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CHARACTERIZING SIDON SETS BY INTERPOLATION PROPERTIES OF SUBSETS

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Abstract. Pisier's characterization of Sidon sets as containing proportional-sized quasi-independent subsets is given a sharper form for groups with only a finite number of elements having orders a power of 2. No such improvement is possible for a general Sidon subset of a group having an infinite number of elements of order 2. The method used also gives several sharper forms of Ramsey's characterization of Sidon sets as containing proportional-sized I_0-subsets in a uniform way, again in groups containing but a finite number of elements of order 2.

1. Introduction. A subset E of a discrete abelian group Γ is called *Sidon* (respectively, I_0) if every bounded E-function is the restriction of the Fourier–Stieltjes transform of a finite (resp., discrete) measure on the dual compact group G. Obviously, I_0 sets are Sidon, but the converse is not true [13].

Sidon and I_0 sets have been extensively studied and examples can be found in every infinite subset of Γ (cf. [3], [7], [9], [10], [12]). Significant efforts have been made to characterize Sidon and I_0 sets in terms of more restricted classes of sets. Pisier [16] obtained an important arithmetic characterization of Sidon sets in terms of quasi-independent sets (1), a notion more general than independence as it includes Hadamard sets (2) with Hadamard ratio greater than 3.

DEFINITION 1. Given two classes of sets $\mathcal{A},\mathcal{B}$ that each contain all finite sets, we say that $E \in \mathcal{A}$ *contains* $\mathcal{B}$ *proportionally* (or is *proportional* $\mathcal{B}$) if there is a constant $C > 0$ such that for every finite $F \subset E$ there exists $H \subset F$ such that $\operatorname{Card} H \geq C \operatorname{Card} F$ and $H \in \mathcal{B}$.

Using probabilistic arguments, Pisier [16] showed a set E not containing the identity character 1 is Sidon if and only if it is proportionally quasi-

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(1) $\{\gamma_i\}$ is *quasi-independent* if whenever $\sum \varepsilon_i\gamma_i = 0$ for $\varepsilon_i = 0, \pm 1$, then $\varepsilon_i = 0$ for all i.
(2) $\{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ is *Hadamard* if there is $q > 1$ (the ratio) such that $qn_j < n_{j+1}$ for all j.

independent. Subsequently, Ramsey [17] showed Sidon sets contain proportional subsets that are I_0 in a uniform way.

Here we characterize Sidon sets in several ways as proportional $\mathcal{B}$ where each of the collections $\mathcal{B}$ considered will have properties stronger than quasi-independence or simple I_0.

We assume throughout the paper that the Sidon set E does not contain the identity character 1. We write $\mathbb{T}$ for the boundary of the unit disk in the complex plane.

We now introduce our collections $\mathcal{B}$ which have properties stronger than than quasi-independence or simple I_0.

DEFINITION 2.

(a) $E \subset \Gamma$ is an *ε-Kronecker set* if for every $\{r_\gamma\}_{\gamma\in E} \subseteq \mathbb{T}^E$ there exists $g \in G$ such that
$$|\gamma(g) - r_\gamma| < \varepsilon \quad \text{for all } \gamma \in E.$$

(b) $E \subset \Gamma$ is an *$RI_0(U)$ set* (respectively, $FZI_0(U)$) if for every Hermitian (defined below) $\phi \in B(\ell^\infty(E))$ there exists a discrete real (resp., non-negative) measure μ supported on U satisfying $\widehat{\mu} = \phi|_H$ on E. When $U = G$ we suppress the U.

Combining Definitions 1 and 2, we have proportional ε-Kronecker, proportional RI_0, etc. sets. In the case of $RI_0(U)$ and $FZI_0(U)$ sets, we actually work with (exhaustive) collections of subclasses of those sets; we give those definitions later, to avoid cluttering the introduction with technicalities.

Sets that are ε-Kronecker for all $\varepsilon > 0$ are independent. Hadamard sets with ratio greater than 3 are ε-Kronecker for some $\varepsilon < \sqrt{2}$, but not all ε-Kronecker sets in $\mathbb{Z}$ are finite unions of Hadamard sets. Moreover, ε-Kronecker sets with $\varepsilon < \sqrt{2}$ are examples of I_0 sets, but $\sqrt{2}$-Kronecker sets need not be. For proofs of these facts and of other properties of ε-Kronecker sets, see, for example, [4], [5], [7], [8], [11], and [19].

In this paper we show that if the discrete group Γ has no elements of order 2 (this includes all duals of compact, connected groups), then $E \subset \Gamma$ is Sidon if and only if it is proportional ε-Kronecker for some $\varepsilon < \sqrt{2}$. This improves Pisier's quasi-independent result for such groups. An example illustrates that if Γ contains infinitely many elements of order 2 then this characterization need not hold. One step in proving our improvement is to first establish that if arbitrary choices of ± 1's can be interpolated on $E \subseteq \Gamma$ to within ε, then E is ε'-Kronecker for any $\varepsilon' > 2\varepsilon$. In particular, if arbitrary choices of ± 1's can be interpolated on E to within any $\varepsilon > 0$, then E is Kronecker. These results are in Sections 3 and 4.

The subclasses where the discrete interpolating measure can be taken to be real (the RI_0 sets), positive (FZI_0) and/or supported on an open set U

($I_0(U)$, $RI_0(U)$ or $FZI_0(U)$ respectively) were investigated in [6] and [7]. There are I_0 sets that are not RI_0 and RI_0 sets that are not FZI_0. One can similarly speak of Sidon(U) sets; it is known [1] that all Sidon sets in duals of connected groups are Sidon (U) for all open U. For the continuous measure analogue of $FZI_0(U)$ sets, see [2]. Whether all I_0 sets are $I_0(U)$ for all open U is unknown (when G is connected), as are the corresponding questions for RI_0 and FZI_0 sets, even on $\mathbb{Z}$.

Here we show that Sidon sets can be characterized as containing proportional FZI_0 sets if Γ has only finitely many elements of order 2, and as proportional $RI_0(U)$ for all open sets U if G is connected. As singletons need not be $FZI_0(U)$, a necessarily slightly weaker result holds for the $FZI_0(U)$ property. These results are in Sections 5 and 6. We remark that there are interpolation constants that appear in these last mentioned proportionality results, so the preceding description is slightly incomplete.

It remains open if all Sidon sets are finite unions of I_0 sets.

2. Definitions and notation. We write $M_{\mathrm{d}}(U)$ for the discrete measures on $U \subseteq G$. A superscript of r or + indicates the real or positive discrete measures on U. We write the set of functions $\phi \in \ell^\infty(E)$ of sup norm at most one as $B(\ell^\infty(E))$, and ϕ is said to be *Hermitian* if $\phi(\gamma) = \overline{\phi(\gamma^{-1})}$ for $\gamma, \gamma^{-1} \in E$.

Every Sidon set has an interpolation constant (the *Sidon constant*) associated with it. Pisier [16] proved that a set E is Sidon if and only if there is a constant C such that E is proportionally Sidon, where proportions have Sidon constant at most C.

Kalton (see [17]) proved that a set E is I_0 if and only if there exists some $0 < \varepsilon < 1$ (equivalently, for every $0 < \varepsilon < 1$) and integer N such that for every $\phi \in B(\ell^\infty(E))$, there exists $\mu = \sum_{j=1}^N a_j \delta_{x_j}$ with $|a_j| \le 1$ and

$$\|\widehat{\mu} - \phi|_E\|_\infty = \sup_{\gamma \in E} |\mu(\gamma) - \phi(\gamma)| \le \varepsilon.$$

We refer to N as the *length* of μ and say that E is $I_0(N, \varepsilon)$. If the x_j may always be chosen in a fixed open set U, we say E is $I_0(U, N, \varepsilon)$. Similar definitions can be made when the interpolating measures are not just discrete, but also real or positive, when we use the notations $RI_0(U, N, \varepsilon)$ and $FZI_0(U, N, \varepsilon)$. Then the functions to be interpolated must be Hermitian, as the Fourier transform of a real measure is Hermitian. Even so, RI_0 sets are I_0. In fact, E is $RI_0(U)$ if and only if $E \cup E^{-1}$ is $I_0(U)$ ([7, Theorem 2.5]).

A subset E of Γ is an $RI_0(U)$ (respectively, $FZI_0(U)$) set if and only if it is $RI_0(U, N, \varepsilon)$ (resp. $FZI_0(U, N, \varepsilon)$) for some $N \ge 1$ and $0 < \varepsilon < 1$ [7, Proposition 2.1], provided U is compact.

