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REGION OF VARIABILITY FOR SPIRAL-LIKE FUNCTIONS WITH RESPECT TO A BOUNDARY POINT

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Abstract. For $\mu \in \mathbb{C}$ such that $\operatorname{Re}\mu > 0$ let $\mathcal{F}_\mu$ denote the class of all non-vanishing analytic functions f in the unit disk $\mathbb{D}$ with $f(0) = 1$ and

$$\operatorname{Re}\left(\frac{2\pi}{\mu}\,\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}\right) > 0 \quad \text{in } \mathbb{D}.$$

For any fixed z_0 in the unit disk, $a \in \mathbb{C}$ with $|a| \leq 1$ and $\lambda \in \overline{\mathbb{D}}$, we shall determine the region of variability $V(z_0, \lambda)$ for $\log f(z_0)$ when f ranges over the class

$$\mathcal{F}_\mu(\lambda) = \left\{ f \in \mathcal{F}_\mu : f'(0) = \frac{\mu}{\pi}(\lambda - 1) \text{ and } f''(0) = \frac{\mu}{\pi}\left(a(1-|\lambda|^2) + \frac{\mu}{\pi}(\lambda-1)^2 - (1-\lambda^2)\right)\right\}.$$

In the final section we graphically illustrate the region of variability for several sets of parameters.

1. Introduction. We denote by $\mathcal{H}(\mathbb{D})$ the class of analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and we think of $\mathcal{H}(\mathbb{D})$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $\mathbb{D}$. Denote by $\mathcal{S}^*$ the subclass of functions $\phi \in \mathcal{H}(\mathbb{D})$ with $\phi(0) = 0$ such that ϕ maps $\mathbb{D}$ univalently onto a domain $\Omega = \phi(\mathbb{D})$ that is starlike with respect to the origin. That is, $t\phi(z) \in \phi(\mathbb{D})$ for each $t \in [0, 1]$. It is well known that for $\phi \in \mathcal{H}(\mathbb{D})$ with $\phi(0) = 0 = \phi'(0) - 1$, $\phi \in \mathcal{S}^*$ if and only if

$$\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right) > 0, \qquad z \in \mathbb{D}.$$

Functions in $\mathcal{S}^*$ are referred to as *starlike functions*. Denote by $\mathcal{C}$ the subclass of functions $\phi \in \mathcal{H}(\mathbb{D})$ with $\phi(0) = 0$ such that ϕ maps $\mathbb{D}$ univalently onto a convex domain. It is well known that for $\phi \in \mathcal{H}(\mathbb{D})$ with $\phi(0) = 0 = \phi'(0)-1$, $\phi \in \mathcal{C}$ if and only if $z\phi' \in \mathcal{S}^*$. Functions in $\mathcal{C}$ are referred to as *convex*

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functions. We refer to the books [2, 5] for a detailed discussion of these two classes. Although the class of starlike functions (with respect to an interior point) has been extensively studied, little was known about starlike functions with respect to a boundary point until the work of Robertson [15]. Motivated by the work in [15] and characterizations of this class of functions, some advancement in this direction has taken place (see [17, 7, 4, 6]). On the other hand, there does not seem to be any development on spiral-like functions with respect to a boundary point until the recent work of Elin et al. [3] (see also [4]). More recently, Aharonov et al. [1] provide a natural geometric approach to spiral-like functions with respect to a boundary point, and the conditions described in [1] cover the results studied by others. On the other hand, several authors have studied region of variability problems for various subclasses of univalent functions in $\mathcal{H}(\mathbb{D})$ (see [8, 9, 12, 13, 14, 18, 19]). For example, it is well known that for each fixed $z_0 \in \mathbb{D}$, the region of variability

$$V(z_0) = \{\log \phi'(z_0) : \phi \in \mathcal{C},\ \phi'(0) = 1\}$$

is the set $\{\log(1-z)^{-2} : |z| \le |z_0|\}$.

For $\mu \in \mathbb{C}$ such that $\operatorname{Re}\mu > 0$, let $\mathcal{F}_\mu$ denote the class of functions $f \in \mathcal{H}(\mathbb{D})$ non-vanishing in $\mathbb{D}$ with $f(0) = 1$ and

$$\operatorname{Re} P_f(z) > 0, \qquad z \in \mathbb{D},$$

where

$$P_f(z) = \frac{2\pi}{\mu}\,\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z}. \tag{1.1}$$

Clearly $P_f(0) = 1$. Basic properties and a number of equivalent characterizations of the class $\mathcal{F}_\mu$ are formulated in [1]. The case $\mu = \pi$ coincides with the class introduced by Robertson [15], who has generated interest in this class, and its associated classes. It is also known that functions in $\mathcal{F}_\pi$ are either close-to-convex or just the constant 1.

For $f \in \mathcal{F}_\mu$, we denote by $\log f$ the single-valued branch of the logarithm of f with $\log f(0) = 0$. The Herglotz representation for analytic functions with positive real part in $\mathbb{D}$ shows that if $f \in \mathcal{F}_\mu$, then there exists a unique positive unit measure ν on $(-\pi, \pi]$ such that

$$\frac{2\pi}{\mu}\,\frac{zf'(z)}{f(z)} + \frac{1+z}{1-z} = \int_{-\pi}^{\pi} \frac{1+ze^{-it}}{1-ze^{-it}}\, d\nu(t),$$

and hence a computation gives

$$\log f(z) = \frac{\mu}{\pi} \int_{-\pi}^{\pi} \log\left(\frac{1-z}{1-ze^{-it}}\right) d\nu(t),$$

or equivalently

$$f(z) = (1-z)^{\mu/\pi} \exp\left\{ \frac{\mu}{\pi} \int_{-\pi}^{\pi} \log\left(\frac{1}{1-ze^{-it}} \right) d\nu(t) \right\}.$$

