. The purpose of this article is to find arithmetic equations that will allow calculating the $(q + 1)$-index-vector from the q-index-vector of P. In Theorem 3.1 we prove the following equality:

(3) $$K(q + 1) = |S^+(q), M(q)|,$$

where $|s, m|$ is the greatest common divisor of the integers $s \ge 0$ and $m \ge 1$ ($|0, m| = m$). Suppose $0 < b \le M(q)/d$ is an integer such that $bS^+(q) \equiv -d$

(mod $M(q)$), where $d = |S^+(q), M(q)|$ (see Remark 3.1). In Theorem 3.2 we prove that

$$S^-(q+1) = bK(q). \tag{4}$$

Notice that every isomorphism between two 3-connected plane graphs is combinatorial (Diestel [3, p. 94]). Hence, if P is simple, then it is determined by any of its index-vectors uniquely up to op-equivalence. Therefore, using equations (1)–(4) we can verify whether simple graphs in $\mathcal{P}$ are op-equivalent.

EXAMPLE 1.1. Let us consider the simple graph S_0 of Fig. 1, and simple graphs S_1, S_2 of Fig. 2. We assume that the edges g_0, g_1, g_2 are of class 0, 1, 2, respectively, and they are incident with a common vertex of degree 3. The graph S_0 has 0-index-vector $(1, 6, 3)$, S_1 has 1-index-vector $(3, 2, 0)$ and S_2 has 2-index-vector $(2, 3, 1)$. Using equations (1)–(4) we check that $\{(1, 6, 3), (3, 2, 0), (2, 3, 1)\}$ is their common orbit (see Example 3.1). Hence, the graph S_q, $q \in Q$, is the q-drawing of the graph S_{q+1} and S_{q+2}. Therefore these graphs are op-equivalent.

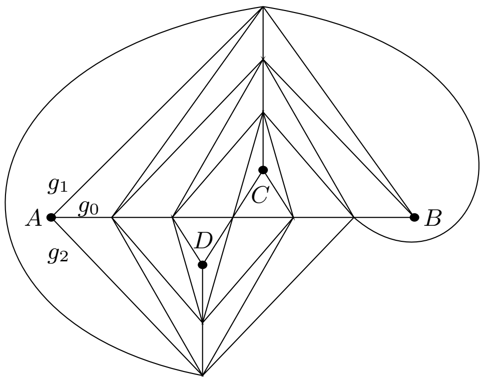


Fig. 1. The graph S_0 with 0-index-vector $(1, 6, 3)$

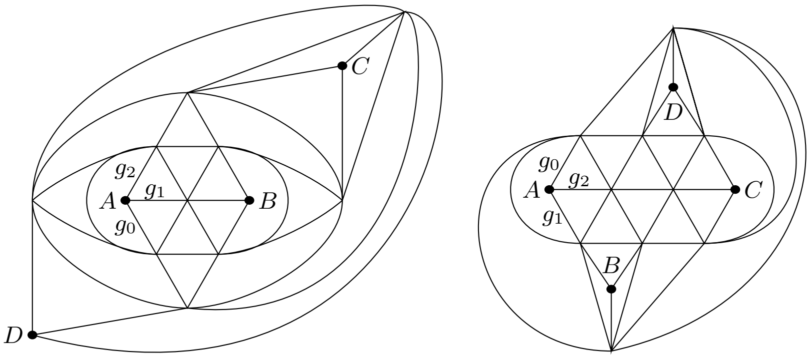


Fig. 2. The graph S_1 with 1-index-vector $(3, 2, 0)$, and S_2 with 2-index-vector $(2, 3, 1)$

We are going to present some applications of equations (1)–(4). From (GM2) it follows that $X = \{(k, m, s) \in \mathbb{Z}^3 : k, m \geq 1,\ 0 \leq s < m\}$ is the set of all index-vectors of graphs in $\mathcal{P}$. In Theorem 4.1 we characterize one-point orbits of graphs in $\mathcal{P}$. Namely, we prove that $(k, m, s) \in X$ is a one-point orbit of a graph in $\mathcal{P}$ if and only if $m = kz$, $s = kx$, where k is a positive integer and $0 \leq x < z$ are integers which satisfy the Diophantine equation $x^2+x+1 = yz$. By a theorem of Gauss, reproved in a relevant special case by Schinzel and Sierpiński [12], the set of all integral solutions of the equation $x^2 + x + 1 = 3y^2$ is infinite. It follows that there is an infinite family of graphs in $\mathcal{P}$ with one-point orbit. In the Theorem of the Appendix, Schinzel has found formulas for positive integers x, $y < z$ which satisfy the equation $x^2 + x + 1 = yz$.

If P has index-vector $(K(q), M(q), S^+(q))$, then its mirror reflection has index-vector $(K(q), M(q), M(q) - S^-(q))$. We say that P is *double mirror symmetric* if there exist $q_1, q_2 \in Q$ such that $S^+(q_i) = M(q_i) - S^-(q_i)$ for $i = 1, 2$. In Theorem 4.2 we show that P is double mirror symmetric if and only if P has a one-point orbit of the form $\{(k, k, 0)\}$ or $\{(k, 3k, k)\}$ for some $k \in \mathbb{N}$.

A graph G has a *2-tree partition* (see [5], [10]) if it has two disjoint induced trees which together cover all vertices of G. It is known that a 2-connected plane graph has a Hamilton cycle if and only if its dual graph has a 2-tree partition (see Stein [14]). Goodey [6] showed that every 2-connected cubic plane graph whose faces are only triangles or hexagons has a Hamilton cycle. Hence, every graph in $\mathcal{P}$ has a 2-tree partition. In Theorem 5.1 we prove that for every 2-tree partition of a graph in $\mathcal{P}$ the two trees have the same order. In Theorem 5.2 we prove a similar result for the family $\mathcal{H}$ of all 2-connected plane triangulations all of whose vertices are of degree at most 6. Namely, for every 2-tree partition of a graph in $\mathcal{H}$ the orders of the two trees differ by at most 3.

A graph G is *equitable k-colorable* if there exists a proper k-coloring of G such that the sizes of any two color classes differ by at most one (see Jensen and Toft [9]). It is easy to see, by condition (GM2), that every graph in $\mathcal{P}$ is equitable 4-colorable. One may guess that every graph in $\mathcal{P}$ has a 2-tree partition such that the two trees have the same order and are equitable 2-colorable. In fact, in Theorem 6.1 we prove (using equations (1)–(3)) that this is the case if P is of order $4n + 2$, $n \in \mathbb{N}$.

2. Index-vector. Let $\mathcal{P}$ be the family of all 2-connected plane triangulations all of whose vertices are of degree 3 or 6. Fix $P \in \mathcal{P}$. Let P be factorable into factors P_0, P_1, P_2 (indexed by elements of the cyclic group $Q = \{0, 1, 2\}$) satisfying condition (GM1). We recall that a subgraph of P is said to be of *class* $q \in Q$ if any edge of the subgraph belongs to the factor P_q. Let $M(q)$ be the length of a maximal path of class q, and $K(q)$ the distance between the two maximal paths of this class in P.