Finite sets are always I_0 and FZI_0 ([7, Proposition 2.9]) and are $RI_0(U)$ sets for all open U if the group G is connected ([7, Corollary 2.6]), with constants depending only on the cardinality of the finite set and on U in the latter case (see Section 7).

Because of that finite set property, we slightly abuse language with the following definition.

DEFINITION 3. E is *proportional* $I_0(U)$, $RI_0(U)$, or $FZI_0(U)$ if E is proportional $I_0(U,N,\varepsilon)$, $RI_0(U,N,\varepsilon)$ or $FZI_0(U,N,\varepsilon)$ for some $N \geq 1$ and $0 < \varepsilon < 1$.

The motivation for considering proportional $RI_0(U)$ and $FZI_0(U)$ sets is a result of Ramsey [17]: Sidon sets are proportional I_0 sets.

Sometimes it is convenient to identify $\mathbb{T}$ with $[0, 2\pi]$:

DEFINITION 4. We say E is *angular ε-Kronecker* if for every $\{r_\gamma\}_{\gamma\in E} \subseteq [0, 2\pi]^E$ there exists $g \in G$ such that

$$d(\arg\gamma(g), r_\gamma) < \varepsilon \quad \text{for all } \gamma \in E.$$

An absence of elements of order 2 in Γ is significant because it allows us to take square roots in G. We remind the reader of some easy facts:

(i) Γ has no elements of order 2 if and only if every element of G is a square;
(ii) Γ has only finitely many elements of order 2 if and only if the quotient of G by the subgroup of squares in G is finite;
(iii) a compact group is connected if and only if it is divisible if and only if the dual has no elements of finite order.

Throughout the paper, we let G_0 be the annihilator of Γ_0, the 2-subgroup of Γ, i.e. $\Gamma_0 = \{\text{characters of order } 2^k \text{ for some } k\}$. Since Γ/Γ_0 has no elements of order 2, every element of G_0 has a square root.

Summary of proportional equivalences for Sidon sets. Here is a summary of what is known to us. E is a subset of Γ, $1 \leq N < \infty$, and $0 < \varepsilon < \sqrt{2}$; the constants may be different in different assertions. The following are equivalent to E being Sidon.

(1) No conditions on E, Γ:
- E is proportional quasi-independent ([16]),
- E is proportional N-Sidon ([16]),
- E is proportional I_0 ([17]).

(2) The 2-subgroup of Γ is finite and E has no elements of order 2:
- E is proportional ε-Kronecker (Theorem 4.4).

(3) Γ has only finitely many elements of order 2:

- E is proportional RI_0 (Theorem 5.1),
- E is cofinitely proportional $RI_0(U)$ for an (all) open $U \subset G$ (Theorem 5.1),
- E is proportional FZI_0 (Theorem 6.2),
- E is cofinitely proportional $FZI_0(U)$ for an (all) open $U \subset G$ (Theorem 6.2).

3. Interpolating arbitrary signs

THEOREM 3.1. *Assume $E \subseteq \Gamma$. Fix an angle $\theta \in [0, \pi]$ and assume that for any $\{r_\gamma\}_{\gamma \in E} \in \{\theta, \theta + \pi\}^E$ there exists a point $x \in G_0$ such that $d(\arg \gamma(x), r_\gamma) \le \varepsilon$ for all $\gamma \in E$. Then E is angular ε'-Kronecker for any $\varepsilon' > 2\varepsilon$.*

Proof. We proceed by induction and show that for each positive integer k and any choice of angles $\{s_\gamma\}_{\gamma \in E}$ that are arguments of 2^kth roots of unity, there exists $x = x(k)$ in G_0 such that for all $\gamma \in E$,

$$d(\arg \gamma(x), (2 - 2^{-(k-1)})\theta + s_\gamma) < (2 - 2^{-(k-1)})\varepsilon.$$

Once this is established we simply choose k such that

$$\pi 2^{-k} - \varepsilon 2^{-(k-1)} + 2\varepsilon < \varepsilon'.$$

Since the angular distance between two adjacent 2^kth roots of unity is $2\pi/2^k$, given any $\{t_\gamma\}_{\gamma \in E} \in [0, 2\pi]^E$, we can choose $\{s_\gamma\}$, arguments of 2^kth roots of unity, such that

$$d(t_\gamma, (2 - 2^{-(k-1)})\theta + s_\gamma) \le \pi 2^{-k} \quad \text{for all } \gamma.$$

Choose $x = x(k)$ as above. Then for all $\gamma \in E$,

$$\begin{aligned} d(\arg \gamma(x), t_\gamma) &\le d(\arg \gamma(x), (2 - 2^{-(k-1)})\theta + s_\gamma) + d(t_\gamma, (2 - 2^{-(k-1)})\theta + s_\gamma) \\ &\le (2 - 2^{-(k-1)})\varepsilon + \pi 2^{-k} < \varepsilon' \end{aligned}$$

as desired.

The result is certainly true for $k = 1$, so assume it is true for k. Let $\{s_\gamma\}$ be the arguments of 2^{k+1}th roots of unity and consider $\{2s_\gamma\}$. These are arguments of 2^kth roots of unity, so by induction we can find $x \in G_0$ such that for all $\gamma \in E$,

$$d(\arg \gamma(x), (2 - 2^{-(k-1)})\theta + 2s_\gamma) < (2 - 2^{-(k-1)})\varepsilon.$$

Since every element of G_0 is a square, we can choose $y \in G_0$ such that $y^2 = x$. Then $\gamma(y)^2 = \gamma(x)$, so the argument of $\gamma(y)$ is either $\arg \gamma(x)/2$ or $\pi + \arg \gamma(x)/2$. Hence either

$$d(\arg \gamma(y), (1 - 2^{-k)})\theta + s_\gamma) < (1 - 2^{-k})\varepsilon$$

or

$$d(\arg \gamma(y), (1 - 2^{-k)})\theta + s_\gamma + \pi) < (1 - 2^{-k})\varepsilon$$

(respectively) for all $\gamma \in E$. In the first case, put $r_\gamma = \theta$; in the second case, put $r_\gamma = \theta + \pi$. Obtain $z \in G_0$ such that $d(\arg\gamma(z), r_\gamma) < \varepsilon$ for all $\gamma \in E$. Let $g = zy \in G_0$. Then we have either

$$\begin{aligned} d(\arg\gamma(g), (2-2^{-k})\theta + s_\gamma) &\le d(\arg\gamma(z), \theta) + d(\arg\gamma(y), (1-2^{-k})\theta + s_\gamma) \\ &< \varepsilon + (1-2^{-k})\varepsilon = (2-2^{-k})\varepsilon, \end{aligned}$$

or

$$\begin{aligned} &d(\arg\gamma(g), (2-2^{-k})\theta + s_\gamma) \\ &\qquad = d(\arg\gamma(g), (2-2^{-k})\theta + s_\gamma + 2\pi) \\ &\qquad \le d(\arg\gamma(z), \theta+\pi) + d(\arg\gamma(y), (1-2^{-k})\theta + s_\gamma + \pi) < (2-2^{-k})\varepsilon. \end{aligned}$$

This completes the induction step. ■

COROLLARY 3.2. *Assume Γ has no elements of order 2 and $E \subseteq \Gamma$. Suppose that given any choice of signs $\{r_\gamma\}_{\gamma\in E} \in \{-1,+1\}^E$ there exists $x \in G$ such that $d(\arg\gamma(x), \arg r_\gamma) < \pi/4$ for all $\gamma \in E$. Then E is ε-Kronecker for some $\varepsilon < \sqrt{2}$ and is FZI_0.*

Proof. As Γ has no elements of order 2, we see that $G_0 = G$, hence we may apply the theorem. Angular ε'-Kronecker for some $\varepsilon' < \pi/2$ is equivalent to ε-Kronecker for some $\varepsilon < \sqrt{2}$. The set E is FZI_0, being ε-Kronecker for some $\varepsilon < \sqrt{2}$. ■

COROLLARY 3.3. *Assume G is connected and $E \subseteq \Gamma$. Suppose that given any choice of signs $\{r_\gamma\}_{\gamma\in E} \in \{-1,+1\}^E$ there exists $x \in G$ such that*

$$d(\arg\gamma(x), \arg r_\gamma) < \pi/4 \quad \text{for all } \gamma \in E.$$

Then E is ε-Kronecker for some $\varepsilon < \sqrt{2}$.

