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NON-NILPOTENT SUBGROUPS OF LOCALLY GRADED GROUPS

BY

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Abstract. We show that a locally graded group with a finite number m of non-(nilpotent of class at most n) subgroups is (soluble of class at most $[\log_2 n] + m + 3$)-by-(finite of order $\leq m!$). We also show that the derived length of a soluble group with a finite number m of non-(nilpotent of class at most n) subgroups is at most $[\log_2 n]+m+1$.

1. Introduction and results. Let G be a group. A non-nilpotent finite group whose proper subgroups are all nilpotent is well-known (called a *Schmidt group*). In 1924, Schmidt [7] studied such groups and proved that they are soluble. Subsequently, Newman and Wiegold [5] discussed infinite non-nilpotent groups whose proper subgroups are all nilpotent. Such groups need not be soluble in general. For example, the *Tarski monsters* are infinite simple groups with all proper subgroups of a fixed prime order.

Following [9] we say that a group G is an $\mathcal{S}^m$*-group* if G has exactly m non-nilpotent subgroups. More recently Zarrin [9] generalized Schmidt's Theorem and proved that every finite $\mathcal{S}^m$-group with $m < 22$ is soluble. Let n be a non-negative integer. We say that a group G is an $\mathcal{S}_n^m$*-group* if G has exactly m non-(nilpotent of class at most n) subgroups. Clearly, every $\mathcal{S}_n^m$-group is an $\mathcal{S}^r$-group for some $r \leq m$. Here, we show that every locally graded group with a finite number m of non-(nilpotent of class at most n) subgroups is soluble-by-finite. Recall that a group G is *locally graded* if every non-trivial finitely generated subgroup of G has a non-trivial finite homomorphic image. This is a rather large class of groups, containing for instance all residually finite groups and all locally (soluble-by-finite) groups.

THEOREM A. *Every locally graded* $\mathcal{S}_n^m$*-group is* (*soluble of class at most* $[\log_2 n] + m + 3$)*-by-*(*finite of order* $\leq m!$).

This result suggests that the behavior of non-(nilpotent of class at most n) subgroups has a strong influence on the structure of the group.

Finding an upper bound for the solubility length of a soluble group is an important problem in the theory of groups, for example see [8]. It is well-known that a nilpotent group of class n (or a group without non-(nilpotent

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of class at most n) subgroups) has derived length $\leq [\log_2 n] + 1$ (see [6, Theorem 5.1.12]). Here, we obtain a result which is of independent interest, namely, the derived length of soluble $\mathcal{S}_n^m$-groups is bounded in terms of m and n. (Note that every nilpotent group of class n is an $\mathcal{S}_n^m$-group with $m = 0$.)

THEOREM B. *Let G be a soluble $\mathcal{S}_n^m$-group and d be the derived length of G. Then $d \leq [\log_2 n] + m + 1$.*

2. Proofs. If G is an arbitrary group, then the *norm* $B_1(G)$ of G is the intersection of the normalizers of all subgroups of G, and $W(G)$ is the intersection of the normalizers of all subnormal subgroups of G. In 1934 and 1958, respectively, those concepts were considered by Baer and Wielandt (see also [1, 2, 3]). Recently Zarrin [10] generalized this concept. Here, we define $A_n(G)$ as the intersection of all the normalizers of non-(nilpotent of class at most n) subgroups of G, i.e.,

$$A_n(G) = \bigcap_{H \in \mathfrak{T}_n(G)} N_G(H),$$

where $\mathfrak{T}_n(G) = \{H \mid H \text{ is a non-(nilpotent of class at most } n) \text{ subgroup of } G\}$ (with the stipulation that $A_n(G) = G$ if all subgroups of G are nilpotent of class at most n). Clearly

$$B_1(G) \leq A_i(G) \leq A_{i+1}(G).$$

Moreover, in view of the proof of Theorem A below, we can see that, for every locally graded group G, we have

$$A_n(G) \text{ is a soluble normal subgroup of } G \text{ of class} \leq [\log_2 n] + 4.$$