DEFINITION 2.1. Let A be a vertex of degree 3 in the graph P, and suppose that $[A, q]$ is a maximal path of class q with an orientation $v_0 v_1 \dots v_{M(q)}$ such that $A = v_0$ is its initial vertex and $A_q = v_{M(q)}$ its terminal vertex. An edge e adjacent to the path $[A, q]$ is called a *left branch* of the path if it is branching off from $[A, q]$ to the left (more precisely, if $v_j v_{j+1}, e$, $0 \leq j < M(q)$, or $e, v_{j-1} v_j$, $0 < j \leq M(q)$, are counterclockwise successive edges incident with the vertex v_j). Otherwise, it is called a *right branch* of the path. We set

$$[A, q](e) = \begin{cases} j & \text{if } e \text{ is a left branch of } [A, q] \text{, incident with } v_j, \\ 2M(q) - j & \text{if } e \text{ is a right branch of } [A, q] \text{, incident with } v_j. \end{cases}$$

REMARK 2.1. Notice that $[A, q] = v_0 v_1 \dots v_{M(q)}$ if and only if $[A_q, q] = v_{M(q)} v_{M(q)-1} \dots v_0$. An edge e is a left branch of $[A, q]$ if and only if it is a right branch of $[A_q, q]$. Moreover,

$$|[A_q, q](e) - [A, q](e)| = M(q).$$

LEMMA 2.1. *Let A, C be the ends of two different maximal paths of class q.*

(1) *If $e, \hat{e}$ and $f, \hat{f}$ are pairs of end-edges of two minimal paths of class $q+1$ so that e, f are adjacent to the path $[A, q]$ and $\hat{e}, \hat{f}$ are adjacent to $[C, q]$, then*

$$[A, q](f) - [A, q](e) \equiv [C, q](\hat{e}) - [C, q](\hat{f}) \pmod{2M(q)}.$$

(2) *Moreover, if e is incident with A, and $\hat{f}$ is incident with C, then*

$$[A, q](f) = [C, q](\hat{e}).$$

Proof. The proof is clearer when we consider the q-drawing of P. Notice that

$$[A', q](f) - [A', q](e) = [C', q](\hat{e}) - [C', q](\hat{f})$$

for some $A' \in \{A, A_q\}$ and $C' \in \{C, C_q\}$. Hence, by Remark 2.1 we obtain (1). If e is incident with A, and $\hat{f}$ is incident with C, then, by (GM1), e is a left branch of $[A, q]$ and $\hat{f}$ is a left branch of $[C, q]$. Hence, $[A, q](e) = 0$ and $[C, q](\hat{f}) = 0$, which yields (2). ■

DEFINITION 2.2. Let A, C be the ends of two different maximal paths of class q in the graph P, and suppose f (or g) is the first edge of $[C, q+1]$

(or $[C, q-1]$, respectively) which is adjacent to $[A, q]$. Let

$$S^+(q) = \begin{cases} [A,q](f) & \text{if } f \text{ is a left branch of } [A,q], \\ [A,q](f) - M(q) & \text{if } f \text{ is a right branch of } [A,q], \end{cases}$$

$$S^-(q) = \begin{cases} [A,q](g) & \text{if } g \text{ is a left branch of } [A,q], \\ [A,q](g) - M(q) & \text{if } g \text{ is a right branch of } [A,q], \end{cases}$$

Notice that by Remark 2.1 and Lemma 2.1(2) the definitions of $S^+(q)$ and $S^-(q)$ do not depend on the choice of ends of two different maximal paths of class q. The following theorem shows that $S^+(q)$ is determined by $S^-(q)$ and vice versa.

THEOREM 2.1.

$$S^-(q) - S^+(q) \equiv K(q) \pmod{M(q)}.$$

Proof. The proof is clearer if one considers the q-drawing of P. Let A, C be the ends of two different maximal paths of class q in P, and suppose f (or g) is the first edge of $[C, q+1]$ (or $[C, q-1]$) which is adjacent to $[A, q]$, say at a vertex E (or F, respectively). If V is the last common vertex of $[C, q+1]$ with a segment CF of $[C, q-1]$, then

$$[A,q](g) - [A,q](f) \equiv |VE| \equiv K(q) \pmod{2M(q)}.$$

Hence,

$$S^-(q) - S^+(q) \equiv [A,q](g) - [A,q](f) \equiv K(q) \pmod{M(q)},$$

which completes the proof. ■

3. Billiards and structure of plane triangulations in $\mathcal{P}$. Let $\mathcal{P}$ be the family of all of 2-connected plane triangulations all of whose vertices are of degree 3 or 6. Fix $P \in \mathcal{P}$ and $q \in Q$ (where $Q = \{0, 1, 2\}$ is the cyclic group). Let $(K(q), M(q), S^+(q))$ be the q-index-vector of P.

If $0 < \theta < 1$, then a *θ-billiard sequence* is a sequence $F(j) \in [0, 1)$, $j \in \mathbb{N}$, which satisfies the following conditions (see [4]): $F(1) = 0$ and

$$F(j) + F(j+1) = \begin{cases} \theta \text{ or } 1+\theta & \text{for } j \text{ odd}, \\ 0 \text{ or } 1 & \text{for } j \text{ even}. \end{cases}$$

We consider a billiard table rectangle with perimeter of length 1 with the bottom left vertex labeled v_0, and the others, in the clockwise direction, v_1, v_2 and v_3. The distance from v_0 to v_1 is $\theta/2$. We describe the position of points on the perimeter by their distance along the perimeter measured in the clockwise direction from v_0, so that v_1 is at position $\theta/2$, v_2 at $1/2$ and v_3 at $(\theta+1)/2$. If a billiard ball is pushed from position $F(1) = 0$ at the angle of $\pi/4$, then it will rebound against the sides of the rectangle consecutively at $F(2)$, $F(3), \ldots$.

The following lemma comes from [4, Theorem 3.3(2) and Example 3.1].

Lemma 3.1. *If* $0 < s/m < 1$ *is a fraction,* $d = |s, m|$ *and* $F(j)$, $j \in \mathbb{N}$, *is the* s/m*-billiard sequence, then:*

(1) $\{2mF(1), 2mF(2), \dots, 2mF(m/d)\} = \{0, 2d, 4d, \dots, 2m - 2d\}$.

(2) $2mF(m/d) = \begin{cases} s & \text{for } s/d \text{ even,} \\ m & \text{for } m/d \text{ even,} \\ s+m & \text{for } s/d \text{ and } m/d \text{ both odd,} \end{cases}$

and $2mF(j) \notin \{s, m, s+m\}$ *for* $1 \le j < m/d$.

(3) *If* a, b *are natural numbers,* $am - bs = d$ *and* $b \le m/d$, *then*

$$2mF(b) = \begin{cases} s+d & \text{for } a \text{ even,} \\ m-d & \text{for } b \text{ even,} \\ s+m+d & \text{for } a \text{ and } b \neq 1 \text{ both odd,} \\ 0 & \text{for } a = b = 1. \end{cases}$$

Remark 3.1. The sequence of all reduced fractions in the interval $[0, 1]$ with denominators not exceeding n, listed in order of size, is called the *Farey sequence* of order n ($0/1$ is the smallest and $1/1$ the greatest fraction of any Farey sequence). Let $0 \le s/m < 1$ be a fraction, $d = |s, m|$, and suppose that $s'/m' = s/m$ is a fraction in lowest terms. Then $s'/m' < a/b$ are consecutive fractions in the Farey sequence of order m' if and only if $am - bs = d$ and $b \le m/d$ (see Schmidt [13]).