Proof. G is connected if and only if Γ has no elements of finite order. ■

COROLLARY 3.4. *Assume Γ has no elements of order 2 and $E \subseteq \Gamma$. Suppose that for any $\varepsilon > 0$ and any choice of signs $\{r_\gamma\}_{\gamma\in E}$, there exists $x \in G$ such that $d(\gamma(x), r_\gamma) < \varepsilon$ for all $\gamma \in E$. Then E is Kronecker.*

Proof. The assumption ensures that E is ε-Kronecker for all $\varepsilon > 0$. ■

When we speak of the "gap" between two intervals or arcs in $\mathbb{T}$, we mean the smaller of the two gaps; when we speak of the "length" of an interval in $\mathbb{T}$, we will mean arc length.

COROLLARY 3.5. *Let $E \subset \Gamma$. Suppose there are two intervals $I_a, I_b \subseteq \mathbb{T}$, each of length $l < \pi$, with gap between them of length $g > 0$, and with the property that for any $A \subseteq E$ there exists $x \in G_0$ such that $\gamma(x) \in I_a$ for all $\gamma \in A$ and $\gamma(x) \in I_b$ for all $\gamma \in E \setminus A$. Then E is angular ε-Kronecker for any $\varepsilon > \pi - g$.*

Proof. Let θ and $\theta+\pi$ be the two points of distance $\pi/2$ from the centre of the gap. Since $g \le \pi$, by symmetry (and without loss of generality) the distance from any point in the interval I_a (or I_b) to θ (or $\theta+\pi$) is at most $(\pi-g)/2$. By Theorem 3.1, E is angular ε-Kronecker for any $\varepsilon > \pi - g$. ∎

REMARK 3.6. Suppose $G = \mathbb{T}$. If the point x in Theorem 3.1 can be chosen from the interval $U = (-a, a)$, then E is ε'-Kronecker$(2U)$ (meaning that the interpolating points can be found in $2U$). To see why this is so, we proceed as in the theorem, but in addition assume inductively that the point $x(k)$, constructed at step k, belongs to $(2-2^{-(k-1)})U$. This is true by assumption for $k = 1$. For the induction step, observe that we can take $y = x/2$, and hence choose $y \in (1-2^{-k})U$. Then $x(k+1) \equiv x(k)y$ belongs to $(1-2^{-k})U + U \subseteq 2U$ for all k.

If Γ has elements of order 2, Corollaries 3.3–3.4 need not be true.

EXAMPLE 3.7. Consider $E = \{\gamma_j\}$, an independent set in $\mathbb{D}_2$. Then we can interpolate ± 1 exactly on E (so E is Sidon), but the set is clearly ε-Kronecker if and only if $\varepsilon > \sqrt{2}$.

EXAMPLE 3.8. Consider $E = \{(j, \gamma_j) : j \in \mathbb{N}\} \subseteq \mathbb{Z} \times \mathbb{D}_2$ where $\{\gamma_j\}$ is again an independent set in $\mathbb{D}_2$. Then E itself has no elements of order 2, but the subgroup it generates does. Of course, we can interpolate ± 1 exactly on E, but as $E \cup E^{-1}$ is not I_0 (though it is Sidon by the union theorem), the set is not RI_0 and hence it is not ε-Kronecker for any $\varepsilon < \sqrt{2}$. In fact, E is not even a finite union of ε-Kronecker sets for any $\varepsilon < \sqrt{2}$. To see this, suppose that it were a finite union of such sets. One of the sets would contain a net $\{(j_\alpha, \gamma_{j\alpha})\}$ with $j_\alpha \to 0$ in the Bohr topology on $\mathbb{Z}$. Suppose

$$|(j_\alpha, \gamma_{j_\alpha})(x,y) - i| = |e^{i2\pi j_\alpha x}\gamma_{j_\alpha}(y) - i| < \varepsilon < \sqrt{2} \quad \text{for all } \alpha.$$

For some α we will have $|e^{i2\pi j_\alpha x} - 1| < \delta$ (for whatever $\delta > 0$ we specify). Since $\gamma_{j_\alpha}(y) = \pm 1$, these inequalities cannot simultaneously hold for suitably small δ.

But E is $\sqrt{2}$-Kronecker. To see this, given $\{t_j\} \subseteq \mathbb{T}$, let $A_j = \{x \in \mathbb{T} : |e^{ijx} - t_j| = \sqrt{2}\}$. Each A_j is a closed set with empty interior, so by the Baire category theorem, $\bigcup_j A_j \neq \mathbb{T}$. Consequently, there exists x such that $|e^{ijx} - t_j| \neq \sqrt{2}$ for any j. Choosing y such that $\gamma_j(y) = -1$ if $|e^{ijx} - t_j| > \sqrt{2}$ and 1 otherwise, we have $|e^{ijx}\gamma_j(y) - t_j| < \sqrt{2}$ for all j.

In the next section we will prove that every Sidon set is a proportional ε-Kronecker set for an $\varepsilon < \sqrt{2}$ depending on the set. That will use the following interpolation result which improves Corollary 3.5 and may be of independent interest.

PROPOSITION 3.9. *Suppose that there are two intervals $I_a, I_b \subseteq \mathbb{T}$, each of length l and having a gap between them of length $g > 0$, with the property that given any $A \subseteq E$ there exists $x \in G_0$ such that $\gamma(x) \in I_a$ for all $\gamma \in A$ and $\gamma(x) \in I_b$ for all $\gamma \in E \setminus A$. Suppose further that $l \leq \pi/32$ and $2l \leq g \leq \pi/2 + \pi/8$. Then E is angular ε-Kronecker for $\varepsilon > \pi/2 - l/2$.*

Proof.

CASE I: Assume that

$$\frac{1}{2}\left(\frac{\pi}{2(k+1)} + \frac{\pi}{2k}\right) \leq g < \frac{1}{2}\left(\frac{\pi}{2k} + \frac{\pi}{2(k-1)}\right) \tag{3.1}$$

for some $k \geq 2$. Obtain x as in the statement of the proposition and suppose $I_a = [a_1, a_2]$ and $I_b = [b_1, b_2]$.

Let θ and $\theta + \pi$ be the points of distance $\pi/2$ from $(k+1)(a_2 + b_1)/2$, the midpoint between the nearest points in $(k+1)I_a$ and $(k+1)I_b$. (The assumptions ensure that these intervals do not overlap.) Then the distance from θ to any point of $(k+1)I_a$ is

$$\begin{aligned}\max\left\{\left|(k+1)a_2 - (k+1)\,\frac{a_2+b_1}{2} + \frac{\pi}{2}\right|, \left|(k+1)\,\frac{a_2+b_1}{2} - \frac{\pi}{2} - (k+1)a_1\right|\right\}\\ = \max\left\{\frac{\pi}{2} - (k+1)\,\frac{g}{2},\ (k+1)l + (k+1)\,\frac{g}{2} - \frac{\pi}{2}\right\}.\end{aligned}$$

The lower bound on g of (3.1) gives the estimate

$$\frac{\pi}{2} - (k+1)\,\frac{g}{2} < \frac{\pi}{4} - \frac{\pi}{8k},$$

and as $2l \leq g \leq \pi/(2(k-1))$, it follows that

$$\frac{\pi}{2} - (k+1)\,\frac{g}{2} \leq \frac{\pi}{4} - \frac{l}{4}. \tag{3.2}$$

Using these estimates and the upper bound in (3.1), we also know

$$\begin{aligned}(k+1)l + \frac{(k+1)g}{2} - \frac{\pi}{2} &\leq (k+1)g - \frac{\pi}{2}\\ &\leq \frac{\pi}{2}\left(1 + \frac{1}{2k} + \frac{1}{k-1}\right) - \frac{\pi}{2}\\ &\leq \frac{11\pi}{48} \qquad \text{if } k \geq 4.\end{aligned} \tag{3.3}$$

But for $2 \leq k < 4$,

$$\begin{aligned}(k+1)l + \frac{(k+1)g}{2} - \frac{\pi}{2} &\leq (k+1)l + \frac{\pi}{4}\left(1 + \frac{1}{2k} + \frac{1}{k-1}\right) - \frac{\pi}{2}\\ &\leq 5l + \frac{\pi}{16} \leq \frac{7\pi}{32}.\end{aligned} \tag{3.4}$$

As $\pi/4 - l/4$ is the greatest of (3.2), (3.3) and (3.4), it dominates the distance from any point of $(k+1)I_a$ to θ. Thus

$$\operatorname{dist}(\arg\gamma(x^{k+1}),\theta) \leq \frac{\pi}{4} - \frac{l}{4} \quad \text{whenever } \gamma \in A.$$

By symmetry, the same statement can be made about $(k+1)I_b$, $\theta + \pi$ and

$$\operatorname{dist}(\arg\gamma(x^{k+1}),\theta+\pi) \quad \text{for } \gamma \in E \setminus A.$$

As $x^{k+1} \in G_0$, we may appeal to the previous theorem to conclude that E is angular ε-Kronecker for $\varepsilon > \pi/2 - l/2$.

