COLLOQUIUM MATHEMATICUM

VOL. 116 2009 NO. 1

ON THE DIOPHANTINE EQUATION $x^2 + 2^\alpha 13^\beta = y^n$

BY

FLORIAN LUCA (Morelia) and ALAIN TOGBÉ (Westville, I)

Abstract. We find all the solutions of the Diophantine equation

$$x^2 + 2^\alpha 13^\beta = y^n$$

in positive integers $x, y, \alpha, \beta, n \geq 3$ with x and y coprime.

1. Introduction. The history of the Diophantine equation

$$x^2 + C = y^n, \quad x \geq 1,\ y \geq 1,\ n \geq 3, \tag{1.1}$$

in integer solutions $x,\ y,\ n$ once C is given is very rich. In 1850, Lebesgue [13] proved that the above equation has no solutions when $C = 1$. In 1965, Chao Ko [10] proved that the only positive integer solution of the above equation with $C = -1$ is $(x, y, n) = (3, 2, 3)$. J. H. E. Cohn [9] solved the above equation for several values of the parameter C in the range $1 \leq C \leq 100$. A couple of the remaining values of C in the above range were covered by Mignotte and de Weger in [17], and the remaining ones in the recent paper [8]. In [19], all solutions of the above equation with $C = B^2$, y^n replaced by $2y^n$ and $B \in \{3, 4, \ldots, 501\}$ were found.

Recently, several authors have become interested in the case when only the prime factors of C are specified. For example, the case when $C = p^k$ with a fixed prime number p was dealt with in [3] and [12] for $p = 2$, in [4], [5] and [14] for $p = 3$, and in [1] for $p = 5$ and k odd. Partial results for a general prime p appear in [6] and [11]. All the positive integer solutions (x, y, n) with x and y coprime were found when $C = 2^a 3^b$, $2^a 5^b$ and $5^a 13^b$ in [15], [16] and [2], respectively. The case when $C = 2^\alpha 3^\beta 5^\gamma 7^\delta$ was dealt with in [18].

In this note, we study the equation

$$x^2 + 2^\alpha 13^\beta = y^n, \quad x, y \geq 1,\ \gcd(x, y) = 1,\ n \geq 3,\ \alpha, \beta \geq 0. \tag{1.2}$$

We prove the following result.

2000 *Mathematics Subject Classification*: 11D61, 11Y50.

Key words and phrases: exponential equations, Diophantine equation.

DOI: 10.4064/cm116-1-7

THEOREM 1.1. *The only solutions of equation* (1.2) *are*:

$$\begin{aligned} n=3, \quad (x,y,\alpha,\beta)\in\{&(5,3,1,0),\ (1,3,1,1),\ (11,5,2,0),\ (25,9,3,1),\\ &(70,17,0,1),\ (47,17,4,2),\ (57,17,7,1),\ (207,35,1,1),\\ &(181,105,9,3),\ (6183,337,8,2),\ (15735,881,25,1),\\ &(18719,705,7,1),\ (27045,901,2,2)\};\end{aligned}$$

$n=4, \quad (x,y,\alpha,\beta)=(7,3,5,0);$

$n=6, \quad (x,y,\alpha,\beta)=(25,3,3,1);$

$n=7, \quad (x,y,\alpha,\beta)=(43,3,1,2).$

For the proof, we apply the method used in [2] to deal with the case when $C=5^a13^b$. Namely, in Sections 2 and 3 we treat the cases $n=3$ and $n=4$, respectively, by reducing the problem of finding all integer solutions of equation (1.2) with those values of n to computing all $\{2,13\}$-integral points on several elliptic curves. Recall that for a finite set $\mathcal{S}$ of primes, an $\mathcal{S}$-*integer* is a rational number a/b, with a and $b>0$ coprime integers, where all the prime factors of b belong to $\mathcal{S}$. In the last section, we may assume that $n\geq 5$ is a prime. Here, we use the theory of primitive divisors for Lucas sequences to deduce that only the case $n=7$ is possible. In this last case, we reduce again the problem to the computation of all $\{2,13\}$-integral points on a few elliptic curves. All the computations have been performed with the software MAGMA.