The following theorem shows that the structure of the graph P is closely related to $S^+(q)/M(q)$-billiard sequences.

Theorem 3.1. *Let* A *be a vertex of degree* 3 *in* P, *and suppose that* $e_1, \dots, e_n$ *is a sequence of all consecutive edges of* $[A, q+1]$ *which are adjacent to* $[A, q]$.

(1) *If* $n > 1$, *then*

$$[A, q](e_j) = 2M(q)F(j) \quad \text{for } 1 \le j \le n,$$

where $F(j)$, $j \in \mathbb{N}$, *is the* $S^+(q)/M(q)$*-billiard sequence.*

(2) $n = M(q)/|S^+(q), M(q)|$.

(3) $K(q+1) = |S^+(q), M(q)|$.

(4) $K(q+1)M(q+1) = K(q)M(q)$.

Proof. Since $2K(q)M(q) + 2$ is the order of P, condition (4) holds.

If $n = 1$, then $S^+(q) = 0$, $M(q+1) = K(q)$ and, by (4), $K(q+1) = M(q)$. Hence, conditions (2) and (3) are satisfied.

Let $n > 1$. Let C be a vertex of degree 3, $C \neq A$, $C \neq A_q$, and suppose that f is the first edge of the path $[C, q+1]$ which is adjacent to $[A, q]$. Without loss of generality we can assume, by Remark 2.1, that f is a left branch of $[A,q]$. Hence, $[A,q](f) = S^+(q)$. Suppose that $\hat{e}_1,\dots,\hat{e}_n$ is a sequence of all consecutive edges of $[A,q+1]$ which are adjacent to $[C,q]$. Note that the edges $\hat{e}_{2j-1}$, $\hat{e}_{2j}$ are incident with the same vertex of $[C,q]$ and that they are on the opposite sides of this path. Hence,

$$[C,q](\hat{e}_{2j-1}) + [C,q](\hat{e}_{2j}) = 2M(q).$$

By Lemma 2.1(1),

$$[A,q](e_j) + [C,q](\hat{e}_j) \equiv [A,q](f) \equiv S^+(q) \pmod{2M(q)} \quad \text{for } 1 \le j \le n.$$

Hence,

$$[A,q](e_{2j-1}) + [A,q](e_{2j}) \equiv 2S^+(q) \pmod{2M(q)}, \quad 2 \le 2j \le n.$$

Since $0 \le [A,q](e_{2j-1}) + [A,q](e_{2j}) < 4M(q)$ and $0 < S^+(q) < M(q)$ we get

(i) $\quad [A,q](e_{2j-1}) + [A,q](e_{2j}) = 2S^+(q)$ or $2M(q) + 2S^+(q), \quad 2 \le 2j \le n.$

By analogy, the edges e_{2j}, e_{2j+1} are incident with the same vertex of $[A,q]$, and therefore they are on the opposite sides of this path. Hence,

(ii) $$[A,q](e_{2j}) + [A,q](e_{2j+1}) = 2M(q), \quad 2 \le 2j \le n-1.$$

From (i) and (ii) we obtain (1).

By definition of A_q, C and C_q, we have

$$A_{q+1} = A_q \text{ and } j = n \;\Leftrightarrow\; [A,q](e_j) = M(q),$$

$$A_{q+1} = C \text{ and } j = n \;\Leftrightarrow\; [C,q](\hat{e}_j) = 0 \;\Leftrightarrow\; [A,q](e_j) = S^+(q),$$

$$A_{q+1} = C_q \text{ and } j = n \;\Leftrightarrow\; [C,q](\hat{e}_j) = M(q) \;\Leftrightarrow\; [A,q](e_j) = M(q) + S^+(q).$$

Accordingly,

(iii) $$\begin{aligned} &[A,q](e_n) \in \{M(q), S^+(q), M(q)+S^+(q)\},\\ &[A,q](e_j) \notin \{M(q), S^+(q), M(q)+S^+(q)\} \quad \text{for } j < n. \end{aligned}$$

By (1) and Lemma 3.1(2), condition (iii) leads to $n = M(q)/|S^+(q), M(q)|$.

Since $n = M(q)/|S^+(q), M(q)|$ condition (4) shows that

$$M(q+1) = nK(q) = \frac{M(q)K(q)}{|S^+(q), M(q)|} = \frac{M(q+1)K(q+1)}{|S^+(q), M(q)|}.$$

Thus $K(q+1) = |S^+(q), M(q)|$ and condition (3) holds. ∎

By analogy, we obtain the following corollary:

Corollary 3.1. *Let A be a vertex of degree 3 in P, and suppose that $e_1,\dots,e_n$ is a sequence of all consecutive edges of $[A,q-1]$ which are adjacent to $[A,q]$.*

(1) *If $n > 1$, then*

$$[A,q](e_j) = 2M(q)F(j) \quad \textit{for } 1 \le j \le n,$$

where $F(j)$, $j \in \mathbb{N}$, is the $S^-(q)/M(q)$-billiard sequence.

(2) $n = M(q)/|S^-(q), M(q)|$.
(3) $K(q-1) = |S^-(q), M(q)|$.

THEOREM 3.2. *Let A be a vertex of degree 3 in P, and suppose that a, b are natural numbers such that $aM(q) - bS^+(q) = d$ and $b \le M(q)/d$, where $d = |S^+(q), M(q)|$. Then:*

(1) $S^-(q+1) = bK(q)$.
(2) $S^+(q+1) \equiv bK(q) - K(q+1) \pmod{M(q+1)}$.

Proof. It suffices to prove (1), because (2) follows from (1) and Theorem 2.1. Suppose that $e_1, \dots, e_n$ is a sequence of all consecutive edges of $[A, q+1]$ which are adjacent to $[A, q]$ at vertices $A = E_1, \dots, E_n$, respectively.

If $n = 1$, then A is the only common vertex of $[A, q+1]$ and $[A, q]$. Hence, $S^+(q) = 0$ and $S^-(q+1) = M(q+1) = K(q)$. Then $a = b = 1$, and condition (1) holds.

Let $n > 1$. Let C be a vertex of degree 3, $C \ne A$, $C \ne A_q$, and suppose that f is the first edge of $[C, q+1]$ which is adjacent to $[A, q]$. Without loss of generality we can assume, by Remark 2.1, that f is a left branch of $[A, q]$. Hence, $[A, q](f) = S^+(q)$. Suppose that $\hat{e}_1, \dots, \hat{e}_n$ is a sequence of all consecutive edges of $[A, q+1]$ which are adjacent to $[C, q]$ at vertices $\hat{E}_1, \dots, \hat{E}_n$, respectively. Note that $E_j = E_{j+1}$ for j even, $\hat{E}_j = \hat{E}_{j+1}$ for j odd, and the segment $E_j\hat{E}_j$ of $[A, q+1]$ has length $|E_j\hat{E}_j| = K(q)$. Hence, the segments AE_b and $A\hat{E}_b$ of $[A, q+1]$ have lengths:

$$\text{(i)} \qquad \begin{cases} |AE_b| = bK(q) & \text{for } b \text{ even}, \\ |A\hat{E}_b| = bK(q) & \text{for } b \text{ odd}. \end{cases}$$