Let $\mathcal{B}_0$ be the class of analytic functions ω in $\mathbb{D}$ such that $|\omega(z)| \leq 1$ in $\mathbb{D}$ and $\omega(0) = 0$. Then for each $f \in \mathcal{F}_\mu$ there exists an $\omega_f \in \mathcal{B}_0$ of the form

$$\omega_f(z) = \frac{P_f(z) - 1}{P_f(z) + 1}, \quad z \in \mathbb{D}, \tag{1.2}$$

and conversely. It is a simple exercise to see that

$$P_f'(0) = 2\omega_f'(0) = 2\left(\frac{\pi}{\mu} f'(0) + 1 \right). \tag{1.3}$$

Suppose that $f \in \mathcal{F}_\mu$. Then a simple application of the classical Schwarz lemma (see for example [2, 10, 11]) shows that

$$|P_f'(0)| = 2|(\pi/\mu) f'(0) + 1| \leq 2,$$

because $|\omega_f'(0)| \leq 1$. Using (1.2), one can compute that

$$\frac{\omega_f''(0)}{2} = \frac{P_f''(0)}{4} - \lambda^2 \quad \text{and} \quad P_f''(0) = \frac{4\pi}{\mu} f''(0) - \frac{4\mu}{\pi}(\lambda-1)^2 + 4,$$

so that

$$\frac{\omega_f''(0)}{2} = \frac{\pi}{\mu} f''(0) - \frac{\mu}{\pi}(\lambda-1)^2 + 1 - \lambda^2.$$

Also if we let

$$g(z) = \frac{\omega_f(z)/z - \lambda}{1 - \overline{\lambda}\omega_f(z)/z} \quad \text{for } |\lambda| < 1,$$

and $g(z) = \lambda z$ for $|\lambda| = 1$, then we see that

$$g'(0) = \begin{cases} \dfrac{1}{1-|\lambda|^2} \left(\dfrac{\omega_f(z)}{z} \right)' \Big|_{z=0} = \dfrac{1}{1-|\lambda|^2} \left(\dfrac{\omega_f''(0)}{2} \right) & \text{for } |\lambda| < 1, \\ \lambda & \text{for } |\lambda| = 1. \end{cases}$$

We note that for $|\lambda| < 1$,

$$\begin{aligned} |g'(0)| \leq 1 \ &\Leftrightarrow \ \frac{|\omega_f''(0)|}{2(1-|\lambda|^2)} \leq 1 \\ &\Leftrightarrow \ \frac{1}{1-|\lambda|^2} \left| \frac{\pi}{\mu} f''(0) - \frac{\mu}{\pi}(\lambda-1)^2 + 1 - \lambda^2 \right| \leq 1 \\ &\Leftrightarrow f''(0) = \frac{\mu}{\pi} \left(a(1-|\lambda|^2) + \frac{\mu}{\pi}(\lambda-1)^2 - (1-\lambda^2) \right) \end{aligned}$$

for some $a \in \mathbb{C}$ with $|a| \leq 1$. Consequently, for $\lambda \in \overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $a \in \mathbb{C}$ with $|a| \leq 1$, and for $z_0 \in \mathbb{D}$ fixed, it is natural to introduce

$$\begin{aligned}\mathcal{F}_\mu(\lambda) = \Big\{ f \in \mathcal{F}_\mu : f'(0) = \frac{\mu}{\pi}(\lambda - 1) \text{ and}\\ f''(0) = \frac{\mu}{\pi}\bigg(a(1-|\lambda|^2) + \frac{\mu}{\pi}(\lambda-1)^2 - (1-\lambda^2)\bigg)\Big\},\\ V(z_0, \lambda) = \{\log f(z_0) : f \in \mathcal{F}_\mu(\lambda)\}.\end{aligned}$$

From (1.3) and the normalization condition introduced in the class $\mathcal{F}_\mu(\lambda)$, we observe that $\omega_f'(0) = \lambda$. The main aim of this paper is to determine the region of variability $V(z_0, \lambda)$ for $\log f(z_0)$ when f ranges over $\mathcal{F}_\mu(\lambda)$. The precise geometric description of the set $V(z_0, \lambda)$ is established in Theorem 2.6.

2. Basic properties of $V(z_0, \lambda)$ and the main result. To state our main theorem, we need some preparation. For a positive integer p, let

$$(\mathcal{S}^*)^p = \{f = f_0^p : f_0 \in \mathcal{S}^*\}$$

and recall the following result from [18].

LEMMA 2.1. *Let f be an analytic function in $\mathbb{D}$ with $f(z) = z^p + \cdots$. If*

$$\mathrm{Re}\bigg(1 + z\,\frac{f''(z)}{f'(z)}\bigg) > 0, \qquad z \in \mathbb{D},$$

then $f \in (\mathcal{S}^*)^p$.

Now, we list some basic properties of $V(z_0, \lambda)$.

PROPOSITION 2.2.

(1) *$V(z_0, \lambda)$ is compact.*
(2) *$V(z_0, \lambda)$ is convex.*
(3) *For $|\lambda| = 1$ or $z_0 = 0$,*

$$V(z_0, \lambda) = \bigg\{\frac{\mu}{\pi}\log\bigg(\frac{1 - z_0}{1 - \lambda z_0}\bigg)\bigg\}. \tag{2.3}$$

(4) *For $|\lambda| < 1$ and $z_0 \in \mathbb{D} \setminus \{0\}$, $V(z_0, \lambda)$ has $(\mu/\pi)\log\big(\frac{1-z_0}{1-\lambda z_0}\big)$ as an interior point.*

Proof. (1) Since $\mathcal{F}_\mu(\lambda)$ is a compact subset of $\mathcal{H}(\mathbb{D})$, it follows that $V(z_0, \lambda)$ is also compact.

(2) If $f_0, f_1 \in \mathcal{F}_\mu(\lambda)$ and $0 \leq t \leq 1$, then the function

$$f_t(z) = \exp\{(1-t)\log f_0(z) + t \log f_1(z)\}$$

is evidently in $\mathcal{F}_\mu(\lambda)$. Also, because of the representation of f_t, we see easily that the set $V(z_0, \lambda)$ is convex.