Case II: Assume that

$$g \in \left[\frac{1}{2}\left(\frac{\pi}{4} + \frac{\pi}{2}\right), \frac{\pi}{2} + \frac{\pi}{8}\right].$$

We take θ and $\theta + \pi$ to be the points of distance $\pi/2$ from the midpoint between $2I_a$ and $2I_b$. The distance from θ to any point of $2I_a$ is

$$\max\left\{\frac{\pi}{2} - g, 2l + g - \frac{\pi}{2}\right\} \leq 2l + \frac{\pi}{8} \leq \frac{3\pi}{16},$$

and we conclude the argument in the same manner as in Case I. ∎

Remark 3.10. A similar argument can be applied if there exists $\delta > 0$ such that $l(1+\delta) \leq g$ provided $l \leq l_0(\delta)$, with the conclusion that E is $\pi/2 - \varepsilon$-Kronecker where the choice of $\varepsilon > 0$ depends on δ and l.

4. Proportional ε-Kronecker sets. Our proportional results all rely upon a variation of a technical construction due to Ramsey, which uses a combinatorial result of Pajor [14]. This construction can be found in the proof of Theorem 15 of [17]. We outline Ramsey's construction in the second subsection and then give the adaptation which we will need in the following subsection. We begin this section with an extension of a result of Pisier, needed for the construction to follow.

4.1. *An extension of a result of Pisier.* Here is a result of Pisier [16].

Theorem 4.1. *If E is a Sidon set (not containing 1) then there exists $\tau > 0$, depending only on the Sidon constant of E, with the property that for any finite subset $F \subseteq E$ there are $2^{\tau|F|}$ points g_j satisfying*

$$\sup_{\gamma \in F} |\gamma(g_j) - \gamma(g_i)| \geq \tau \quad \text{if } i \neq j.$$

We will need the following improvement of Theorem 4.1.

Corollary 4.2. *Suppose E is a Sidon set (not containing 1) and that finitely many translates of a subset $V \subseteq G$ cover G. There are constants $\delta = \delta(E) > 0$ and $\alpha = \alpha(E, V)$ with the property that for any finite subset*

$F \subseteq E$ having cardinality at least α, there is a set $S_0 = \{g_j\} \subset V$ having at least $2^{\delta|F|}$ points satisfying

$$\sup_{\gamma\in F} |\gamma(g_j) - \gamma(g_i)| \geq \delta \quad \textit{for } i \neq j. \tag{4.1}$$

Proof. Assume $G = \bigcup_{k=1}^n a_k V$. By Pisier's theorem there are $2^{\tau|F|}$ points $g_j \in G$ such that for $i \neq j$ we have $\sup_{\gamma\in F} |\gamma(g_i) - \gamma(g_j)| \geq \tau$ where τ depends only on E. At least $2^{\tau|F|}/n$ of these points belong to one set $a_k V$. For such points we have $g_j = a_k v_j$ for some $v_j \in V$ and, of course, $|\gamma(g_i) - \gamma(g_j)| = |\gamma(v_i) - \gamma(v_j)|$.

Thus given any $F \subseteq E$ there are at least $2^{\tau|F|}/n$ points g_j of V such that $\sup_{\gamma\in F} |\gamma(g_j) - \gamma(g_i)| \geq \tau$ for $i \neq j$.

Pick α such that $n \leq 2^{\alpha\tau/2}$ and put $\delta = \tau/2$. If $\operatorname{Card} F \geq \alpha$, then there will be at least $2^{\delta|F|}$ points $g_j \in V$ satisfying (4.1). ■

Notice that when $V = G$ we may take $\alpha = 0$.

4.2. *Ramsey's technical construction, I.* We start with the situation of Corollary 4.2 in mind, and in particular (4.1). Suppose F is a finite set with the property that there is a set $S_0 = \{g_j\}$ of $2^{\delta|F|}$ elements satisfying (4.1). Choose any $p \in 4\mathbb{Z}$ such that $\lambda \equiv 2\pi/p < \delta/2$ and put $\lambda' = \lambda/Q$ where $Q = \lceil (1 - 2^{-\delta/2})^{-1} \rceil$. Enumerate F as $\{\gamma_i\}_{i=1}^{|F|}$.

As in Ramsey's argument, partition $\mathbb{T}$ into disjoint arcs

$$T_k = \{e^{i\theta} : k\lambda \leq \theta < (k+1)\lambda\}$$

for $0 \leq k < p$ and partition each T_k into disjoint arcs $U_{k,m}$ of the form

$$U_{k,m} = \{e^{i\theta} : k\lambda + m\lambda' \leq \theta < k\lambda + (m+1)\lambda'\}$$

for $0 \leq m < Q$.

Define S_i inductively, as follows: Let

$$S_k^i = \{g \in S_{i-1} : \gamma_i(g) \in T_k\} \quad \text{and} \quad S_{k,m}^i = \{g \in S_{i-1} : \gamma_i(g) \in U_{k,m}\}$$

for $0 \leq k < p$, $0 \leq m < Q$. For each pair i, k pick the index $m = m(i,k)$ for which $\operatorname{Card} S_{k,m}^i$ is minimal. Clearly, $\operatorname{Card} S_{k,m}^i \leq Q^{-1} \operatorname{Card} S_k^i$. Put

$$S_i = S_{i-1} \setminus \bigcup_{k=0}^{p-1} S_{k,m(i,k)}^i.$$

One can easily check that $\operatorname{Card} S_n \geq 2^{n\delta/2}$.

Let $I_{i,k}$ be the arc between $U_{k-1,m(i,k-1)}$ and $U_{k,m(i,k)}$. These arcs have lengths at most $2\lambda - 2\lambda'$ and are separated by a gap of length at least λ'. Moreover, the choice of $p \in 4\mathbb{Z}$ ensures that the points $0, \pm\pi/2, \pi$ are either boundary points of $I_{i,k}$, or at distance at least λ' from the boundary.

For $g \in S_n$, define $h_g \in \ell^\infty(F)$ by $h_g(\gamma_i) = k_i$ where $\gamma_i(g) \in I_{i,k_i}$. The construction of S_n ensures that these functions are all distinct, hence there are at least $2^{n\delta/2}$ such functions, the cardinality of S_n.

By [14, Corollary 2] or [15, Corollary 1.6, p. 11] there is a constant $c_1 > 0$ depending only on δ, a subset H of F with cardinality at least $c_1 \operatorname{Card} F$ and natural numbers $a < b$ satisfying

$$\{a,b\}^H \subseteq \{h_g|_H : g \in S_n\}.$$

Thus this construction identifies a fixed subset $H \subset F$, and, for each index i, two arcs, $I_{i,a}$ and $I_{i,b}$ such that

$$A \subseteq H \;\Rightarrow\; \exists g \in S_n \text{ with } \gamma_i(g) \in \begin{cases} I_{i,a} & \text{if } \gamma_i \in A, \\ I_{i,b} & \text{if } \gamma_i \in H \setminus A. \end{cases}$$

If $b - a \geq 2$, then

$$I \equiv \{e^{i\theta} : (a-1)\lambda + \lambda' \leq \theta < (a+1)\lambda - \lambda'\} \supset I_{i,a}, \tag{4.2}$$

$$J \equiv \{e^{i\theta} : (b-1)\lambda + \lambda' \leq \theta < (b+1)\lambda - \lambda'\} \supset I_{i,b}. \tag{4.3}$$

The two arcs on the circle, I, J, are separated by a gap of size at least $(b-a-2)\lambda + 2\lambda' \geq 2\lambda'$ and have length $2\lambda - 2\lambda'$ each. Moreover, for some $g \in S_0$, $\gamma(g)$ belongs to I if $\gamma \in A$ and belongs to J if $\gamma \in H \setminus A$.

If, instead, $b - a = 1$, then for suitable y_i,

$$I_{i,a} \subseteq [(a-1)\lambda + \lambda', y_i], \qquad I_{i,b} \subseteq [y_i + \lambda', (b+1)\lambda - \lambda'].$$

Hence

$$I \equiv [-3\lambda + 2\lambda', -\lambda'] \supset I_{i,a} I_{i,b}^{-1}, \tag{4.4}$$

$$J \equiv [\lambda', 3\lambda - 2\lambda'] \supset I_{i,a}^{-1} I_{i,b}. \tag{4.5}$$

The arcs I and J are of length $3\lambda - 3\lambda'$ each and are separated by at least $2\lambda'$. By choosing appropriate $g_1, g_2 \in S_0$ and putting $g = g_1 g_2^{-1} \in S_0 S_0^{-1}$, we have $\gamma(g) \in I$ if $\gamma \in A$ and $\gamma(g) \in J$ if $\gamma \in H \setminus A$.