By Remark 2.1 and Lemma 2.1(1), we have

$$\begin{aligned}[][A_q, q](e_j) - [A_q, q](e_i) &\equiv [A, q](e_j) - [A, q](e_i) \equiv [C, q](\hat{e}_i) - [C, q](\hat{e}_j) \\ &\equiv [C_q, q](\hat{e}_i) - [C_q, q](\hat{e}_j) \pmod{2M(q)} \quad \text{for } 1 \le i, j \le n.\end{aligned}$$

From Theorem 3.1(2) it follows that $n = M(q)/d$. Hence, by Theorem 3.1(1) and Lemma 3.1(1), we obtain

$$\begin{aligned}\text{(ii)} \quad [A_q, q](e_j) - [A_q, q](e_i) &\equiv [C, q](\hat{e}_i) - [C, q](\hat{e}_j) \\ &\equiv [C_q, q](\hat{e}_i) - [C_q, q](\hat{e}_j) \equiv 0 \pmod{2d} \quad \text{for } 1 \le i, j \le n.\end{aligned}$$

By Lemma 3.1(3) we get

$$[A, q](e_b) = \begin{cases} S^+(q) + d & \text{for } a \text{ even}, \\ M(q) - d & \text{for } b \text{ even}, \\ S^+(q) + M(q) + d & \text{for } a \text{ and } b \ne 1 \text{ both odd}. \end{cases}$$

Since $[A, q](f) = S^+(q)$, Lemma 2.1(1) shows that

$$[C, q](\hat{e}_b) \equiv [A, q](f) - [A, q](e_b) \equiv S^+(q) - [A, q](e_b) \pmod{2M(q)}.$$

Accordingly, by Remark 2.1, we obtain

$$\text{(iii)}\qquad \begin{cases} [C,q](\hat{e}_b) = 2M(q) - d & \text{for } a \text{ even},\\ [A_q,q](e_b) = 2M(q) - d & \text{for } b \text{ even},\\ [C_q,q](\hat{e}_b) = 2M(q) - d & \text{for } a \text{ and } b \neq 1 \text{ both odd}. \end{cases}$$

Hence, for b even (resp. b odd and $b \neq 1$), e_b (resp. $\hat{e}_b$) is a right branch of $[A_q, q]$ (resp. $[C,q]$ or $[C_q,q]$). For b even (resp. b odd and $b \neq 1$), suppose that g is the first arc of the directed path $[A_q,q]$ (resp. $[C,q]$ or $[C_q,q]$) which is adjacent to the directed path $[A, q+1]$. By (ii)–(iii), E_b (resp. $\hat{E}_b$) is the common head of the arcs g and e_b (resp. $\hat{e}_b$). Hence, g is a left branch of $[A, q+1]$. Thus, by (i),

$$S^-(q+1) = [A, q+1](g) = |AE_b| \text{ (resp. } |A\hat{E}_b|) = bK(q),$$

and condition (1) holds. ■

Example 3.1. Let $\{a_j\}$ be the Fibonacci sequence:

$$a_1 = a_2 = 1 \quad \text{and} \quad a_{j+2} = a_j + a_{j+1} \quad \text{for } j \in \mathbb{N}.$$

We will check that

$$\{(1, a_{2n+1}a_{2n+2}, a_{2n}a_{2n+2}), (a_{2n+2}, a_{2n+1}, 0), (a_{2n+1}, a_{2n+2}, a_{2n})\}$$

is the orbit of a graph in $\mathcal{P}$. Notice that for $n = 1$ we obtain the orbit

$$\{(1,6,3), (3,2,0), (2,3,1)\}.$$

Proof. Since a_j/a_{j+1} is the jth convergent to $(\sqrt{5}-1)/2$, $j \in \mathbb{N}$, we have the following conditions (see Schmidt [13, Lemmas 3C, 3D]):

$$\text{(1)}\qquad a_{j+1}^2 - a_j a_{j+2} = (-1)^j,$$

$$\text{(2)}\qquad a_{j+3}a_j - a_{j+2}a_{j+1} = (-1)^{j+1}.$$

If $(K(1), M(1), S^+(1)) = (1, a_{2n+1}a_{2n+2}, a_{2n}a_{2n+2})$ then, by (1), for $j = 2n-1$,

$$a_{2n-1}M(1) - a_{2n}S^+(1) = a_{2n+2}.$$

Hence, by Theorem 3.1(3–4) we have

$$K(2) = a_{2n+2}, \qquad M(2) = a_{2n+1},$$

and, by Theorem 3.2(2),

$$S^+(2) \equiv a_{2n}K(1) - K(2) = a_{2n} - a_{2n+2} = -a_{2n+1} \equiv 0 \pmod{a_{2n+1}}.$$

If $(K(2), M(2), S^+(2)) = (a_{2n+2}, a_{2n+1}, 0)$, then $M(2) - S^+(2) = a_{2n+1}$. Hence, by Theorem 3.1(3–4) we obtain

$$K(3) = a_{2n+1}, \qquad M(3) = a_{2n+2},$$

and, by Theorem 3.2(2),

$$S^+(3) \equiv K(2) - K(3) = a_{2n+2} - a_{2n+1} = a_{2n} \pmod{a_{2n+2}}.$$

If $(K(3), M(3), S^+(3)) = (a_{2n+1}, a_{2n+2}, a_{2n})$, then, by (2), for $j = 2n-1$,

$$a_{2n-1}M(3) - a_{2n+1}S^+(3) = 1.$$

Hence, by Theorem 3.1(3–4) we have

$$K(1) = 1, \qquad M(1) = a_{2n+1}a_{2n+2},$$

and, by Theorem 3.2(2) and (1), for $j = 2n$,

$$S^+(1) \equiv a_{2n+1}K(3) - K(1) = a_{2n+1}^2 - 1 = a_{2n}a_{2n+2} \pmod{a_{2n+1}a_{2n+2}},$$

and the proof is complete. ∎

4. One-point orbits of plane triangulations in $\mathcal{P}$. We recall that $X = \{(k, m, s) \in \mathbb{Z}^3 : k, m \geq 1,\ 0 \leq s < m\}$ is the set of all index-vectors of graphs in $\mathcal{P}$. In the following theorem we characterize one-point orbits.

THEOREM 4.1. *$\{(k, m, s)\} \in X$ is a one-point orbit of a graph in $\mathcal{P}$ if and only if $m = kz$, $s = kx$, where $0 \leq x < z$ are integers such that z is a divisor of $x^2 + x + 1$.*

Proof. Let (k, m, s) be an index-vector of a graph in $\mathcal{P}$. It is easy to prove that the following conditions are equivalent ((ii)⇔(iii) follows from Theorems 3.1(3–4) and 3.2(2)):

(i) $\{(k, m, s)\}$ is a one-point orbit,
(ii) $(k, m, s) = (K(q), M(q), S^+(q)) = (K(q+1), M(q+1), S^+(q+1))$,
(iii) $k = |s, m|$ and $s = bk - k$, where b is an integer such that $0 < b \leq m/k$ and $bs \equiv -k \pmod{m}$,
(iv) $m = kz$, $s = kx = bk - k$, where $z \geq 1$, $x \geq 0$ and $0 < b \leq z$ are integers such that $bx \equiv -1 \pmod{z}$,
(v) $m = kz$, $s = kx$, where $0 \leq x < z$ are integers such that z is a divisor of $x^2 + x + 1$. ∎