(3) If $z_0 = 0$, (2.3) trivially holds. If $|\lambda| = |\omega_f'(0)| = 1$, then it follows from the classical Schwarz lemma that $\omega_f(z) = \lambda z$, which implies

$$P_f(z) = \frac{1+\lambda z}{1-\lambda z} \quad \text{and} \quad f(z) = \left(\frac{1-z}{1-\lambda z}\right)^{\mu/\pi}.$$

Consequently,

$$V(z_0, \lambda) = \left\{\frac{\mu}{\pi}\log\left(\frac{1-z_0}{1-\lambda z_0}\right)\right\}.$$

(4) For $|\lambda| < 1$ and $a \in \overline{\mathbb{D}}$, we define

$$\delta(z, \lambda) = \frac{z+\lambda}{1+\overline{\lambda}z}$$

and

$$H_{a,\lambda}(z) = \exp\left(\frac{\mu}{\pi}\int_0^z \frac{\delta(a\zeta, \lambda) - 1}{(1-\delta(a\zeta,\lambda)\zeta)(1-\zeta)}\, d\zeta\right), \qquad z \in \mathbb{D}. \tag{2.4}$$

First we claim that $H_{a,\lambda} \in \mathcal{F}_\mu(\lambda)$. For this, we compute

$$\frac{2\pi}{\mu}\,\frac{zH'_{a,\lambda}(z)}{H_{a,\lambda}(z)} = \frac{2z(\delta(az,\lambda)-1)}{(1-\delta(az,\lambda)z)(1-z)} = \frac{2z\delta(az,\lambda)}{1-\delta(az,\lambda)z} - \frac{2z}{1-z},$$

and so we see easily that

$$\frac{2\pi}{\mu}\,\frac{zH'_{a,\lambda}(z)}{H_{a,\lambda}(z)} + \frac{1+z}{1-z} = \frac{1+\delta(az,\lambda)z}{1-\delta(az,\lambda)z}.$$

As $\delta(az,\lambda)$ lies in the unit disk $\mathbb{D}$, $H_{a,\lambda} \in \mathcal{F}_\mu(\lambda)$ and the claim follows. Also we observe that

$$\omega_{H_{a,\lambda}}(z) = z\delta(az,\lambda). \tag{2.5}$$

Next we claim that the mapping $\mathbb{D} \ni a \mapsto \log H_{a,\lambda}(z_0)$ is a non-constant analytic function of a for each fixed $z_0 \in \mathbb{D} \setminus \{0\}$ and $\lambda \in \mathbb{D}$. To see this, we put

$$h(z) = \frac{2\pi}{\mu(1-|\lambda|^2)}\,\frac{\partial}{\partial a}\{\log H_{a,\lambda}(z)\}\bigg|_{a=0}.$$

A computation gives

$$h(z) = 2\int_0^z \frac{\zeta}{(1-\lambda\zeta)^2}\, d\zeta = z^2 + \cdots,$$

from which it is easy to see that

$$\mathrm{Re}\left\{\frac{zh''(z)}{h'(z)}\right\} = \mathrm{Re}\left\{\frac{1+\lambda z}{1-\lambda z}\right\} > 0, \qquad z \in \mathbb{D}.$$

By Lemma 2.1 there exists a function $h_0 \in \mathcal{S}^*$ with $h = h_0^2$. The univalence of h_0 together with the condition $h_0(0) = 0$ implies that $h(z_0) \neq 0$ for

$z_0 \in \mathbb{D} \setminus \{0\}$. Consequently, the mapping $\mathbb{D} \ni a \mapsto \log H_{a,\lambda}(z_0)$ is a non-constant analytic function of a, and hence it is an open mapping. Thus, $V(z_0, \lambda)$ contains the open set $\{\log H_{a,\lambda}(z_0) : |a| < 1\}$. In particular,

$$\log H_{0,\lambda}(z_0) = (\mu/\pi) \log\left(\frac{1 - z_0}{1 - \lambda z_0}\right)$$

is an interior point of $\{\log H_{a,\lambda}(z_0) : a \in \mathbb{D}\} \subset V(z_0, \lambda)$. ∎

We remark that, since $V(z_0, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has non-empty interior, the boundary $\partial V(z_0, \lambda)$ is a Jordan curve and $V(z_0, \lambda)$ is the union of $\partial V(z_0, \lambda)$ and its inner domain.

Now we state our main result; its proof will be presented in Section 3.

THEOREM 2.6. *For $\lambda \in \mathbb{D}$ and $z_0 \in \mathbb{D} \setminus \{0\}$, the boundary $\partial V(z_0, \lambda)$ is the Jordan curve given by*

$$(-\pi, \pi] \ni \theta \mapsto \log H_{e^{i\theta},\lambda}(z_0) = \frac{\mu}{\pi} \int_0^{z_0} \frac{\delta(e^{i\theta}\zeta, \lambda) - 1}{(1 - \delta(e^{i\theta}\zeta, \lambda)\zeta)(1 - \zeta)}\, d\zeta.$$

If $\log f(z_0) = \log H_{e^{i\theta},\lambda}(z_0)$ for some $f \in \mathcal{F}_\mu(\lambda)$ and $\theta \in (-\pi, \pi]$, then $f(z) = H_{e^{i\theta},\lambda}(z)$.

3. Proof of Theorem 2.6

PROPOSITION 3.1. *For $f \in \mathcal{F}_\mu(\lambda)$ we have*

$$\left|\frac{f'(z)}{f(z)} - \frac{\mu}{\pi}\, c(z, \lambda)\right| \le \frac{|\mu|}{\pi}\, r(z, \lambda), \quad z \in \mathbb{D}, \tag{3.2}$$

where

$$c(z, \lambda) = \frac{|z|^2(\overline{z} - \lambda)(1 - \overline{\lambda}) - (1 - \lambda)(1 - \overline{\lambda}\overline{z})}{(1 - z)(1 - |z|^2)(1 + |z|^2 - 2\,\mathrm{Re}(\lambda z))},$$
$$r(z, \lambda) = \frac{(1 - |\lambda|^2)|z|}{(1 - |z|^2)(1 + |z|^2 - 2\,\mathrm{Re}(\lambda z))}.$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = H_{e^{i\theta},\lambda}$ for some $\theta \in \mathbb{R}$.