Remarks 4.3. (i) We emphasize that the proportionality constant c_1, and the interval and gap lengths, λ and λ', all depend (effectively) only on δ, which depends in turn on the Sidon constant of E.

(ii) Any Sidon set in a discrete group with a finite 2-subgroup is proportional ε-Kronecker for some $\varepsilon < 2$. That is a straightforward use of §4.2, Corollary 3.5, and Corollary 4.2. However, we do not know that if that proportional property for $\varepsilon < 2$ is equivalent to Sidonicity, although [5, Theorem 4.1] shows that if E is ε-Kronecker for some $\varepsilon < 2$, then E does not contain arbitrarily large squares, for example. Using the extended technical construction to follow, we will obtain proportional ε-Kronecker with an $\varepsilon < \sqrt{2}$ (depending on E), which is equivalent to Sidonicity.

4.3. *Ramsey's technical construction, II.* We continue with the notation of §4.2. Put $l = \lambda'/4$. Then l effectively depends only on δ (and the Sidon constant). A review of how l is found shows that

$$\frac{\delta}{2} \le \frac{1}{2}, \quad Q \ge 4, \quad p \ge 8, \quad \lambda < 2\pi/8 < 0.8, \quad \lambda' < \frac{1}{5}, \quad l < \frac{1}{20}. \tag{4.6}$$

Displays (4.2)–(4.5) give us two intervals I, J, of lengths at most 3λ and gap at least λ', and a subset $H \subseteq F$, with $\operatorname{Card} H \ge c_1 \operatorname{Card} F$, having the property that for all $A \subseteq H$ there is some $g \in S_0 \cup S_0 S_0^{-1}$ such that $\gamma(g) \in I$ for $\gamma \in A$ and $\gamma(g) \in J$ for $\gamma \in H \setminus A$.

Partition each of I and J into $s = 12Q$ equal sized, disjoint subintervals, $I_{a_1}, \ldots, I_{a_s}$ and $J_{b_1}, \ldots, J_{b_s}$, with lengths at most $l = \lambda'/4$. We reduce the set of g slightly and let

$$S = \{g \in S_0 \cup S_0 S_0^{-1} : \gamma(g) \in I \cup J \text{ for all } \gamma \in H\}.$$

Let $X^+ = \{a_1, \ldots, a_s\}$, $X^- = \{b_1, \ldots, b_s\}$ and $X = X^+ \cup X^-$.

View S as a subset of X^H by identifying g with $(z_\gamma^{(g)})_{\gamma \in H}$ according to the rule $\gamma(g) \in I_{z_\gamma^{(g)}}$ for $\gamma \in H$. Define $\Pi : X^H \to \{-1, 1\}^H$ by

$$\Pi(z_\gamma^{(g)}) = (r_\gamma)_{\gamma \in H} \quad \text{where} \quad r_\gamma = \begin{cases} 1 & \text{if } z_\gamma \in X^+, \\ -1 & \text{if } z_\gamma \in X^-, \end{cases}$$

i.e. $r_\gamma = 1$ if $\gamma(g) \in I_a$ and $r_\gamma = -1$ if $\gamma(g) \in I_b$. By taking suitable choices of g we can obtain all elements of $\{\pm 1\}^H$. Hence $\Pi(S) = \{\pm 1\}^H$.

By [14, Theorem 2] there exist $a_j \in X^+$, $b_k \in X^-$, $c_2 = c_2(\delta)$ and

$$H_1 \subseteq H \quad \text{with } \operatorname{Card} H_1 \ge c_2 \operatorname{Card} H \tag{4.7}$$

such that $\{a_j, b_k\}^{H_1} \subseteq P^{H_1}(S)$ (where $P^{H_1}(f) = f|_{H_1}$). In other words, for every $A \subseteq H_1$ there exists $g \in S_0 \cup S_0 S_0^{-1}$ with $\gamma(g) \in I_{a_j}$ if $\gamma \in A$ and $\gamma(g) \in J_{b_k}$ for $\gamma \in H_1 \setminus A$.

By construction, the gap between the intervals I_{a_j} and J_{b_k} is at least four times their lengths, and the interval lengths and gap size depend only on E. Moreover, $\operatorname{Card} H_1 \ge c_1 c_2 \operatorname{Card} F$. The (new) proportionality constant $C = c_1 c_2$ depends only on E.

THEOREM 4.4. *Suppose the* 2*-subgroup of* Γ *is finite and that* E *has no elements of order* 2*. Then* E *is proportional* ε*-Kronecker for some* $\varepsilon < \sqrt{2}$ *if and only if* E *is Sidon.*

Proof. Suppose E is Sidon. Obtain $\delta(E)$ and $\alpha(E, G_0)$ of Corollary 4.2, where G_0 is, as usual, the annihilator of the 2-subgroup of Γ.

As E has no elements of order 2, a singleton subset of E is ε-Kronecker for each $\varepsilon > 1$. Hence, given $F \subseteq E$ of cardinality less than α, take H to be any singleton in F to get a subset of size $\ge \operatorname{Card} F/\alpha$ that is ε-Kronecker for any $\varepsilon > 1$.

If $\operatorname{Card} F \geq \alpha$, apply the refinement of the technical construction in the beginning of this subsection taking as S_0 the $2^{\delta|F|}$ points in G_0 identified in §4.2. The proportionality constant C and interval length l depend only on δ and therefore only on E.

Let H_1 be the proportional-sized subset of F that arises from the construction, i.e. from (4.7). If the gap, g, between the two identified intervals is at least $\pi/2+\pi/8$, by Corollary 3.5 we see that H is angular $3\pi/8$-Kronecker. Otherwise, since $g \geq 4l$, we can appeal to Proposition 3.9 to conclude that H is ε-Kronecker for some $\varepsilon < \sqrt{2}$ depending only on l, and therefore only on the Sidon set E. Replacing C if necessary by the minimum of C and $1/\alpha$, we deduce that E is proportional ε-Kronecker for some $\varepsilon < \sqrt{2}$.

For the converse, note that ε-Kronecker sets with $\varepsilon < \sqrt{2}$ are $I_0(N, \delta)$ for some N and δ depending only on ε ([5]). In particular, proportional ε-Kronecker sets are proportional Sidon sets and so Sidon by [16, Corollary 2.3]. ∎

Corollary 4.5. *Suppose that E has no elements of order 2 and that the subgroup it generates has a finite 2-subgroup. Then E is proportional ε-Kronecker for some $\varepsilon < \sqrt{2}$ if and only if E is Sidon.*

Proof. Apply Theorem 4.4 with Γ the subgroup generated by E. ∎

Remark 4.6. In contrast, Sidon sets and even ε-Kronecker sets need not be "proportional Hadamard" sets, meaning finite subsets contain proportional-sized subsets that are Hadamard with ratios bounded away from 1. Example 5.2 of [5] provides such a counterexample.

We now show that the restriction on elements of order 2 is necessary in the statement of Theorem 4.4.

Example 4.7. Let $E = \{(j, \gamma_j) : j \in \mathbb{N}\} \subseteq \mathbb{Z} \times \mathbb{D}_2$ be the set of Example 3.8. Then E is Sidon but not proportional ε-Kronecker for any $\varepsilon < \sqrt{2}$ (equivalently, not proportional angular $\pi/2 - \varepsilon$-Kronecker for any $\varepsilon > 0$). However, E is $\sqrt{2}$-Kronecker.

Proof. Suppose E were proportional angular $\pi/2-\varepsilon$-Kronecker for some $\varepsilon > 0$ with proportionality constant C.

For any fixed N (to be specified later) Szemerédi's theorem [18] says there exists $M = M(C, N)$ such that any subset of $[1, \ldots, M]$, with density at least C, contains an arithmetic progression of length N. By assumption, the set $\{(j, \gamma_j) : j = 1, \ldots, M\}$ contains a subset $X = \{(j, \gamma_j) : j \in F_M\}$ that is $\pi/2 - \varepsilon$-Kronecker, where F_M is a subset of $[1, \ldots, M]$ of density C. Hence F_M contains an arithmetic progression of length N.

Pick mth roots of unity (for m even) that are $\varepsilon/2$-dense in $\mathbb{T}$, say $\{w_1, \ldots, w_m\}$. Pick another $\varepsilon/2$-dense set $\{s_j\}$ and let p_j be any mth root of s_j.