2. The case $n=3$

LEMMA 2.1. *When* $n=3$, *the only solutions to equation* (1.2) *are*

$$\begin{aligned} (x,y,\alpha,\beta)\in\{&(5,3,1,0),\ (1,3,1,1),\ (11,5,2,0),\ (25,9,3,1),\\ &(70,17,0,1),\ (47,17,4,2),\ (57,17,7,1),\ (207,35,1,1),\\ &(181,105,9,3),\ (6183,337,8,2),\ (15735,881,25,1),\\ &(18719,705,7,1),\ (27045,901,2,2)\}.\end{aligned} \tag{2.1}$$

In particular, *for* $n=6$, *the only solution is* $(x,y,\alpha,\beta)=(25,3,3,1)$.

Proof. We rewrite equation (1.2) as

$$\left(\frac{x}{z^3}\right)^2 + A = \left(\frac{y}{z^2}\right)^3, \tag{2.2}$$

where A is sixth power free and defined implicitly by $2^\alpha 13^\beta = Az^6$ with some integer z. One can see that $A=2^{\alpha_1}13^{\beta_1}$ with $\alpha_1,\beta_1\in\{0,1,2,3,4,5\}$. We thus get the equation

$$V^2 = U^3 - 2^{\alpha_1}13^{\beta_1} \tag{2.3}$$

with $U = y/z^2$, $V = x/z^3$ and $\alpha_1, \beta_1 \in \{0, 1, 2, 3, 4, 5\}$. We need to determine all the $\{2, 13\}$-integral points on the above 36 elliptic curves. To do that, we use MAGMA. Here are a few remarks about the computations:

(1) We discard the solutions with $U \leq 0$ or $V = 0$ because they lead to $x \leq 0$ or $y = 0$, which we do not consider.
(2) We do not consider the solutions having the numerators of U and V not coprime.
(3) If U and V are integers, then $z = 1$, therefore $\alpha_1 = \alpha$ and $\beta_1 = \beta$.
(4) If U and V are rational numbers which are not integers, then z is determined by the denominators of U and V. The numerators of these rational numbers give x and y. Thus, α and β are computed from the formula $2^\alpha 13^\beta = Az^6$.

MAGMA showed that all solutions to equation (2.3) subject to the above restrictions are:

$$\begin{aligned}(U, V, \alpha_1, \beta_1) = {} & (17, 70, 0, 1),\ (3, 5, 1, 0),\ (3, 1, 1, 1),\ (705/4, 18719/8, 1, 1),\\ & (17/4, 57/8, 1, 1),\ (881/256, 15735/4096, 1, 1),\ (5, 11, 2, 0),\\ & (901, 27045, 2, 2),\ (337/4, 6183/8, 2, 2),\ (9, 25, 3, 1),\\ & (105/4, 181/8, 3, 3),\ (17, 47, 4, 2).\end{aligned}$$

In turn, they lead to the solutions (x, y, α, β) listed in (2.1).

For $n = 6$, the equation

$$x^2 + 2^\alpha 13^\beta = y^6 \tag{2.4}$$

can be rewritten as

$$x^2 + 2^\alpha 13^\beta = (y^2)^3. \tag{2.5}$$

We look at the list of solutions of equation (2.1) and observe that the only solution whose second component is a perfect square is $(25, 9, 3, 1)$. Therefore, the only solution (x, y, α, β) to equation (2.4) is $(25, 3, 3, 1)$. This concludes the proof for the case $n = 3$. ∎

3. The case when $n = 4$

LEMMA 3.1. *When $n = 4$, the only solution to equation* (1.2) *is*

$$(x, y, \alpha, \beta) = (7, 3, 5, 0). \tag{3.1}$$

Proof. Here, we rewrite equation (1.2) as

$$\left(\frac{x}{z^2}\right)^2 + A = \left(\frac{y}{z}\right)^4, \tag{3.2}$$

where A is fourth power free and defined implicitly by $2^\alpha 13^\beta = Az^4$ with some integer z. One can see that $A = 2^{\alpha_1} 13^{\beta_1}$ with $\alpha_1, \beta_1 \in \{0, 1, 2, 3\}$.