Proof. Let $f \in \mathcal{F}_\mu(\lambda)$. Then there exists $\omega_f \in \mathcal{B}_0$ satisfying (1.2). As noticed in the introduction, through (1.3) and the normalization of f, we have $\omega_f'(0) = \lambda$. It follows from the Schwarz lemma (see for example [2, 10, 11]) that

$$\left|\frac{\omega_f(z)/z - \lambda}{1 - \overline{\lambda}\omega_f(z)/z}\right| \le |z|, \quad z \in \mathbb{D}. \tag{3.3}$$

From (1.1) and (1.2) this is equivalent to

$$\left|\frac{\frac{f'(z)}{f(z)} - \frac{\mu}{\pi}A(z, \lambda)}{\frac{f'(z)}{f(z)} + \frac{\mu}{\pi}B(z, \lambda)}\right| \le |z|\, |\tau(z, \lambda)|, \tag{3.4}$$

where

$$\text{(3.5)} \qquad \begin{cases} A(z,\lambda) = \dfrac{\lambda-1}{(1-\lambda z)(1-z)}, \\ B(z,\lambda) = \dfrac{1-\overline{\lambda}}{(1-z)(z-\overline{\lambda})}, \\ \tau(z,\lambda) = \dfrac{z-\overline{\lambda}}{1-\lambda z}. \end{cases}$$

A simple calculation shows that the inequality (3.4) is equivalent to

$$\text{(3.6)} \qquad \left| \frac{f'(z)}{f(z)} - \frac{\mu}{\pi} \frac{A(z,\lambda) + |z|^2 \, |\tau(z,\lambda)|^2 B(z,\lambda)}{1 - |z|^2 \, |\tau(z,\lambda)|^2} \right| \\ \leq \frac{|\mu|}{\pi} \frac{|z| \, |\tau(z,\lambda)| \, |A(z,\lambda) + B(z,\lambda)|}{1 - |z|^2 \, |\tau(z,\lambda)|^2}.$$

Using (3.5) we can easily see that

$$1 - |z|^2 \, |\tau(z,\lambda)|^2 = \frac{(1-|z|^2)(1+|z|^2 - 2\operatorname{Re}(\lambda z))}{|1-\lambda z|^2},$$

$$A(z,\lambda) + B(z,\lambda) = \frac{1-|\lambda|^2}{(1-\lambda z)(z-\overline{\lambda})}$$

and

$$A(z,\lambda) + |z|^2 |\tau(z,\lambda)|^2 B(z,\lambda) = \frac{(\lambda-1)(1-\overline{\lambda}\overline{z}) + |z|^2(\overline{z}-\lambda)(1-\overline{\lambda})}{(1-z)|1-\lambda z|^2}.$$

Thus, by a simple computation, we see that

$$\frac{A(z,\lambda) + |z|^2 |\tau(z,\lambda)|^2 B(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2} = c(z,\lambda),$$

$$\frac{|z| \, |\tau(z,\lambda)| \, |A(z,\lambda) + B(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2} = r(z,\lambda).$$

Now the inequality (3.2) follows from these equalities and (3.6).

It is easy to see that equality occurs in (3.2) for a $z \in \mathbb{D}$ when $f = H_{e^{i\theta},\lambda}$ for some $\theta \in \mathbb{R}$. Conversely, if equality occurs for some $z \in \mathbb{D} \setminus \{0\}$ in (3.2), then equality must hold in (3.3). Thus from the Schwarz lemma there exists a $\theta \in \mathbb{R}$ such that $\omega_f(z) = z\delta(e^{i\theta}z, \lambda)$ for all $z \in \mathbb{D}$. This implies $f = H_{e^{i\theta},\lambda}$. ■

The choice of $\lambda = 0$ gives the following result which may deserve a special mention.

Corollary 3.7. *For $f \in \mathcal{F}_\mu(0)$ we have*

$$\left| \frac{f'(z)}{f(z)} - \frac{\mu(|z|^2\overline{z} - 1)}{\pi(1-z)(1-|z|^4)} \right| \leq \frac{|\mu| \, |z|}{\pi(1-|z|^4)}, \qquad z \in \mathbb{D}.$$

For each $z \in \mathbb{D} \setminus \{0\}$, equality holds if and only if $f = H_{e^{i\theta},0}$ for some $\theta \in \mathbb{R}$.

Corollary 3.8. *Let $\gamma : z(t)$, $0 \le t \le 1$, be a C^1-curve in $\mathbb{D}$ with $z(0) = 0$ and $z(1) = z_0$. Then*

$$V(z_0, \lambda) \subset \left\{ w \in \mathbb{C} : \left| w - \frac{\mu}{\pi} C(\lambda, \gamma) \right| \le \frac{|\mu|}{\pi} R(\lambda, \gamma) \right\},$$

where

$$C(\lambda, \gamma) = \int_0^1 c(z(t), \lambda) z'(t)\, dt \quad \textit{and} \quad R(\lambda, \gamma) = \int_0^1 r(z(t), \lambda)|z'(t)|\, dt.$$

Proof. For $f \in \mathcal{F}_\mu(\lambda)$, it follows from Proposition 3.1 that

$$\begin{aligned} \left| \log f(z_0) - \frac{\mu}{\pi} C(\lambda, \gamma) \right| &= \left| \int_0^1 \left\{ \frac{f'(z(t))}{f(z(t))} - \frac{\mu}{\pi} c(z(t), \lambda) \right\} z'(t)\, dt \right| \\ &\le \int_0^1 \left| \frac{f'(z(t))}{f(z(t))} - \frac{\mu}{\pi} c(z(t), \lambda) \right| |z'(t)|\, dt \\ &\le \frac{|\mu|}{\pi} \int_0^1 r(z(t), \lambda)|z'(t)|\, dt = \frac{|\mu|}{\pi} R(\lambda, \gamma). \end{aligned}$$

Since $\log f(z_0) \in V(z_0, \lambda)$ was arbitrary, the conclusion follows. ∎

For the proof of our next result, we need the following lemma.