Define numbers $t_i^{(k)}$ of modulus 1 recursively as follows:

(1) Let $t_1^{(1)} = 1$.
(2) Define $t_{i+1}^{(k)} = t_i^{(k)}/(p_k w_j)$ for $i = (j-1)m+1, \dots, jm$ and $j,k = 1, \dots, m$.
(3) Define $t_1^{(k+1)} = t_{m^2+1}^{(k)}$.

We think of these as ordered by fixing k and ordering in index i, and then letting k vary.

Put $N = m^3$. Assume $\{(jk_0 + d_0, \gamma_{jk_0+d_0}) : j = 1, \dots, N\} \subset X$ and that at the character $(jk_0 + d_0, \gamma_{jk_0+d_0})$ we are to interpolate t_j. Then there exists a point (x,y) and error term ε_j such that $|\varepsilon_j| \le \pi/2 - \varepsilon$ so that $t_j = e^{i(jk_0+d_0)x}\gamma_j(y)e^{i\varepsilon_j}$. Thus, for suitable $r_j = \pm 1$,

$$\frac{t_j}{t_{j+1}} = r_j e^{-ik_0 x} e^{i(\varepsilon_j - \varepsilon_{j+1})}.$$

Let I be the union of the two subintervals of $\mathbb{T}$ of length 2ε which are at angular distance at least $\pi/2 - \varepsilon$ away from both $\pm e^{-ik_0 x}$. For each k, the numbers $\{p_k w_j : j = 1, \dots, m\}$ are $\varepsilon/2$-dense, hence there is some $j = j(k)$ such that $p_k w_j \in I$. This compels all the pairs $(t_l^{(k)}, t_{l+1}^{(k)})$ with ratio $p_k w_j$ (there are m such pairs, those with $l = (j-1)m+1, \dots, jm$) to have error terms $(\varepsilon_l^{(k)}, \varepsilon_{l+1}^{(k)})$ opposite in sign.

Now

$$\frac{t_{(j-1)m+1}^{(k)}}{t_{jm+1}^{(k)}} = (p_k w_j)^m = s_k = r'_{k,j} e^{-imk_0 x} e^{i(\varepsilon'_{k,j})}$$

where $r'_{k,j} = \pm 1$ and $\varepsilon'_{k,j} = \varepsilon_{(j-1)m+1}^{(k)} - \varepsilon_{jm+1}^{(k)}$. Since m is even, $\varepsilon_{(j-1)m+1}^{(k)}$ and $\varepsilon_{jm+1}^{(k)}$ have the same sign, so $|\varepsilon'_k| \le \pi/2 - \varepsilon$.

As $\{s_k\}$ are $\varepsilon/2$-dense, at least one s_k belongs to one of the intervals of width 2ε that are at angular distance at least $\pi/2 - \varepsilon$ from both $\pm e^{-imk_0 x}$, and this is a contradiction. ∎

5. Proportional RI_0 subsets. Since ε-Kronecker sets with $\varepsilon < \sqrt{2}$ are $FZI_0(N(\varepsilon), \delta(\varepsilon))$ sets [7, Theorem 3.1], Theorem 4.4 implies that Sidon sets in duals of groups with a finite 2-subgroup are proportional FZI_0. In the remainder of the paper we improve upon this fact.

First, we consider the problem of interpolating with real measures. We begin with a further definition.

DEFINITION 5. We call $E \in \mathcal{A}$ *cofinitely proportional* $\mathcal{B}$ if there exist constants $C, \alpha > 0$ such that for every finite $F \subset E$ with $\operatorname{Card} F > \alpha$ there exists $H \subset F$ such that $\operatorname{Card} H \ge C \operatorname{Card} F$ and $H \in \mathcal{B}$.

The "cofinite" restriction is irrelevant when $U = G$ but crucial when U is small.

THEOREM 5.1. *Suppose Γ has only finitely many elements of order* 2. *The following are equivalent*:

(1) *E is a Sidon set*;
(2) *E is a proportional RI_0 set*;
(3) *for each open set $U \subseteq G$, E is cofinitely proportional $RI_0(U)$.*

Proof. (1)⇒(3). As a set is $RI_0(U)$ if and only if it is $RI_0(Ug)$, there is no loss of generality in assuming U is a neighbourhood of e. Obtain a symmetric e-neighbourhood V such that $V^{10} \subseteq U$. Let $V_1 = \{g^2 : g \in V\}$. By compactness, G is the union of finitely many translates of V. Since Γ has only finitely many elements of order 2, G is the union of finitely many translates of G_2, the set of squares of G. Clearly, G_2 is the union of the same number of translates of V_1, and consequently $G = \bigcup_{j=1}^{n} a_j V_1$ for suitable $a_j \in G$ and n.

By Corollary 4.2 there exist $\delta = \delta(E) > 0$ and $\alpha = \alpha(E, V_1)$ such that given a finite subset F of E of cardinality at least α, there are at least $2^{\delta|F|}$ points $g_j \in V_1$ satisfying (4.1).

Given such a subset F, we apply §4.3, taking the set S_0 to be these $2^{\delta|F|}$ points in V_1. This construction produces two arcs, I, J, with lengths at most $l \leq \pi/16$ and separated by a gap of at least $4l$, and a subset $H \subseteq F$ with $\operatorname{Card} H \geq C \operatorname{Card} F$ having the property that given any $A \subseteq H$ there is a point $g \in V_1^2$ with $\gamma(g) \in I$ if $\gamma \in A$ and $\gamma(g) \in J$ if $\gamma \in H \setminus A$. The numbers l and C depend only on δ and hence on E. It is convenient to replace I, J by their closures. This will, of course, not change their lengths or the size of the gap between them.

Indeed, because the gap between the two arcs is double the lengths of the arcs, $d > 0$ can be chosen so that either the real parts (of the elements of the arcs) differ by at least d (Case I), or their imaginary parts differ by at least d and lie on opposite sides of 0 (Case II).

CASE I: The real parts differ by at least d. In this case we can assume there are constants $\varepsilon_1, \varepsilon_2$ such that (without loss of generality) $\Re I \geq \varepsilon_1$, $\Re J \leq \varepsilon_2$ and $\varepsilon_1 - \varepsilon_2 \geq d$.

Given $\phi \in B(\ell^\infty(H))$, ϕ real-valued, put $A = \{\gamma \in H : \phi(\gamma) \geq 0\}$. Obtain the corresponding $g \in V_1^2 \subseteq V^4$ such that

$$\Re\gamma(g) \begin{cases} \geq \varepsilon_1 & \text{if } \gamma \in A, \\ \leq \varepsilon_2 & \text{if } \gamma \in H \setminus A, \end{cases}$$

and set

$$\mu = \frac{1}{4}\,(\delta_g + \delta_{g^{-1}} - (\varepsilon_1 + \varepsilon_2)\delta_e) \in M_{\mathrm{d}}^{\mathrm{r}}(V_1^2) \subseteq M_{\mathrm{d}}^{\mathrm{r}}(V^4).$$

It is a routine calculation to check that for all $\gamma \in H$,

$$|\widehat{\mu}(\gamma) - \phi(\gamma)| \leq 1 - d/4 < 1.$$

Furthermore, length $\mu \leq 3$ and μ has real-valued transform. With m iterations we can obtain a measure $\mu \in M_{\mathrm{d}}^{\mathrm{r}}(V^4)$ of length at most $3m$ and satisfying $|\widehat{\mu}(\gamma) - \phi(\gamma)| \leq (1 - d/4)^m$ for all $\gamma \in H$. (Of course, in the limit we can interpolate real ϕ by real μ with real-valued transform and supported on the closure of the set V^4.)

Next, we see how to interpolate i.

First, suppose one of the two arcs, I, J, does not intersect the x axis. By construction, the boundaries of the arcs are at integer multiples of l. Thus one of the arcs must be at distance at least l from either 0 or π. Choose $g \in V_1^2$ such that $\gamma(g)$ belongs to this arc for all $\gamma \in H$. Then, for the appropriate choice of sign,

$$\left| \pm \frac{1}{2} (\delta_g - \delta_{g^{-1}})^\wedge(\gamma) - i \right| \leq 1 - l/2 \quad \text{ on } H.$$

Either way, this produces a measure in $M_{\mathrm{d}}^{\mathrm{r}}(V^4)$, of length 2, that approximates i to within $1 - l/2$ on H.