REMARK 4.1. Notice that if $\{(k, m, s)\}$ is a one-point orbit of a graph $G \in \mathcal{P}$, then, by Theorem 2.1, $\{(k, m, m - s - k)\}$ is a one-point orbit of the mirror reflection of G. Hence, by Theorem 4.1, z is a divisor of $x^2 + x + 1$ if and only if z is a divisor of $(z - x - 1)^2 + (z - x - 1) + 1$, which is confirmed by the following equivalence:

$$x^2 + x + 1 = yz \ \Leftrightarrow\ (z - x - 1)^2 + (z - x - 1) + 1 = (z - 2x - 1 + y)z.$$

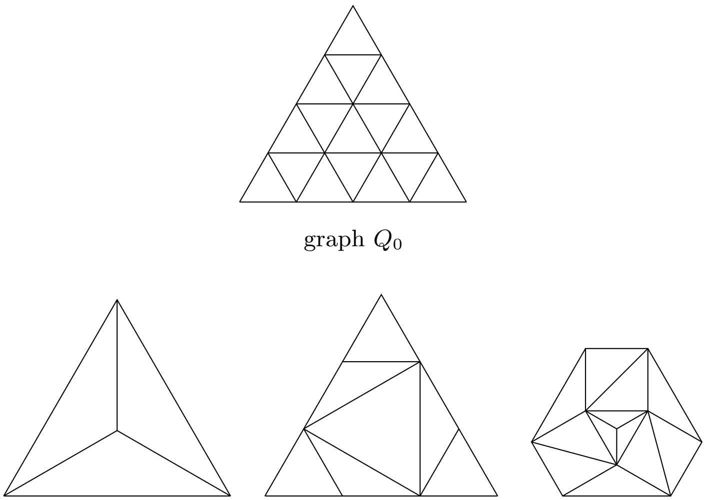
EXAMPLE 4.1. Notice that $(y, z, x) = (1, 1, 0), (1, 3, 1), (1, 7, 2)$ and $(1, 13, 3)$ are all integral solutions of the Diophantine equation

$$x^2 + x + 1 = yz \quad \text{for } 0 \leq x \leq 3 \ \text{ and } \ x < z.$$

Hence, by Theorem 4.1, $\{(k, k, 0)\}$, for $k \in \mathbb{N}$, $\{(1, 3, 1)\}$, $\{(1, 7, 2)\}$ and $\{(1, 13, 3)\}$ are all one-point orbits with $s \leq 3$. Notice that K^4 (tetrahedron) has one-point orbit $\{(1,1,0)\}$. Let G_0, G_1, G_2 and G_3 be graphs in $\mathcal{P}$ with one-point orbits

$$\{(4,4,0)\}, \quad \{(1,3,1)\}, \quad \{(1,7,2)\}, \quad \{(1,13,3)\},$$

respectively. Consider a solid regular tetrahedron with closed 3-faces f_1, f_2, f_4, f_4. We leave it to the reader to verify that G_j, $j = 0,1,2,3$, can be embedded in the sphere of the solid regular tetrahedron in such a way that all four induced plane graphs $G_j[V_j \cap f_1], \ldots, G_j[V_j \cap f_4]$ are op-equivalent to the plane graph Q_j shown in Fig. 3.



graph Q_0

graph Q_1 graph Q_2 graph Q_3

Fig. 3

We conjecture that each graph $G \in \mathcal{P}$ with one-point orbit and vertex set V can be embedded in the sphere of the solid regular tetrahedron in such a way that all four induced plane graphs $G[V \cap f_1], \ldots, G[V \cap f_4]$ are op-equivalent.

THEOREM 4.2. *$G \in \mathcal{P}$ is double mirror symmetric if and only if G has a one-point orbit of the form $\{(k,k,0)\}$ or $\{(k,3k,k)\}$ for some $k \in \mathbb{N}$.*

Proof. Let $G \in \mathcal{P}$ and suppose that $\{(K(q), M(q), S^+(q)) : q \in Q\}$ is the orbit of G. First we prove that if $S^+(q) + S^-(q) = M(q)$ for $q = 1,2$, then G has a one-point orbit of the form $\{(k, 2s+k, s)\}$. If $S^+(q) + S^-(q) = M(q)$ for $q = 1,2$, then by Theorem 3.1(3) and Corollary 3.1(3) we conclude that $K(0) = K(1) = K(2) = k$. Hence, $M(0) = M(1) = M(2) = m$, by Theorem 3.1(4). Suppose that a_q, b_q for $q \in Q$ are integers such that $a_q m - b_q S^+(q) = k$ and $1 \le b_q \le m/k$. By Theorem 3.2(1–2), we deduce that $S^+(q+1) = b_q k - k$ and $S^-(q+1) = b_q k$. Since $S^+(q+1) + S^-(q+1) = m$ for $q = 0, 1$, we see that $b_0 = b_1$, $S^+(1) = S^+(2) = s$, and $s + (s+k) = m$. Since $(K(1), M(q), S^+(1)) = (K(2), M(2), S^+(2)) = (k, 2s+k, s)$, we have $(K(0), M(0), S^+(0)) = (k, 2s+k, s)$. This completes the proof of the claimed implication. The opposite implication follows from Theorem 2.1.

It is easy to see that the following conditions are equivalent ((i)⇔(ii) follows from Theorem 4.1):

(i) $\{(k, 2s+k, s)\}$ is a one-point orbit of G,

(ii) $m = 2s+k = kz$, $s = kx$, where $0 \le x < z$ are integers such that z is a divisor of $x^2 + x + 1$,

(iii) $m = k(2x+1)$, $s = kx$, where integers $x \ge 0$ and $y > 0$ are solutions of the equation $x^2 + x + 1 = y(2x+1)$.

Let D be the determinant of the quadratic equation $x^2 + x(1-2y) + 1 - y = 0$. Since $D = 4y^2 - 3$ is the square of an integer, it follows that $y = 1$. Hence, $x = 0$ or $x = 1$, which completes the proof. ∎

5. 2-tree partitions with trees of the same order. Suppose that G is a 2-connected plane triangulation which has a 2-tree partition, that is, G has two disjoint induced trees S, T which together cover all vertices of G. Denote by f_i^S and f_i^T the number of vertices of degree i contained in S and T, respectively. Tutte [15] proved the following identity, which is the dual version of the well-known Grinberg theorem [7]:

$$\sum_i (i-2) f_i^S = \sum_i (i-2) f_i^T. \tag{1}$$

Let us denote by f_i the number of vertices of degree i of the graph G. Euler's equation becomes

$$\sum_i (6-i) f_i = 12. \tag{2}$$

Recall that $\mathcal{P}$ (resp. $\mathcal{H}$) is the family of all 2-connected plane triangulations all of whose vertices are of degree 3 or 6 (at most 6, respectively).