Lemma 3.9. *For $\theta \in \mathbb{R}$ and $\lambda \in \mathbb{D}$, the function*

$$G(z) = \frac{\mu}{\pi} \int_0^z \frac{e^{i\theta}\zeta}{\{1 + (\overline{\lambda} e^{i\theta} - \lambda)\zeta - e^{i\theta}\zeta^2\}^2}\, d\zeta, \qquad z \in \mathbb{D},$$

has a double zero at the origin and no zeros elsewhere in $\mathbb{D}$. Furthermore, there exists a starlike univalent function G_0 in $\mathbb{D}$ with $G = (\mu/(2\pi))e^{i\theta} G_0^2$ and $G_0(0) = G_0'(0) - 1 = 0$.

Proof. Let $b = \mathrm{Im}(\overline{\lambda} e^{i\theta/2}) \in \mathbb{R}$. Then a computation shows that

$$1 + (\overline{\lambda} e^{i\theta} - \lambda) z - e^{i\theta} z^2 = (1 - z/z_1)(1 - z/z_2),$$

where

$$z_1 = e^{-i\theta/2}(ib + \sqrt{1 - b^2}) \quad \text{and} \quad z_2 = e^{-i\theta/2}(ib - \sqrt{1 - b^2}).$$

From this we have

$$\frac{G''(z)}{G'(z)} - \frac{1}{z} = \frac{d}{dz}\left\{ \log \frac{G'(z)}{z} \right\} = \frac{2/z_1}{1 - z/z_1} + \frac{2/z_2}{1 - z/z_2}.$$

Since $|z_1| = |z_2| = 1$, for $z \in \mathbb{D}$ we have

$$\mathrm{Re}\left\{ 1 + \frac{zG''(z)}{G'(z)} \right\} = \mathrm{Re}\left\{ \frac{1 + z/z_1}{1 - z/z_1} \right\} + \mathrm{Re}\left\{ \frac{1 + z/z_2}{1 - z/z_2} \right\} > 0.$$

Lemma 2.1 applied to $(2\pi/\mu)e^{-i\theta}G(z)$ with $p=2$ yields a $G_0 \in S^*$ such that $G = (\mu/(2\pi))e^{i\theta}G_0^2$. ■

PROPOSITION 3.10. *Let $z_0 \in \mathbb{D} \setminus \{0\}$. Then for $\theta \in (-\pi, \pi]$ we have $\log H_{e^{i\theta},\lambda}(z_0) \in \partial V(z_0, \lambda)$. Furthermore, if $\log f(z_0) = \log H_{e^{i\theta},\lambda}(z_0)$ for some $f \in \mathcal{F}_\mu(\lambda)$ and $\theta \in (-\pi, \pi]$, then $f = H_{e^{i\theta},\lambda}$.*

Proof. From (2.4) we have

$$\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} = \frac{\mu}{\pi}\,\frac{\delta(az,\lambda)-1}{(1-\delta(az,\lambda)z)(1-z)} = \frac{\mu}{\pi}\,\frac{(\lambda-1)+(1-\overline{\lambda})az}{(1-z)(1+(\overline{\lambda}a-\lambda)z-az^2)}.$$

Using (3.5) we compute

$$\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} - \frac{\mu}{\pi}\,A(z,\lambda) = \frac{\mu(1-|\lambda|^2)az}{\pi(1-\lambda z)(1+(\overline{\lambda}a-\lambda)z-az^2)},$$
$$\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} + \frac{\mu}{\pi}\,B(z,\lambda) = \frac{\mu(1-\lambda^2)}{\pi(z-\lambda)(1+(\overline{\lambda}a-\lambda)z-az^2)}$$

and hence

$$\begin{aligned}
\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} - \frac{\mu}{\pi}\,c(z,\lambda) &= \frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} - \frac{\mu}{\pi}\,\frac{A(z,\lambda)+|z|^2|\tau(z,\lambda)|^2B(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2} \\
&= \frac{1}{1-|z|^2|\tau(z,\lambda)|^2}\bigg\{\bigg(\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} - \frac{\mu}{\pi}\,A(z,\lambda)\bigg) \\
&\qquad - |z|^2|\tau(z,\lambda)|^2\bigg(\frac{H'_{a,\lambda}(z)}{H_{a,\lambda}(z)} + \frac{\mu}{\pi}\,B(z,\lambda)\bigg)\bigg\} \\
&= \frac{\mu(1-|\lambda|^2)z[a(1-\overline{\lambda}\overline{z})-\overline{z}(\overline{z}-\lambda)]}{\pi(1-|z|^2)(1+|z|^2-2\operatorname{Re}(\lambda z))(1+(\overline{\lambda}a-\lambda)z-az^2)} \\
&= r(z,\lambda)\,\frac{\mu}{\pi}\,\frac{az}{|z|}\,\frac{|1+(\overline{\lambda}a-\lambda)z-az^2|^2}{(1+(\overline{\lambda}a-\lambda)z-az^2)^2}.
\end{aligned}$$

Now by substituting $a = e^{i\theta}$ we easily see that

$$\frac{H'_{e^{i\theta},\lambda}(z)}{H_{e^{i\theta},\lambda}(z)} - \frac{\mu}{\pi}\,c(z,\lambda) = r(z,\lambda)\,\frac{\mu}{\pi}\,\frac{e^{i\theta}z}{|z|}\,\frac{|1+(\overline{\lambda}e^{i\theta}-\lambda)z-e^{i\theta}z^2|^2}{(1+(\overline{\lambda}e^{i\theta}-\lambda)z-e^{i\theta}z^2)^2}.$$

For $G(z)$ as in Lemma 3.9, we get

$$\frac{H'_{e^{i\theta},\lambda}(z)}{H_{e^{i\theta},\lambda}(z)} - \frac{\mu}{\pi}\,c(z,\lambda) = \frac{|\mu|}{\pi}\,r(z,\lambda)\,\frac{G'(z)}{|G'(z)|} \tag{3.11}$$

and there exists a starlike univalent function G_0 in $\mathbb{D}$ such that $G = (\mu/(2\pi))e^{i\theta}G_0^2$ and $G_0(0) = G_0'(0) - 1 = 0$. As G_0 is starlike, for any $z_0 \in \mathbb{D} \setminus \{0\}$ the linear segment joining 0 and $G_0(z_0)$ entirely lies in $G_0(\mathbb{D})$.

