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AN IDENTITY RAMANUJAN PROBABLY MISSED

BY

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Abstract. "Ramanujan's 6-10-8 identity" inspired Hirschhorn to formulate his "3-7-5 identity". Now, we give a new "6-14-10 identity" which we suppose Ramanujan would have discovered but missed to mention in his notebooks.

1. Introduction. In his third Notebook [5], Ramanujan writes: "If $a/b = c/d$, then

$$\begin{aligned} &64\{(a+b+c)^6+(b+c+d)^6-(c+d+a)^6 \\ &\qquad -(d+a+b)^6+(a-d)^6-(b-c)^6)\} \\ &\times\{(a+b+c)^{10}+(b+c+d)^{10}-(c+d+a)^{10} \\ &\qquad -(d+a+b)^{10}+(a-d)^{10}-(b-c)^{10})\} \\ &=45\{(a+b+c)^8+(b+c+d)^8-(c+d+a)^8 \\ &\qquad -(d+a+b)^8+(a-d)^8-(b-c)^8)\}^2." \end{aligned}$$

This is "Ramanujan's 6-10-8 identity" which Berndt [1] describes to be "an amazing identity". He replicates a proof due to Berndt and Bhargava [2] and refers to another proof by Nanjundiah [4]. Inspired by "Ramanujan's 6-10-8 identity", Hirschhorn [3] found a "3-7-5 identity" as

$$\begin{aligned} &25\{(a+b+d)^3+(b+c+d)^3-(a+b+c)^3 \\ &\qquad -(a+c+d)^3+(a-d)^3-(b-c)^3)\} \\ &\times\{(a+b+d)^7+(b+c+d)^7-(a+b+c)^7 \\ &\qquad -(a+c+d)^7+(a-d)^7-(b-c)^7))\} \\ &=21\{(a+b+d)^5+(b+c+d)^5-(a+b+c)^5 \\ &\qquad -(a+c+d)^5+(a-d)^5-(b-c)^5)\}^2, \end{aligned}$$

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where it is assumed that $ad = bc$. But, a similar identity which the great master probably missed to mention is:

$$\begin{aligned} 25\{(m^2+n^2)^6 - (m^2-n^2)^6 - (2mn)^6\} \\ \times\{(m^2+n^2)^{14} - (m^2-n^2)^{14} - (2mn)^{14}\} \\ = 21\{(m^2+n^2)^{10} - (m^2-n^2)^{10} - (2mn)^{10}\}^2, \end{aligned} \tag{1.1}$$

which is true for any real values of m and n. In the next section, we will prove this "6-14-10 identity" by using very elementary steps, which will naturally inspire us to re-look at the "remarkable identity of Ramanujan" in the hope of discovering similar simpler steps involved in its proof—the proofs due to Berndt and Bhargava [2] and Nanjundiah [4] are not that elementary.

2. The key identity. We need the following lemma to prove the identity (1.1):

LEMMA 2.1. *For any non-zero real values of a and b we have*

$$\begin{aligned} 25\{(a+b)^3 - (a-b)^3 - (2b)^3\} \\ \times\{(a+b)^7 - (a-b)^7 - (2b)^7\} \\ = 21\{(a+b)^5 - (a-b)^5 - (2b)^5\}^2. \end{aligned} \tag{2.1}$$

Proof. By direct algebraic manipulation we get

$$\frac{(a+b)^3 - (a-b)^3 - (2b)^3}{(a+b)^5 - (a-b)^5 - (2b)^5} = \frac{3}{5(a^2+3b^2)} \tag{2.2}$$

and

$$\frac{(a+b)^7 - (a-b)^7 - (2b)^7}{(a+b)^5 - (a-b)^5 - (2b)^5} = \frac{7(a^2+3b^2)}{5}. \tag{2.3}$$

So, LHS of (2.2)$\times$ LHS of (2.3)= RHS of (2.2)$\times$ RHS of (2.3), which, after simplification, gives (2.1). ∎

Proof of (1.1). Taking $a = (m^4+n^4)$, $b = 2m^2n^2$ in (2.1), and observing that $(a+b) = (m^2+n^2)^2$, $(a-b) = (m^2-n^2)^2$ and $2b = (2mn)^2$, we find that the identity (2.1) transforms to the identity (1.1). ∎

3. Conclusion. At this moment, we do not know on which route Ramanujan discovered his "6-10-8 identity". Keeping in mind his humble background, and his unconventional way of approach to a problem, we expect the route to be straight and smooth.

Our "3-7-5 identity" of (2.1) and "6-14-10 identity" of (1.1) are not exactly similar to that of Ramanujan. But they are meant to motivate the search for undiscovered simpler steps involved in the proof of "Ramanujan's 6-10-8 identity".

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