Proof of Theorem A. The group G acts on the set $\mathfrak{T}_n(G)$ by conjugation. By assumption $|\mathfrak{T}_n(G)| = m$. It is easy to see that the subgroup $A_n(G)$ is the kernel of this action, and so $A_n(G)$ is normal in G and $G/A_n(G)$ is embedded in S_m, the symmetric group of degree m. Hence

$$|G/A_n(G)| \leq m!.$$

Therefore to complete the proof it is enough to show that $H = A_n(G)$ is soluble of class at most $[\log_2 n] + 4$. To see this, it is enough to show that $K = H^{(3)}$ is nilpotent of class at most n. Suppose on the contrary that K is not nilpotent of class at most n. It follows that every subgroup containing K is not nilpotent of class at most n and so, by definition of $A_n(G)$, it is a normal subgroup of H. Therefore every subgroup of H/K is normal. That is, H/K is a Dedekind group, and hence it is well-known (see [6, Theorem 5.3.7]) that H/K is metabelian. Consequently,

$$(\bullet) \qquad H^{(2)} = H^{(3)} = K.$$

We proceed through the following conclusions.

Step 1: Every proper normal subgroup of K is nilpotent of class at most n. Suppose on the contrary that there exists a proper normal subgroup M of $K = H^{(2)}$ such that M is not nilpotent of class at most n. Then we can deduce, by definition of $A_n(G)$, that $H^{(2)}/M$ is a Dedekind group (so it is metabelian) and hence, in view of ($\bullet$), $H^{(2)} = M$, a contradiction.

Step 2: The product of all proper normal subgroups of K, say R, is a proper nilpotent subgroup of K of class at most n. Suppose that $M_1, \ldots, M_t$ are proper normal subgroups of $H^{(2)}$. Then, by Step 1, every M_i is soluble and so $M_1 \ldots M_t$ is soluble. Now, by ($\bullet$), we conclude that $H^{(2)} \neq M_1 \ldots M_t$. Therefore $M_1 \ldots M_t$ is a proper normal subgroup of $H^{(2)}$ and so, by Step 1 again, it is nilpotent of class at most n. Therefore R is locally nilpotent of class at most n, and so nilpotent of class at most n (note that the class of nilpotent groups of class at most n is locally closed). Also, because of ($\bullet$), we have $R \neq H^{(2)}$.

Step 3: Finishing the proof. We note that, by definition of $A_n(G)$, every subgroup of $H^{(2)}$ which is not nilpotent of class at most c is a normal subgroup of $H^{(2)}$. It follows, as $H^{(2)}/R$ is a simple group, that all proper subgroups of $H^{(2)}/R$ are nilpotent of class at most n. Since $H^{(2)}$ is locally graded, by the main result of [4], $H^{(2)}/R$ is locally graded. Therefore if $H^{(2)}/R$ is finitely generated then it must be finite. Thus, by Schmidt's Theorem, $H^{(2)}/R$ is soluble, contradicting ($\bullet$). If $H^{(2)}/R$ is not finitely generated, then $H^{(2)}/R$ is locally nilpotent of class at most n and so $H^{(2)}/R$ is nilpotent of class at most n, a contradiction.

Proof of Theorem B. Assume that a soluble group G has derived length $> [\log_2 n] + 1 + m$ for some $n, m \geq 1$. Then obviously the $m+1$ derived subgroups $G, G', \ldots, G^{(m)}$ are all pairwise distinct and have solubility length $> [\log_2 n] + 1$. Therefore they cannot be nilpotent of class at most n. This shows that G cannot be an $\mathcal{S}_n^m$-group, a contradiction.

Finally, as every $\mathcal{S}_n^m$-group is an $\mathcal{S}^r$-group for some $r \leq m$, and by the main result in [9], we can see that every $\mathcal{S}_n^m$-group with $m \leq 21$ is soluble. Hence the following question arises naturally:

Question 1. *Assume that G is an $\mathcal{S}_n^m$-group. What relations between m and n guarantee that G is soluble?*

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