Now, we define γ_0 by

$$\gamma_0 : \; z(t) = G_0^{-1}(tG_0(z_0)), \quad 0 \le t \le 1. \tag{3.12}$$

Since $G(z(t)) = (\mu/(2\pi))e^{i\theta}(G_0(z(t)))^2 = (\mu/(2\pi))e^{i\theta}(tG_0(z_0))^2 = t^2 G(z_0)$, we have

$$G'(z(t))z'(t) = 2tG(z_0), \quad t \in [0,1]. \tag{3.13}$$

Using (3.13) and (3.11) we have

$$\begin{aligned}
&\log H_{e^{i\theta},\lambda}(z_0) - \frac{\mu}{\pi} C(\lambda, \gamma_0) \\
&= \int_0^1 \left\{ \frac{H'_{e^{i\theta},\lambda}(z(t))}{H_{e^{i\theta},\lambda}(z(t))} - \frac{\mu}{\pi} c(z(t), \lambda) \right\} z'(t)\, dt \\
&= \frac{|\mu|}{\pi} \int_0^1 r(z(t), \lambda) \frac{G'(z(t))z'(t)}{|G'(z(t))z'(t)|} |z'(t)|\, dt \\
&= \frac{G(z_0)}{|G(z_0)|} \frac{|\mu|}{\pi} \int_0^1 r(z(t), \lambda)|z'(t)|\, dt = \frac{G(z_0)}{|G(z_0)|} \frac{|\mu|}{\pi} R(\lambda, \gamma_0),
\end{aligned} \tag{3.14}$$

where $C(\lambda, \gamma_0)$ and $R(\lambda, \gamma_0)$ are defined as in Corollary 3.8. Thus, we have

$$\log H_{e^{i\theta},\lambda}(z_0) \in \partial \overline{\mathbb{D}}\left(\frac{\mu}{\pi} C(\lambda, \gamma_0), \frac{|\mu|}{\pi} R(\lambda, \gamma_0) \right).$$

Also, from Corollary 3.8, we have

$$\log H_{e^{i\theta},\lambda}(z_0) \in V(z_0, \lambda) \subset \overline{\mathbb{D}}\left(\frac{\mu}{\pi} C(\lambda, \gamma_0), \frac{|\mu|}{\pi} R(\lambda, \gamma_0) \right).$$

Hence, we conclude that $\log H_{e^{i\theta},\lambda}(z_0) \in \partial V(z_0, \lambda)$.

Finally, we prove the uniqueness of the curve. Suppose that

$$\log f(z_0) = \log H_{e^{i\theta},\lambda}(z_0)$$

for some $f \in \mathcal{F}_\mu(\lambda)$ and $\theta \in (-\pi, \pi]$. We introduce

$$h(t) = \frac{\overline{G(z_0)}}{|G(z_0)|} \left\{ \frac{f'(z(t))}{f(z(t))} - \frac{\mu}{\pi} c(z(t), \lambda) \right\} z'(t),$$

where $\gamma_0 : z(t)$, $0 \le t \le 1$, is given by (3.12). Then $h(t)$ is a continuous function in $[0, 1]$ and satisfies

$$|h(t)| \le \frac{|\mu|}{\pi} r(z(t), \lambda)|z'(t)|.$$

Furthermore, from (3.14) we have

$$\begin{aligned}\int_0^1 \operatorname{Re} h(t)\,dt &= \int_0^1 \operatorname{Re}\left\{ \frac{\overline{G(z_0)}}{|G(z_0)|}\left\{ \frac{f'(z(t))}{f(z(t))} - \frac{\mu}{\pi}\, c(z(t),\lambda)\right\} z'(t)\right\} dt \\ &= \operatorname{Re}\left\{ \frac{\overline{G(z_0)}}{|G(z_0)|}\left\{ \log f(z_0) - \frac{\mu}{\pi}\, C(\lambda,\gamma_0)\right\}\right\} \\ &= \operatorname{Re}\left\{ \frac{\overline{G(z_0)}}{|G(z_0)|}\left\{ \log H_{e^{i\theta},\lambda}(z_0) - \frac{\mu}{\pi}\, C(\lambda,\gamma_0)\right\}\right\} \\ &= \frac{|\mu|}{\pi}\int_0^1 r(z(t),\lambda)|z'(t)|\,dt.\end{aligned}$$

Thus, we have

$$h(t) = \frac{|\mu|}{\pi}\, r(z(t),\lambda)|z'(t)| \qquad \text{for all } t \in [0,1].$$

From (3.11) and (3.13), it follows that

$$\frac{f'}{f} = \frac{H'_{e^{i\theta},\lambda}}{H_{e^{i\theta},\lambda}} \qquad \text{on } \gamma_0.$$

By applying the identity theorem for analytic functions, we get

$$\frac{f'}{f} = \frac{H'_{e^{i\theta},\lambda}}{H_{e^{i\theta},\lambda}} \qquad \text{in } \mathbb{D},$$

and hence, by normalization, $f = H_{e^{i\theta},\lambda}$ in $\mathbb{D}$. ■

Proof of Theorem 2.6. We need to prove that the closed curve

$$(-\pi,\pi] \ni \theta \mapsto \log H_{e^{i\theta},\lambda}(z_0)$$

is simple. Suppose that

$$\log H_{e^{i\theta_1},\lambda}(z_0) = \log H_{e^{i\theta_2},\lambda}(z_0)$$

for some $\theta_1, \theta_2 \in (-\pi,\pi]$ with $\theta_1 \neq \theta_2$. Then, from Proposition 3.10, we have

$$H_{e^{i\theta_1},\lambda} = H_{e^{i\theta_2},\lambda}.$$

From (2.5) this gives a contradiction that

$$e^{i\theta_1} z = \tau\left(\frac{\omega_{H_{e^{i\theta_1},\lambda}}}{z}, \lambda\right) = \tau\left(\frac{\omega_{H_{e^{i\theta_2},\lambda}}}{z}, \lambda\right) = e^{i\theta_2} z.$$

Thus, the curve must be simple.