Hence, we have reduced the problem to determining all the $\{2, 13\}$-integral points (U, V) on the totality of the 16 elliptic curves

$$V^2 = U^4 - 2^{\alpha_1} 13^{\beta_1} \tag{3.3}$$

with $U = y/z$, $V = x/z^2$ and $\alpha_1, \beta_1 \in \{0, 1, 2, 3\}$. Using MAGMA we find that the only convenient solutions are

$$(U, V, \alpha_1, \beta_1) = (1, 0, 0, 0), (3/2, 7/4, 1, 0).$$

With the conditions on x and y and the definition of U and V, one can see that the only acceptable solution is $(x, y, \alpha, \beta) = (7, 3, 5, 0)$. This concludes the proof for the case $n = 4$. ∎

From now on, we may assume that $n \neq 3, 4, 6$. If (x, y, α, β, n) is a solution of the Diophantine equation (1.2) and d is any proper divisor of n, then $(x, y^d, \alpha, \beta, n/d)$ is also a solution of the same equation. Since $n \geq 5$, it follows that it suffices to look at the solutions n for which $p \mid n$ for some odd prime $p \geq 5$. In this case, we may replace n by p, and thus assume for the rest of the paper that $n \geq 5$ is prime.

4. The case $n \geq 5$ prime

LEMMA 4.1. *The Diophantine equation* (1.2) *has no solution with $n \geq 5$ prime except for $n = 7$ when the only solution is* $(x, y, \alpha, \beta) = (43, 3, 1, 2)$.

Proof. We rewrite the Diophantine equation (1.2) as $x^2 + dz^2 = y^n$, where $d = 1, 2, 13, 26$ according to the parities of the exponents α and β. Here, $z = 2^a 13^b$ for some nonnegative integers a and b. Let $\mathbb{K} = \mathbb{Q}[i\sqrt{d}]$. We factor the above equation in $\mathbb{K}$ getting

$$(x + i\sqrt{d}\, z)(x - i\sqrt{d}\, z) = y^n. \tag{4.1}$$

Note that y is odd. Indeed, if y is even, then since x and y are coprime, we see that both x and dz^2 are odd. But in this case, $x^2 \equiv 1 \pmod 4$ and dz^2 is a power of 13, so it is also congruent to 1 modulo 4. Thus, $x^2 + dz^2 \equiv 2 \pmod 4$, which is impossible. Hence, y is odd. A standard argument applied to the factorization (4.1) shows that the ideals generated by $x + i\sqrt{d}\, z$ and $x - i\sqrt{d}\, z$ in the ring $\mathcal{O}_{\mathbb{K}}$ of algebraic integers of $\mathbb{K}$ are coprime. By unique factorization for ideals, the ideal $(x + i\sqrt{d}\, z)\mathcal{O}_{\mathbb{K}}$ is an nth power of some ideal in $\mathcal{O}_{\mathbb{K}}$. A short calculation shows that the class number of $\mathbb{K}$ belongs to $\{1, 2, 6\}$. In particular, it is coprime to n. Thus, again by a standard argument, it follows that $x + i\sqrt{d}\, z$ is associated to an nth power in $\mathcal{O}_{\mathbb{K}}$. Since the group of units of $\mathbb{K}$ is of order 2 or 4 (hence, coprime to n), it follows that we may assume that the equation