Otherwise, both arcs intersect the x axis, and hence one must contain π. Choose $g \in V_1^2$ such that $\gamma(g)$ belongs to the arc containing π for all $\gamma \in H$. By definition of V_1, there exists some $g_0 \in V^2$ such that $g_0^2 = g$. As $\gamma(g_0)^2 \in \{e^{i\theta} : \theta \in [\pi - l, \pi + l]\}$, it must be the case that for all $\gamma \in H$,

$$\gamma(g_0) \in \left\{ e^{i\theta} : \theta \in \left[\frac{\pi - l}{2}, \frac{\pi + l}{2}\right] \cup \left[\frac{3\pi - l}{2}, \frac{3\pi + l}{2}\right] \right\}.$$

Let $A = \{\gamma \in H : \gamma(g_0) = e^{i\theta} \text{ with } \theta \in [\pi/2 - l/2, \pi/2 + l/2]\}$ and put $\phi = 1$ on A and $\phi = 1$ on $H \setminus A$. Pick $m = m(l)$ such that $(1 - d/4)^m < l$ and obtain $\mu \in M_{\mathrm{d}}^{\mathrm{r}}(V_1^2)$, with real transform, such that length $\mu \leq 3m$ and $|\widehat{\mu}(\gamma) - \phi(\gamma)| < l$ for all $\gamma \in H$. Then $\mu * \delta_{g_0}$ interpolates i on H to within $2l$ and $\mu * \delta_{g_0}$ is supported on $V_1^2 V^2 \subseteq V^6$.

Standard arguments now show that if $\phi \in B(\ell^\infty(H))$, then there is a measure $\mu \in M_{\mathrm{d}}^{\mathrm{r}}(V^{10}) \subseteq M_{\mathrm{d}}^{\mathrm{r}}(U)$, of length at most $M = M(l)$, such that $\|\widehat{\mu} - \phi|_H\|_\infty \leq 1 - \varepsilon$ for some $\varepsilon = \varepsilon(l) > 0$. That completes the argument if the real parts of the arcs I and J are separated by at least d.

Case II: The real parts do not differ by at least d. Since the gap is double the length of the intervals and the intervals are short, the imaginary parts must be strictly on opposite sides of 0. This means the arcs are at distance at least l from both 0 and π, and hence their imaginary parts are bounded away from 0 by at least $l/2$. Without loss of generality assume the imaginary part of I is positive.

Given real-valued $\phi \in B(\ell^\infty(H))$, let $A = \{\gamma \in H : \phi(\gamma) \geq 0\}$. Choose $g \in V_1^2$ such that $\gamma(g) \in I$ if $\gamma \in A$ and $\gamma(g) \in J$ if $\gamma \in H \setminus A$. Next, put $\mu = (\delta_g - \delta_{g^{-1}})/4 \in M_{\mathrm{d}}^{\mathrm{r}}(V_1^2)$. Then μ has length at most 2, $\widehat{\mu}$ is purely imaginary and $\|\widehat{\mu} - i\phi|_H\|_\infty \leq 1 - l/4 < 1$. With the usual iteration argument we can obtain $\mu \in M_{\mathrm{d}}^{\mathrm{r}}(V^4)$ with length $\leq 2m$, $\widehat{\mu}$ purely imaginary and $\|\widehat{\mu} - \psi|_H\|_\infty \leq (1 - l/4)^m$ for any ψ purely imaginary.

Given any $\phi \in B(\ell^\infty(H))$, we write $\phi = -i(i\phi_1) + i\phi_2$ where $i\phi_1, i\phi_2$ are purely imaginary, and then consider $\mu = \mu_0 * \mu_1 + \mu_2 \in M_{\mathrm{d}}^{\mathrm{r}}(V^8) \subseteq M_{\mathrm{d}}^{\mathrm{r}}(U)$, where $\widehat{\mu}_j$ approximates $i\phi_j$ on H for $j = 1, 2$ and $\widehat{\mu}_0$ approximates $-i$. We have

$$\|\widehat{\mu_0 * \mu_1} + \widehat{\mu}_2 - (-i(i\phi_1) + i\phi_2)|_H\|_\infty \leq 4(1 - l/4)^m < 1$$

(for suitably large m) and the length of μ is at most $6m^2$.

This completes the argument in Case II and thus E is cofinitely proportional $RI_0(U)$.

(3)⇒(2). In Corollary 7.3 we show that finite sets are RI_0 with constants depending only on their cardinality. Thus if E is cofinitely proportional $RI_0(G)$, then it is also proportional RI_0.

(2)⇒(1). This follows from Ramsey's work as proportional RI_0 sets are proportional I_0. ∎

COROLLARY 5.2. *If G is connected and E is Sidon, then E is proportional $RI_0(U)$ for all open U.*

Proof. When G is connected, Γ has no elements of finite order; thus any Sidon set is cofinitely proportional $RI_0(U)$ for all open sets U. Since finite sets are $RI_0(U)$ when the group is connected, with constants depending only on the cardinality of the set (Corollary 7.3), the result follows. ∎

REMARK 5.3. It would be interesting to know if an alternative proof that all Sidon sets in duals of connected groups are Sidon(U) could be derived from this characterization.

EXAMPLE 5.4. Consider $G = \prod G_j$, where G_1 is finite and none of the groups has elements of order 2. Then Γ has no elements of order 2. Let E be the set of projections π_j onto the factors G_j. This set is independent and hence Sidon. Let $U = e \times \prod_{j \neq 1} G_j$. Then E is not proportional $RI_0(U)$ since the singleton $\{\pi_1\}$ is not $RI_0(U)$. Thus without connectedness we can only be sure of Sidon sets being cofinitely proportional $RI_0(U)$.

DEFINITION 6. We call a set E a *real* $RI_0(U, N, \varepsilon)$ (respectively, *real* $FZI_0(U, N, \varepsilon)$) *set* if for every real-valued Hermitian $\phi \in B(\ell^\infty(E))$ there exists $\mu \in M_{\mathrm{d}}^{\mathrm{r}}(U)$ (resp., $\mu \in M_{\mathrm{d}}^{+}(U)$) of length at most N and satisfying $\|\widehat{\mu} - \phi|_H\|_\infty < \varepsilon$. We suppress the N, ε in practice.

Real RI_0 (respectively, real FZI_0) sets E are precisely those for which every real-valued, bounded Hermitian function on E is the restriction of the Fourier transform of a real, discrete (resp., positive and discrete) measure to E. We can define proportional real RI_0 and proportional real FZI_0 in the obvious fashion.

Notice that in the proof of the theorem, the restriction on the number of order 2 elements (used to ensure the ability to take square roots) was only needed to interpolate i. Thus we have the following corollary:

COROLLARY 5.5. *$E \subseteq \Gamma$ is Sidon if and only if it is proportional real RI_0.*

6. Proportional FZI_0 subsets. To upgrade the results from RI_0 to FZI_0 we need to show how to interpolate -1 with the transform of a positive discrete measure.

LEMMA 6.1. *Suppose E is Sidon and V is a neighbourhood of e. There are constants $k_0 = k_0(E)$, $\alpha = \alpha(E, V)$ and $C = C(E, V)$ such that, given any finite $F \subseteq E$ with $\operatorname{Card} F \geq \alpha$, there are $H \subseteq F$ with $\operatorname{Card} H \geq C \operatorname{Card} F$, and a measure $\varrho \in M_{\mathrm{d}}^{+}(V^{k_0})$ of real transform and length two, which satisfy $|\widehat{\varrho}(\gamma) + 1| < 1 - \sin \pi/8$ for all $\gamma \in H$.*

In Lemma 6.1, the independence of k_0 from V is surprising at first. This independence can occur because if characters whose values are near 1 on V were included in F, the large $\alpha(E, V)$ would mean that they could not be all of F, so those characters would then be excluded from H by the small $C(E, V)$. It is crucial, of course, that this is a cofinitely proportional result.

Proof of Lemma 6.1. Apply §4.3 to find the number $l = l(\delta)$ where $\delta = \delta(E)$ arises from Corollary 4.2. Put $k_0 = \lfloor 2\pi/3l \rfloor$ (and notice that k_0 is independent of the choice of V, but not of E).

Now choose a symmetric e-neighbourhood W such that $W^{k_0} \subseteq V$. Get $\alpha(E, W)$ from Corollary 4.2, so that if $\operatorname{Card} F \geq \alpha$, then there are points $g_1, \ldots, g_n \in W$, with $n \geq 2^{\delta |F|}$, such that (4.1) holds.