Since $V(z_0,\lambda)$ is a compact convex subset of $\mathbb{C}$ and has non-empty interior, the boundary $\partial V(z_0,\lambda)$ is a simple closed curve. From Proposition 3.1, the curve $\partial V(z_0,\lambda)$ contains the curve $(-\pi,\pi] \ni \theta \mapsto \log H_{e^{i\theta},\lambda}(z_0)$. Recall that a simple closed curve cannot contain any simple closed curve other than itself. Thus, $\partial V(z_0,\lambda)$ is given by $(-\pi,\pi] \ni \theta \mapsto \log H_{e^{i\theta},\lambda}(z_0)$.

4. Geometric view of Theorem 2.6. Using Mathematica 4.1 (see [16]), we describe the boundary of the set $V(z_0, \lambda)$. In the program below, "z0" stands for z_0, "lam" for λ, and "mu" for μ.

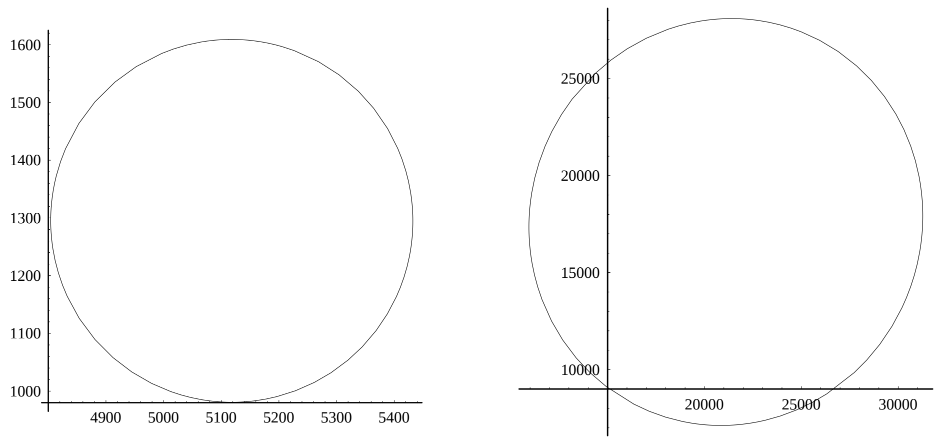
```
Remove["Global‘*"];

z0 = Random[]Exp[I* Random[Real, {-Pi, Pi}]]
lam = Random[]Exp[I *Random[Real, {-Pi, Pi}]]
mu = Random[Real, {0, 10^3}] + I *Random[Real, {-10^3, 10^3}]

Q[lam_, the_] := ((lam - 1) + (1 - Conjugate[lam])Exp[I*the]z)/
                 ((1 - z)((1 + ( Conjugate[lam]*Exp[I*the] - lam)*z )
                 - Exp[I*the]*z*z));

myf2[lam_, the_, z0_] := mu/Pi NIntegrate[Q[lam, the], {z, 0, z0}];
image = ParametricPlot[{Re[myf2[lam, the, z0]],
        Im[myf2[lam, the, z0]]}, {the, -Pi, Pi},
        AspectRatio -> Automatic];
(*Clear[z0, lam, mu];*)
```

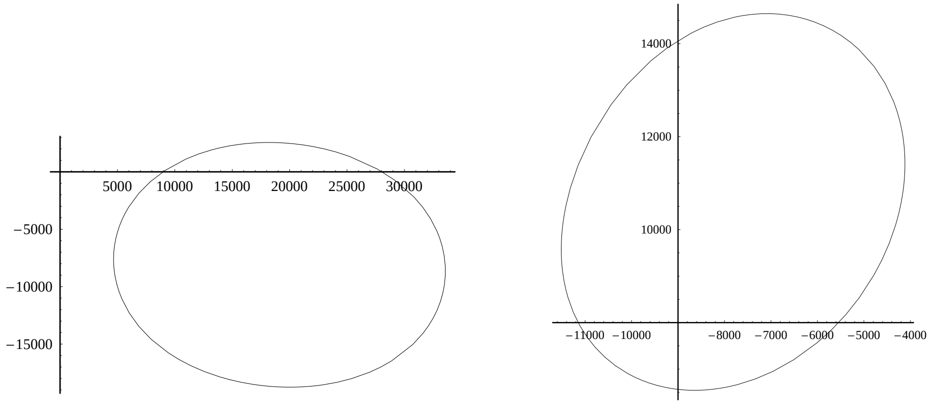
The following figures show the boundary of $V(z_0, \lambda)$ for the values of $z_0 \in \mathbb{D} \setminus \{0\}$, $\lambda \in \mathbb{D}$ and $\mu \in \mathbb{C}$ with $\operatorname{Re} \mu > 0$, given below the pictures. Note that according to Proposition 2.2 the region bounded by the curve $\partial V(z_0, \lambda)$ is compact and convex.



$z_0 = -0.173777 + 0.0869191i$
$\lambda = -0.196029 + 0.480913i$
$\mu = 32796 + 64560.2i$

$z_0 = -0.713811 - 0.0997298i$
$\lambda = -0.225338 + 0.323073i$
$\mu = 69097.4 + 83886.6i$

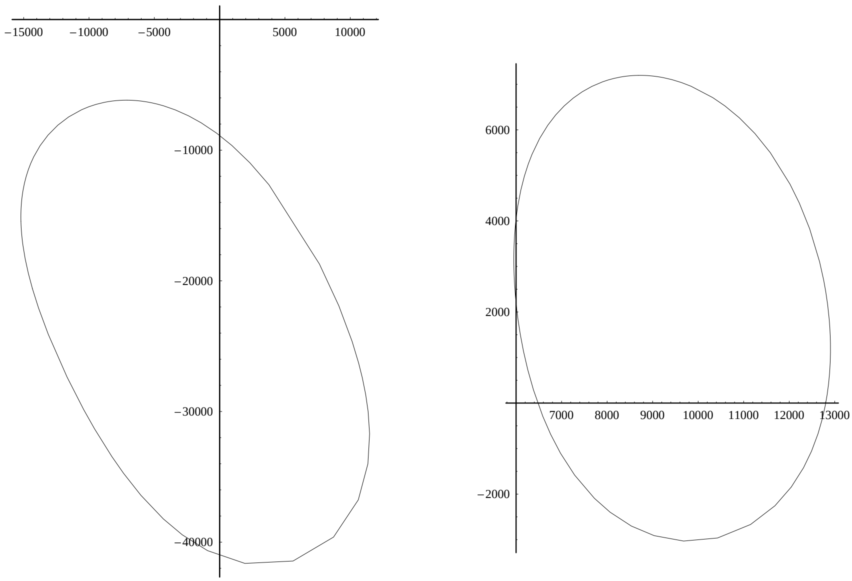
Fig. 1



$z_0 = -0.734426 + 0.61942i$
$\lambda = -0.0564481 - 0.00656122i$
$\mu = 54025 - 5108.28i$