$$x + i\sqrt{d}\, z = \gamma^n \tag{4.2}$$

holds with some algebraic integer $\gamma \in \mathcal{O}_{\mathbb{K}}$. Finally, since the discriminant of $\mathbb{K}$ is $-4d$, it follows that $\{1, i\sqrt{d}\}$ is a base for $\mathcal{O}_{\mathbb{K}}$. In conclusion, we can write $\gamma = u + i\sqrt{d}\,v$. Taking complex conjugates in (4.2) and subtracting the two relations, we get

$$2i\sqrt{d}\,2^a 13^b = \gamma^n - \overline{\gamma}^n. \tag{4.3}$$

The right hand side of the above equation is a multiple of $2i\sqrt{d}\,v = \gamma - \overline{\gamma}$. We deduce that $v \mid 2^a 13^b$, and that

$$\frac{2^a 13^b}{v} = \frac{\gamma^n - \overline{\gamma}^n}{\gamma - \overline{\gamma}} \in \mathbb{Z}. \tag{4.4}$$

Let $\{L_m\}_{m \geq 0}$ be the sequence given by

$$L_m = \frac{\gamma^m - \overline{\gamma}^m}{\gamma - \overline{\gamma}} \quad \text{for all } m \geq 0.$$

This is a *Lucas sequence* and it consists of integers. For a nonzero integer k, we write $P(k)$ for the largest prime factor of k. Equation (4.4) leads to the conclusion that

$$P(L_n) = P\left(\frac{2^a 13^b}{v}\right). \tag{4.5}$$

At this step, we recall that the Primitive Divisor Theorem for Lucas sequences ensures that if $n \geq 5$ is prime, then L_n has a *primitive* prime factor except for finitely many pairs $(\gamma, \overline{\gamma})$, all of which appear in Table 1 of [7]. These exceptional Lucas numbers are called *defective*. A primitive prime factor q of L_n has (among others) the properties that $q \nmid -4dv^2 = (\gamma - \overline{\gamma})^2$ and $q \equiv \pm 1 \pmod{n}$. More precisely, $q \equiv e \pmod{n}$, where $e = \left(\frac{-4d}{q}\right)$. Here and in what follows, $\left(\frac{a}{q}\right)$ stands for the Legendre symbol of a with respect to the odd prime q.

Since $\mathbb{K} = \mathbb{Q}[i\sqrt{d}]$ with $d \in \{1, 2, 13, 26\}$, a quick inspection of Table 1 in [7] reveals that our number L_n cannot be defective. Thus, L_n must have a primitive divisor q. Clearly, $q \in \{2, 13\}$ and $q \equiv \pm 1 \pmod{n}$. Hence, the only possibility is $q = 13$, and we conclude that $n \mid 12$ or $n \mid 14$. Since $n \geq 5$ is prime, the only possibility is $n = 7$, and since $13 \equiv -1 \pmod{7}$, we must have $\left(\frac{-4d}{13}\right) = -1$. Since $d \in \{1, 2, 13, 26\}$, we conclude that $d = 2$. Looking now again at equation (4.3) with $n = 7$, we obtain the equation

$$v(7u^6 - 70u^4v^2 + 84u^2v^2 - 8v^6) = 2^a 13^b. \tag{4.6}$$

Since u and v are coprime, we have the possibilities

$$v = \pm 2^a 13^b, \quad v = \pm 13^b, \quad v = \pm 2^a, \quad v = \pm 1. \tag{4.7}$$

The first two cases lead to the conclusion that $P(L_n) = P(2^a 13^b / v) \leq 2$, which is impossible because it leads again to the conclusion that L_n has no primitive divisors, so we look at the last two possibilities.