THEOREM 5.1. *If $G \in \mathcal{P}$, then for every 2-tree partition of G the trees have the same number of vertices of degree 6, and the same number of vertices of degree 3 in G.*

Proof. Let S and T be two disjoint induced trees which together cover all vertices of G. By (1) we have $4f_6^S + f_3^S = 4f_6^T + f_3^T$. Hence, $f_3^S \equiv f_3^T \pmod 4$. In view of $f_3^S + f_3^T = 4$ we have two cases: $f_3^S = 4$ or $f_3^S = 2 = f_3^T$. In the first case, $4f_6^S + 4 = 4f_6^T$. Accordingly, f_6 is odd. Hence, we have a contradiction, because the order of G is even. In the second case we have $f_3^S = f_3^T$ and we obtain $f_6^S = f_6^T$. ∎

Theorem 5.2. *If $G \in \mathcal{H}$, then for every 2-tree partition of G the orders of the trees differ by at most 3.*

Proof. Let S and T be two disjoint induced trees which together cover all vertices of G. By (1) and (2) we obtain

$$\Big|\sum_{i=3}^{6} f_i^T - \sum_{i=3}^{6} f_i^S\Big| = \Big|\sum_{i=3}^{5}(f_i^T - f_i^S) - \sum_{i=3}^{5} \frac{i-2}{4}(f_i^T - f_i^S)\Big|$$
$$= \Big|\frac{1}{4}\sum_{i=3}^{5}(6-i)(f_i^T - f_i^S)\Big| \le \frac{1}{4}\sum_{i=3}^{5}(6-i)f_i = 3,$$

which completes the proof. ■

6. 2-tree partitions with trees which are equitable 2-colorable. Let $\mathcal{P}$ be the family of all 2-connected plane triangulations all of whose vertices are of degree 3 or 6. We recall that a graph $P \in \mathcal{P}$ is factorable into factors P_0, P_1, P_2 (indexed by elements of the cyclic group $Q = \{0,1,2\}$) satisfying condition (GM1). We will give an example of a graph in $\mathcal{P}$ which has a 2-tree partition, but the trees are not equitable 2-colorable. In Theorem 6.1 we will prove that if $P \in \mathcal{P}$ has order $4n+2$, $n \in \mathbb{N}$, then it has a 2-tree partition such that the two trees are equitable 2-colorable.

Example 6.1. Let $G \in \mathcal{P}$ be the graph of Fig. 4. Notice that G contains two disjoint induced trees whose vertices together span all of G. However, the induced trees are not equitable 2-colorable.

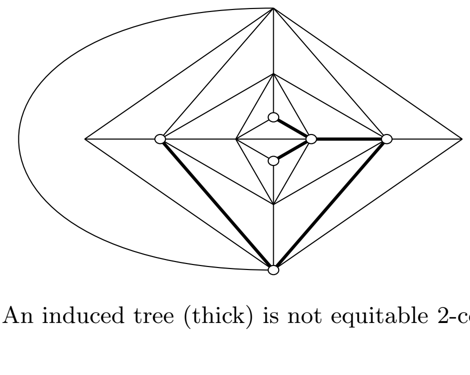


Fig. 4. An induced tree (thick) is not equitable 2-colorable.

A *k-caterpillar*, $k \ge 1$, is a tree T which contains a path T_0 such that $T - V(T_0)$ is a family of independent paths of the same order k. The path T_0 is referred to as the *spine* of T (see Chartrand and Lesniak [1]). Paths and k-caterpillars, for k even, are called *even caterpillars*. Notice that even caterpillars are equitable 2-colorable.

Goodey [6] constructed a Hamiltonian cycle in every 2-connected cubic plane graph whose faces are only triangles or hexagons. Suppose that $H \in \mathcal{P}$ has a unique cycle of class q for some $q \in Q$. In Lemma 6.1 we use a dual version of Goodey's construction to partition the vertex set of H into two subsets so that each induces an even caterpillar.

LEMMA 6.1. *Suppose that $H \in \mathcal{P}$ has a unique cycle of class q, say γ_1, for some $q \in Q$. Then H contains two disjoint, induced even caterpillars T and S (T is a $(2d-2)$-caterpillar, where $d = |S^+(q)+1, M(q)|$, and S is a path) whose vertices together span all of H. Moreover,*

(1) *$T \cap \gamma_1$ is a family of independent paths in H with the same order $2d-1$, and $S \cap \gamma_1$ is an independent set of vertices.*

Proof. Let $\gamma = v_0 v_1 \dots v_{M(q)}$ and γ' be two maximal paths of class q, and suppose that $\gamma_1 = t_0 t_1 \dots t_{2M(q)-1}$ is the clockwise oriented cycle of class q in H. Without loss of generality we can assume that the vertices t_0, t_1 are adjacent to v_1 (see Fig. 6).

Suppose that $S^+(q) < M(q) - 1$. In the graph $H - V(\gamma)$ we identify successive vertices and edges of the path $t_0 t_1 \dots t_{M(q)}$ with successive vertices and edges of the path $t_0 t_{2M(q)-1} t_{2M(q)-2} \dots t_{M(q)}$. After the identification we obtain a path $\delta = w_0 w_1 \dots w_{M(q)}$ and a graph $H_\gamma \in \mathcal{P}$ (see Fig. 5). We can assume that δ and γ' are two maximal paths of the same class q in H_γ. Since $K(q) = 2$, $(K_\gamma(q), M_\gamma(q), S^+_\gamma(q)) = (1, M(q), S^+(q))$ is the q-index-vector of the graph H_γ. Let $e_1, \dots, e_n$ be a sequence of all consecutive edges of the path $[w_0, q-1]$ which are adjacent to the path δ (see Fig. 5).

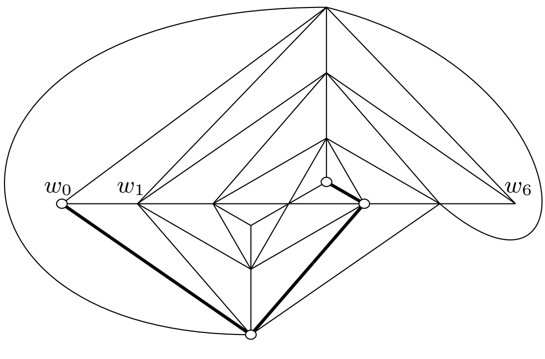


Fig. 5. A $[w_0, q-1]$ path (thick) in the graph H_γ; $I = \{0, 4\}$ (see Lemma 6.1).

Since $S^-_\gamma(q) = S^+_\gamma(q) + 1 = S^+(q) + 1 < M(q)$, we have $n > 1$. By Lemma 3.1(1) and Corollary 3.1(1–2), we obtain

(2) $\{[w_0, q](e_1), [w_0, q](e_2), \dots, [w_0, q](e_n)\} = \{0, 2d, 4d, \dots, 2M(q) - 2d\}$,

where $d = |S^-_\gamma(q), M_\gamma(q)| = |S^+(q)+1, M(q)|$. Let

$$I = \{0 \le i \le M(q) : w_i \in V([w_0, q-1])\}.$$

We can consider $V_0 = V([w_0, q-1]) \cap V(\gamma')$ as a set of vertices in H. It is not difficult to see that the set

$$V_1 = V_0 \cup \bigcup_{i\in I}\{v_i, t_i\} \cup \bigcup_{i\in I\setminus\{0,M(q)\}} \{t_{2M(q)-i}\}$$

induces a path T_0 in H (see Fig. 6). Accordingly, by (2), the set

$$V_2 = V_1 \cup \bigcup_{i\in I}\{t_{i+1}, t_{i+2}, \ldots, t_{i+2d-2}\} \cup \bigcup_{i\in I\setminus\{0,M(q)\}} \{t_{2M(q)-i+1}, t_{2M(q)-i+2}, \ldots, t_{2M(q)-i+2d-2}\}$$

induces a $(2d-2)$-caterpillar T in H with spine T_0 (see Fig. 6). Notice that $V(H) - V_2$ induces a path S in H, and condition (1) holds.