$z_0 = -0.69693 - 0.601351i$
$\lambda = -0.0416728 - 0.683999i$
$\mu = 23944.2 + 50613.5i$

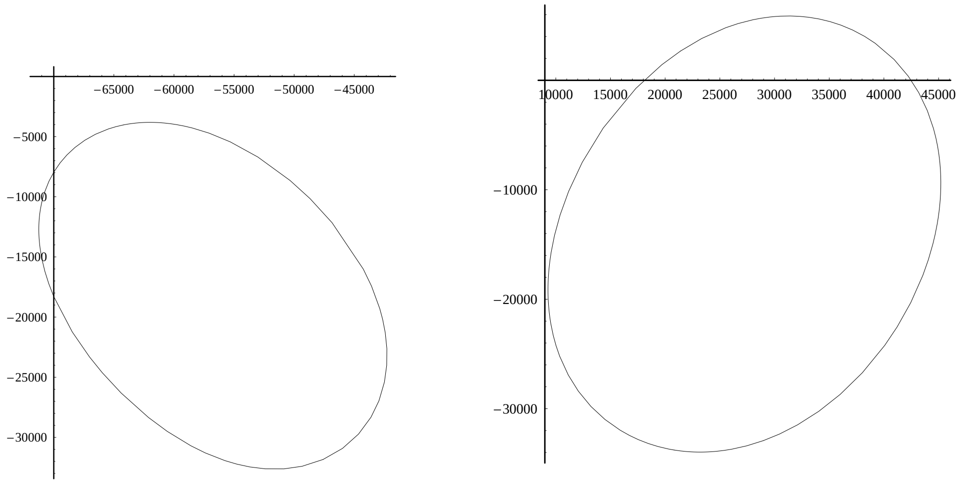
Fig. 2



$z_0 = 0.0150249 + 0.994594i$
$\lambda = -0.219752 - 0.256693i$
$\mu = 16828.1 - 35690.8i$

$z_0 = 0.378332 - 0.90135i$
$\lambda = 0.366791 - 0.600223i$
$\mu = 5006.59 - 46769.8i$

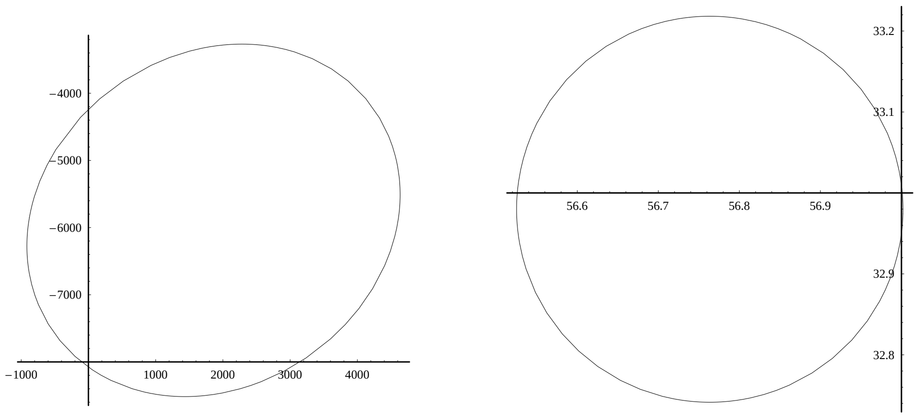
Fig. 3



$z_0 = 0.80351 + 0.549035i$
$\lambda = -0.55886 + 0.0419296i$
$\mu = 83278.8 - 90464.3i$

$z_0 = 0.691568 + 0.644823i$
$\lambda = 0.126172 + 0.137643i$
$\mu = 47178.4 + 83497.8i$

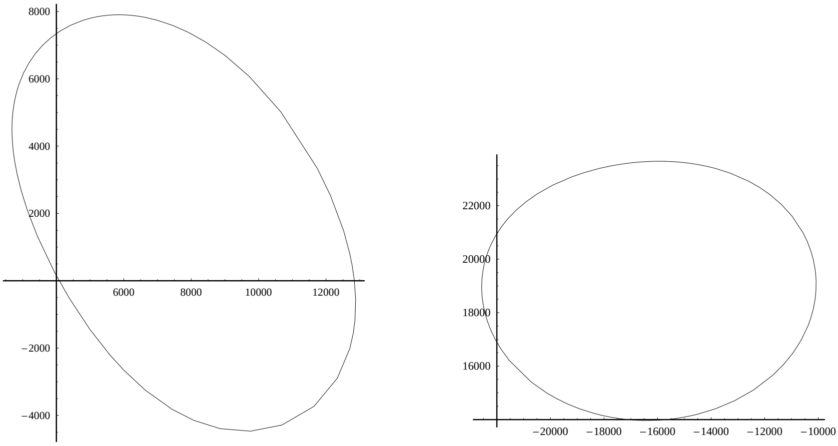
Fig. 4



$z_0 = 0.737135 + 0.496542i$
$\lambda = -0.00646307 - 0.0167039i$
$\mu = 14038.5 + 9544.66i$

$z_0 = -0.00588894 - 0.00496324i$
$\lambda = -0.0472837 + 0.0970889i$
$\mu = 25447.1 - 2011.7i$

Fig. 5



$z_0 = 0.556307 - 0.814404i$
$\lambda = 0.226895 - 0.384635i$
$\mu = 13589.3 - 25797.8i$

$z_0 = 0.880992 - 0.328223i$
$\lambda = -0.0326596 + 0.656304i$
$\mu = 39935.5 + 11412i$

Fig. 6

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