CASE 1: $v = \pm 2^a$. In this case, the Diophantine equation (4.6) is

$$7u^6 - 70u^4v^2 + 84u^2v^2 - 8v^6 = \pm 13^b. \tag{4.8}$$

Dividing by v^6, we obtain the elliptic equations

$$7X^3 - 70X^2 + 84X - 8 = D_1Y^2, \tag{4.9}$$

where

$$X = \frac{u^2}{v^2}, \quad Y = \frac{13^{b_1}}{v^3}, \quad b_1 = \left\lfloor \frac{b}{2} \right\rfloor, \quad D_1 = \pm 1, \pm 13.$$

• In the case $D_1 = \pm 1$ (changing X to $-X$ when $D_1 = -1$), we need to find the $\{2\}$-integral points on the elliptic curve

$$7X^3 + \eta 70X^2 + 84X + \eta 8 = Y^2, \qquad \eta \in \{-1, 1\}. \tag{4.10}$$

We multiply both sides of (4.10) by 7^2 to obtain

$$U^3 + \eta 70U^2 + 588U + \eta 392 = V^2, \tag{4.11}$$

where $(U, V) = (\eta 7X, 7Y)$ are $\{2\}$-integral points on the above elliptic curve. Using MAGMA we found only $(U, V) = (7, 91)$, for $\eta = 1$. This gives $(X, Y) = (1, 13)$; then $a = 0$, $b = 2$, $u = v = 1$, leading to the solution $(x, y, \alpha, \beta) = (43, 3, 1, 2)$ of the original equation (1.2).

• When $D = \pm 13$, we multiply both sides of (4.9) by $7^2 13^3$ and obtain the elliptic curves

$$U^3 + \eta 910U^2 + 99372U + \eta 861224 = V^2, \qquad \eta \in \{-1, 1\}, \tag{4.12}$$

with

$$U = \eta 91X, \qquad V = 1183Y,$$

for which we again need to determine the $\{2\}$-integral points. In the same way, using MAGMA, we find nine solutions, but only the solution (U, V) $= (91, 1183)$ leads to $(X, Y) = (1, 1)$, leading once more to the solution $(x, y, \alpha, \beta) = (43, 3, 1, 2)$.

CASE 2: $v = \pm 1$. Here, we obtain the equation

$$7u^6 - 70u^4 + 84u^2 - 8 = 2^a 13^b. \tag{4.13}$$

By the same method, we can rewrite the above equation as

$$7X^3 - 70X^2 + 84X - 8 = D_1Y^2, \tag{4.14}$$

where

$$X = u^2, \;\; Y = 2^{a_1}13^{b_1}, \;\; a_1 = \lfloor a/2 \rfloor, \;\; b_1 = \lfloor b/2 \rfloor, \;\; D_1 = \pm 1, \pm 2, \pm 13, \pm 26.$$

When $D_1 = \pm 1$, ± 13, we again get the curves (4.10) and (4.12), except that now we need only their integral points, which have already been computed by MAGMA.

• When $D_1 = \pm 2$, we multiply both sides of (4.14) by $7^2 13^3$ to get the two elliptic curves

$$U^3 + \eta 910 U^2 + 99372U + \eta 861224 = V^2, \qquad \eta \in \{-1, 1\}, \tag{4.15}$$

where $U = \eta 91X$, $V = 1183Y$, and we need again their integral points. We used MAGMA to find seven integral points but only the integral point $(U, V) = (91, 1183)$ gives the solution $(x, y, \alpha, \beta) = (43, 3, 1, 2)$.

• Finally, when $D_1 = \pm 26$, we multiply both sides of (4.14) by $7^2 2^3 13^3$ to obtain

$$U^3 + \eta 1820 U^2 + 397488U + \eta 6889792 = V^2, \qquad \eta \in \{-1, 1\}, \tag{4.16}$$

with $U = 182X$, $V = 4732Y$, whose integral solutions (U, V) we need to compute. We used MAGMA to find two integral solutions when $\eta = -1$ and eight when $\eta = 1$. None of them leads to a solution of (1.2). This completes the proof of the lemma and of the theorem. ■

Acknowledgments. The authors express their gratitude to the anonymous referee for constructive suggestions to improve the quality of the paper. Research of the first author was supported in part by grants SEP-CONACyT 79685 and PAPIIT 100508. The second author was partially supported by Purdue University North Central.