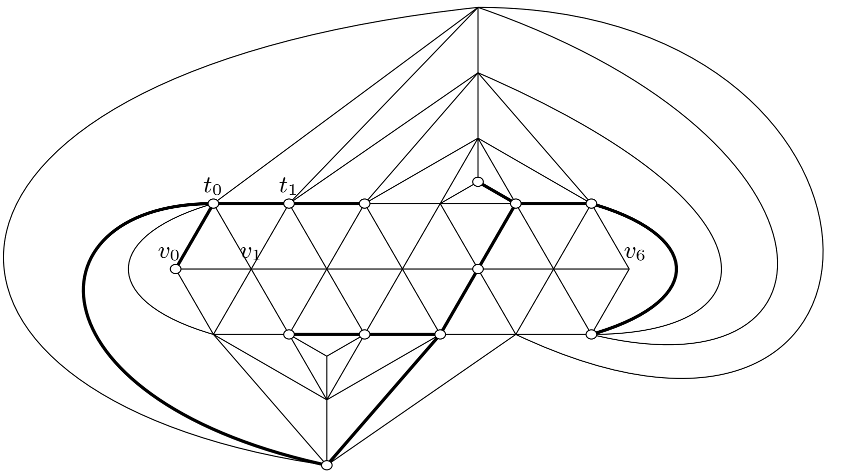


Fig. 6. A 2-caterpillar T (thick) in the graph H (see Lemma 6.1); $J = \{3, 7, 11\}$ (see Theorem 6.1).

If $S^+(q) = M(q) - 1$, then there exists a vertex $u \ne v_0$ of degree 3 which is adjacent to $t_{2M(q)-1}$ and t_0. Then the set $W = \{u, v_0, t_0, t_1, \ldots, t_{2M(q)-2}\}$ induces a $(2M(q)-2)$-caterpillar T with spine ut_0v_0, and $V(H) - W$ induces a path S satisfying condition (1). ∎

THEOREM 6.1. *Let $P \in \mathcal{P}$. If P has order $4n+2$, $n \in \mathbb{N}$, then P contains two disjoint, induced even caterpillars whose vertices together span all of P.*

Proof. Let $P \in \mathcal{P}$ have order $4n+2$, $n \in \mathbb{N}$. Let $(K(q), M(q), S^+(q))$ be the q-index-vector of P, $q \in Q$. First we prove that $K(q)$ is even for some

$q \in Q$. We know that $2K(q)M(q) + 2 = 4n + 2$ for every $q \in Q$. Suppose that $K(q)$ is odd for some $q \in Q$. Hence, $M(q)$ is even. By Theorem 2.1, $S^-(q) - S^+(q) \equiv K(q) \pmod{M(q)}$, whence $S^+(q)$ or $S^-(q)$ is even. By Theorem 3.1(3) and Corollary 3.1(3), $K(q \pm 1) = |S^\pm(q), M(q)|$, whence $K(q+1)$ or $K(q-1)$ is even.

Let now $K(q) = k$ be even, and suppose that γ_0, γ' are maximal paths of class q, and $\gamma_1, \ldots, \gamma_{k-1}$ are clockwise oriented cycles of class q in P such that vertices of γ_j are adjacent to vertices of γ_{j-1}, $1 \le j < k$. We will prove that P contains two disjoint, induced even caterpillars T_k and S_k whose vertices together span all of P, and the following condition is satisfied:

$$(3) \quad \begin{cases} \{T_k \cap \gamma_j : j \text{ odd}, 1 \le j < k\} \cup \{S_k \cap \gamma_j : j \text{ even}, 1 < j < k\} \\ \text{is a family of independent paths in } P \text{ with the same odd order,} \\ \text{and } \{T_k \cap \gamma_j : j \text{ even}, 1 < j < k\} \cup \{S_k \cap \gamma_j : j \text{ odd}, 1 \le j < k\} \\ \text{is an independent set of vertices in P.} \end{cases}$$

We proceed by induction on the even number $K(q) = k$. By Lemma 6.1, we can assume that $k \ge 4$. Let

$$\gamma_{k-3} = x_0 x_1 \ldots x_{M(q)-1}, \quad \gamma_{k-2} = y_0 y_1 \ldots y_{2M(q)-1}, \quad \gamma_{k-1} = z_0 z_1 \ldots z_{2M(q)-1}.$$

Without loss of generality we can assume that y_0, y_1 are adjacent to x_1, and z_0, z_1 are adjacent to y_0 (see Fig. 7). In the graph $P - V(\gamma_{k-2})$ we

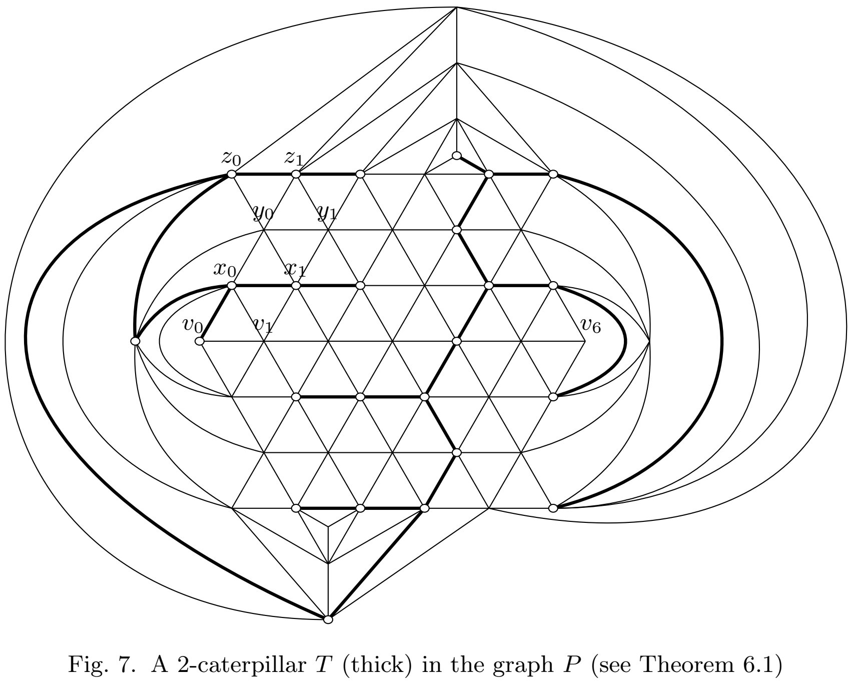


Fig. 7. A 2-caterpillar T (thick) in the graph P (see Theorem 6.1)

identify successive vertices and edges of the cycle γ_{k-1} with successive vertices and edges of the cycle γ_{k-3}. After the identification we obtain a cycle $\delta = t_0 t_1 \cdots t_{2M(q)-1}$ and a graph $H \in \mathcal{P}$ (see Fig. 6). We can assume that γ_0, γ' (or γ_j, for $1 \le j < k-3$, and δ) are maximal paths (or cycles, respectively) of class q in H. By induction H contains two disjoint, induced even caterpillars T_{k-2} and S_{k-2} whose vertices together span all of H, and condition (3) holds (for k replaced with $k-2$, and P replaced with H). Let

$$I = \{0 \le i < 2M(q) : t_i \in V(T_{k-2})\},$$
$$J = \{0 \le i < 2M(q) : t_i \in V(S_{k-2})\}.$$

We can consider $V_T = V(T_{k-2}) \setminus V(\delta)$ and $V_S = V(S_{k-2}) \setminus V(\delta)$ as sets of vertices in the graph P. Hence, the sets

$$V_T \cup \{x_i : i \in I\} \cup \{z_i : i \in I\} \cup \{y_i : i \in J\},$$
$$V_S \cup \{x_i : i \in J\} \cup \{z_i : i \in J\} \cup \{y_i : i \in I\}$$

induce (respectively) two disjoint even caterpillars T_k and S_k whose vertices together span all of P, and condition (3) holds. ■

7. Orbits of non-simple plane triangulations in $\mathcal{P}$. In the following theorem we characterize orbits of plane triangulations in $\mathcal{P}$ which are not simple.