REFERENCES

[1] F. S. Abu Muriefah and S. A. Arif, *The Diophantine equation* $x^2 + 5^{2k+1} = y^n$, Indian J. Pure Appl. Math. 30 (1999), 229–231.

[2] F. S. Abu Muriefah, F. Luca and A. Togbé, *On the equation* $x^2 + 5^a \cdot 13^b = y^n$, Glasgow Math. J. 50 (2008), 175–181.

[3] S. A. Arif and F. S. Abu Muriefah, *On the Diophantine equation* $x^2 + 2^k = y^n$, Int. J. Math. Sci. 20 (1997), 299–304.

[4] —, —, *The Diophantine equation* $x^2 + 3^m = y^n$, ibid. 21 (1998), 619–620.

[5] —, —, *On a Diophantine equation*, Bull. Austral. Math. Soc. 57 (1998), 189–198.

[6] —, —, *On the Diophantine equation* $x^2 + q^{2k+1} = y^n$, J. Number Theory 95 (2002), 95–100.

[7] Yu. Bilu, G. Hanrot and P. M. Voutier, *Existence of primitive divisors of Lucas and Lehmer numbers* (with an appendix by M. Mignotte), J. Reine Angew. Math. 539 (2001), 75–122.

[8] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and modular approaches to exponential Diophantine equations. II. The Lebesgue–Nagell equation*, Compos. Math. 142 (2006), 31–62.

[9] J. H. E. Cohn, *The Diophantine equation* $x^2+c = y^n$ Acta Arith. 65 (1993), 367–381.

[10] C. Ko, *On the Diophantine equation* $x^2 = y^n + 1$, $xy \neq 0$, Sci. Sinica 14 (1965), 457–460.

[11] M. Le, *An exponential Diophantine equation*, Bull. Austral. Math. Soc. 64 (2001), 99–105.

[12] M. Le, *On Cohn's conjecture concerning the Diophantine equation* $x^2 + 2^m = y^n$, Arch. Math. (Basel) 78 (2002), 26–35.

[13] V. A. Lebesgue, *Sur l'impossibilité en nombres entiers de l'équation* $x^m = y^2 + 1$ Nouv. Ann. Math. 9 (1850), 178–181.

[14] F. Luca, *On a Diophantine equation*, Bull. Austral. Math. Soc. 61 (2000), 241–246.

[15] —, *On the equation* $x^2 + 2^a \cdot 3^b = y^n$, Int. J. Math. Math. Sci. 29 (2002), 239–244.

[16] F. Luca and A. Togbé, *On the equation* $x^2 + 2^a \cdot 5^b = y^n$, Int. J. Number Theory 4 (2008), 973–979.

[17] M. Mignotte and B. M. M. de Weger, *On the Diophantine equations* $x^2 + 74 = y^5$ *and* $x^2 + 86 = y^5$, Glasgow Math. J. 38 (1996), 77–85.

[18] I. Pink, *On the diophantine equation* $x^2 + 2^\alpha \cdot 3^\beta \cdot 5^\gamma \cdot 7^\delta = y^n$, Publ. Math. Debrecen 70 (2006), 149–166.

[19] Sz. Tengely, *On the Diophantine equation* $x^2 + a^2 = 2y^p$, Indag. Math. (N.S.) 15 (2004), 291–304.

Instituto de Matemáticas UNAM
Campus Morelia
Apartado Postal 27-3 (Xangari)
C.P. 58089
Morelia, Michoacán, Mexico
E-mail: fluca@matmor.unam.mx

Mathematics Department
Purdue University North Central
1401 S, U.S. 421
Westville, IN 46391 U.S.A.
E-mail: atogbe@pnc.edu

Received 30 July 2007;
revised 5 January 2009 (4952)