THEOREM 7.1. *$G \in \mathcal{P}$ is not simple if and only if G has an orbit of the form*

$$\{(n,1,0),(1,n,n-1),(1,n,0)\} \quad \textit{for some integer } n > 1.$$

Proof. Let $G \in \mathcal{P}$. It is easy to prove that the following conditions are equivalent ((iv)⇔(v) follows from Theorems 3.1(3–4) and 3.2(2)):

(i) G is not simple,
(ii) G has a cycle of class q and length 2 for some $q \in Q$,
(iii) $G \ne K_4$ and it has two edges of class q with ends of degree 3 for some $q \in Q$,
(iv) G has an index-vector of the form $(n,1,0)$ for some $n > 1$,
(v) G has an orbit of the form $\{(n,1,0),(1,n,n-1),(1,n,0)\}$ for some $n > 1$.

This completes the proof. ■

Appendix: On the Diophantine equation $x^2 + x + 1 = yz$ (by A. Schinzel). Let us adopt the notation introduced in the classical book

[11, pp. 5–6]:

$$\begin{aligned} &A_{-1}=1, \quad A_0=b_0, \quad A_\nu=b_\nu A_{\nu-1}+A_{\nu-2} \quad (\nu\ge 1),\\ &B_{-1}=0, \quad B_0=1, \quad B_\nu=b_\nu B_{\nu-1}+B_{\nu-2} \quad (\nu\ge 1), \end{aligned}$$

where b_ν ($\nu \ge 0$) is an arbitrary sequence of integers.

We shall prove

THEOREM. *For every even $k \ge 0$ and all positive integers $b_0, \dots, b_k$, the positive integers*

$$\begin{aligned} x &= A_{k-1}A_k + B_{k-1}B_k + A_kB_{k-1},\\ y &= A_{k-1}^2 + A_{k-1}B_{k-1} + B_{k-1}^2,\\ z &= A_k^2 + A_kB_k + B_k^2 \end{aligned}$$

satisfy the equation $x^2+x+1=yz$ and the inequality $y<z$.

Proof. We have (see [11, p. 16, formula (30)])

$$A_\lambda B_{\lambda-1} - A_{\lambda-1}B_\lambda = (-1)^{\lambda-1},$$

which for k even gives

$$A_kB_{k-1} - A_{k-1}B_k = -1,$$

hence

$$\begin{aligned} x^2+x+1-yz &= A_{k-1}^2A_k^2 + 2A_{k-1}A_kB_{k-1}B_k + 2A_{k-1}A_k^2B_{k-1}\\ &\quad + B_{k-1}^2B_k^2 + 2A_kB_{k-1}^2B_k + A_k^2B_{k-1}^2 + A_{k-1}A_k + B_{k-1}B_k + A_kB_{k-1} + 1\\ &\quad - A_{k-1}^2A_k^2 - A_{k-1}^2A_kB_k - A_{k-1}^2B_k^2 - A_{k-1}A_k^2B_{k-1} - A_{k-1}A_kB_{k-1}B_k\\ &\quad - A_{k-1}B_{k-1}B_k^2 - A_k^2B_{k-1}^2 - A_kB_{k-1}^2B_k - B_{k-1}^2B_k^2\\ &= A_{k-1}A_kB_{k-1}B_k + A_{k-1}A_k^2B_{k-1} + A_kB_{k-1}^2B_k\\ &\quad + A_{k-1}A_k + B_{k-1}B_k + A_kB_{k-1} + 1\\ &\quad - A_{k-1}^2A_kB_k - A_{k-1}^2B_k^2 - A_{k-1}B_{k-1}B_k^2\\ &= (A_{k-1}B_k + A_{k-1}A_k + B_{k-1}B_k)(A_kB_{k-1} - A_{k-1}B_k)\\ &\quad + A_{k-1}A_k + B_{k-1}B_k + A_kB_{k-1} + 1\\ &= -A_{k-1}B_k + A_kB_{k-1} + 1 = 0. \end{aligned}$$

Moreover, since b_i are positive integers, we have

$$0 < A_{k-1} < A_k, \quad 0 \le B_{k-1} \le B_k, \quad \text{hence } y < z.$$

Using [11, Chapter II, Theorem 13] and [2, Theorem 131] one can prove that all solutions of the equation $x^2+x+1=yz$ in positive integers x, y, z satisfying the condition $y<z$ can be obtained from the formula given in the Theorem for some integer b_0 and some positive integers b_i ($i=1,\dots,k$). ∎

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REFERENCES

[1] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman and Hall, New York, 2005.

[2] L. E. Dickson, *Modern Elementary Theory of Numbers*, Univ. Press, Chicago, 1939.

[3] R. Diestel, *Graph Theory*, Springer, Berlin, 2005.

[4] J. Florek, *Billiard and Diophantine approximation*, Acta Arith. 134 (2008), 317–327.

[5] J. Florek, *On Barnette's conjecture*, Discrete Math. 310 (2010), 1531–1535.

[6] P. R. Goodey, *A class of Hamiltonian polytopes*, J. Graph Theory 1 (1977), 181–185.

[7] E. J. Grinberg, *Plane homogeneous graphs of degree three without Hamiltonian circits*, in: Latvian Math. Yearbook 4, Izdat. "Zinatne", 1968, 51–58 (in Russian).

[8] B. Grünbaum and T. S. Motzkin, *The number of hexagons and the simplicity of geodesics on certain polyhedra*, Canad. J. Math. 15 (1963), 744–751.

[9] T. R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley, New York, 1995.

[10] X. Lu, *A note on Barnette's conjecture*, Discrete Math. 311 (2011), 2711–2715.

[11] O. Perron, *Die Lehre von den Kettenbrüchen*, 2nd ed., reprint Chelsea, New York, 1950.

[12] A. Schinzel et W. Sierpiński, *Sur l'équation* $x^2+x+1=3y^2$, Colloq. Math. 4 (1956), 71–73.

[13] W. M. Schmidt, *Diophantine Approximation*, Springer, New York, 1980.

[14] S. K. Stein, *B-sets and planar maps*, Pacific J. Math. 37 (1971), 217–224.

[15] W. T. Tutte, *Graph Theory*, Encyclopedia Math. Appl. 21, Addison-Wesley, 1984.

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