Obtain a proportionality constant $C = C(\delta) > 0$, a subset H of F with $\operatorname{Card} H \geq C \operatorname{Card} F$, and two arcs on the circle of lengths at most l and gap between them of size at least $4l$, as in §4.2. At least one of these arcs is separated from the angle 0 by at least $4l$. Choose such an arc and call it K. The construction ensures that there is some $g = g_i g_j^{-1} \in W^2$ such that $\chi(g)$ belongs to K for all $\chi \in H$. We can assume $K = [\theta, \theta + l]$ with (without loss of generality) $4l \leq \theta \leq \pi$. (We may replace K by $-K \pmod{2\pi}$ if needed.) If $\pi \in K$ then $\varrho = (\delta_g + \delta_{g^{-1}})/2$ will satisfy

$$|\widehat{\varrho}(\chi) + 1| < 1 - \cos l \leq 1 - \cos(1/20) < 1 - \sin \pi/8 \quad \text{ for all } \chi \in H.$$

So we can assume, without loss of generality, that K lies entirely within quadrants 1 and 2.

We consider several cases.

If $\theta \geq \pi/2 + \pi/8$, then $|\widehat{\varrho}(\chi) + 1| < 1 - \sin \pi/8$ for all $\chi \in H$.

If $\theta \in [\pi/3, \pi/2 + \pi/8)$, then $\chi(g^2) \in [2\pi/3, 5\pi/4]$, and consequently $\varrho = (\delta_{g^2} + \delta_{g^{-2}})/2$ also satisfies $|\widehat{\varrho}(\chi) + 1| \leq 1/2 < 1 - \sin \pi/8$ for all $\chi \in H$.

Otherwise, $\theta \in [4l, \pi/3)$. Put $k_\theta = \lceil 2\pi/3\theta \rceil \leq k_0$. Then for all $\chi \in H$,

$$\chi(g^{k_\theta}) \in [k_\theta \theta, k_\theta(\theta + l)] \subseteq \left[\frac{2\pi}{3}, \left(\frac{2\pi}{3\theta} + 1\right)(\theta + l)\right].$$

As $4l \leq \theta \leq \pi/3$ and $l \leq \pi/24$,

$$\left(\frac{2\pi}{3\theta} + 1\right)(\theta + l) = \frac{2\pi}{3} + \theta + l + \frac{2\pi}{3}\,\frac{l}{\theta} \leq \pi + l + \frac{\pi}{6} \leq \frac{3\pi}{2} - \frac{\pi}{8}.$$

Thus the positive, discrete measure $\varrho = (\delta_{g^{k_\theta}} + \delta_{g^{-k_\theta}})/2$ will satisfy

$$|\widehat{\varrho}(\chi) + 1| < 1 - \sin \pi/8 \quad \text{ for all } \chi \in H.$$

Moreover, in all cases, ϱ has length 2, real transform and is supported on $W^{k_\theta} \subseteq W^{k_0} \subseteq V$. ■

The independence of k_0 from V is important in the application of this lemma.

THEOREM 6.2. *Assume Γ has only finitely many elements of order* 2. *The following are equivalent*:

(1) *$E \subset \Gamma$ is Sidon*;
(2) *E is proportional FZI_0*;
(3) *E is cofinitely proportional $FZI_0(U)$ for all open sets U.*

Proof. (1)⇒(3). Pick a symmetric e-neighbourhood U_1 with $U_1^2 \subseteq U$. By Theorem 5.1, E is cofinitely proportional $RI_0(U_1)$ with, say, constants $C', n, \varepsilon, \alpha_1$. Choose an even integer k_1 such that

$$(1 - \sin \pi/8)^{k_1} \leq (1 - \varepsilon)/(2n)$$

and take $k_0 = k_0(E)$ as in Lemma 6.1. Select a symmetric neighbourhood V of the identity such that $V^{k_0 k_1} \subset U_1$. Let $C = C(E, V)$ and $\alpha = \alpha(E, V)$ also be given by Lemma 6.1. Put $\alpha_0 = \max(\alpha, \alpha_1/C)$.

Suppose F is a finite subset of E with $\operatorname{Card} F \geq \alpha_0$. Apply Lemma 6.1 to find $H \subseteq F$ with $\operatorname{Card} H \geq C \operatorname{Card} F$ and a measure $\varrho \in M_{\mathrm{d}}^+(V^{k_0})$ of length 2 with real transform and satisfying $|\widehat{\varrho}(\gamma) + 1| < 1 - \sin \pi/8$ for all $\gamma \in H$. Let

$$\nu = \sum_{j=1}^{k_1} \binom{k_1}{j} \varrho^j \in M_{\mathrm{d}}^+(V^{k_0 k_1}) \subseteq M_{\mathrm{d}}^+(U_1).$$

Then the length of ν is at most $k_1 2^{2k_1}$ and

$$|\widehat{\nu}(\gamma)+1| = \left|\sum_{j=0}^{k_1}\binom{k_1}{j}\widehat{\varrho}^j(\gamma)\right| = |\widehat{\varrho}(\gamma)+1|^{k_1} \le \frac{1-\varepsilon}{2n}.$$

As $\operatorname{Card} H \ge \alpha_1$, a further subset $H' \subseteq H$ is $RI_0(U_1,n,\varepsilon)$ and satisfies

$$\operatorname{Card} H' \ge C' \operatorname{Card} H \ge C\,C' \operatorname{Card} F.$$

Given Hermitian $\phi \in B(\ell^\infty(H'))$, let $\mu \in M_{\mathrm{d}}^{\mathrm{r}}(U_1)$ have length at most n (as in the first paragraph of the proof) and satisfy $\|\widehat{\mu} - \phi|_{H'}\| < \varepsilon$. Write $\mu = \sum_{k=1}^n (a_k^+ - a_k^-)\delta_{x_k}$ where $0 \le a_k^+, a_k^- \le 1$.

Assume $\nu = \sum b_j \delta_{y_j}$ and put

$$\omega = \sum_k a_k^+ \delta_{x_k} + \sum_k \sum_j a_k^- b_j \delta_{x_k y_j}.$$

Clearly, $\omega \in M_{\mathrm{d}}^+(U)$, the length of ω is $\le 2nk_1 2^{2k_1} \equiv N$, and one can easily check that for $\chi \in H'$,

$$\begin{aligned}|\widehat{\omega}(\chi) - \phi(\chi)| &\le |\widehat{\mu}(\chi) - \phi(\chi)| + \sum a_k^- |\widehat{\nu}(\chi)+1| \\ &< \varepsilon + \frac{n(1-\varepsilon)}{2n} = \frac{1+\varepsilon}{2} = \varepsilon' < 1.\end{aligned}$$

Thus H' is $FI_0(U,N,\varepsilon')$ and so E is cofinitely proportional $FZI_0(U)$.

(3)⇒(2) follows from the fact (Corollary 7.7) that finite sets not containing the identity are FZI_0 with constants depending on their cardinality.

(2)⇒(1) holds as proportional FZI_0 sets are proportional Sidon. ∎

REMARKS 6.3. (i) We note that even in the connected group case, Sidon sets need not be proportional $FZI_0(U)$ for all open sets since singletons fail to be $FZI_0(U)$ for U sufficiently small; that is, "cofinitely" is essential.

(ii) It is known that if U is an open subset of a connected group and E is ε-Kronecker for some $\varepsilon < \sqrt{2}$, then a cofinite subset of E is $FZI_0(U)$ ([6]), but it is not clear if this fact can be used to show that Sidon sets are cofinitely proportional $FZI_0(U)$.

Since the assumption on the number of order 2 elements was not needed to interpolate -1, these arguments imply that all Sidon sets are proportional real FZI_0. Indeed, we have the following improvement on Ramsey's proportional I_0 result:

COROLLARY 6.4. *$E \subseteq \Gamma$ is Sidon if and only if there exist positive constants $C, N, \varepsilon < 1$ such that whenever $F \subseteq E$ is finite, then there is a subset $H \subseteq F$ with $\operatorname{Card} H \ge C \operatorname{Card} F$ and having the property that whenever $\phi \in B(\ell^\infty(H))$ there is a measure $\mu = \mu_1 + i\mu_2$ such that μ_1, μ_2*

are positive discrete measures of length at most N and

$$\sup_{\gamma\in H}|\widehat{\mu}(\gamma)-\phi(\gamma)|<\varepsilon.$$

7. Interpolation on finite sets

7.1. *I_0 and RI_0 properties.* In this subsection we prove that all finite sets are RI_0 sets (or $RI_0(U)$ sets in the connected group case) with constants depending only on the cardinality of the set (and U).

LEMMA 7.1. *Assume $V^m = G$ for some m. Then there exists a constant $n = n(m)$ such that for each $\gamma_1 \neq \gamma_2$ there is some $\mu \in M_{\mathrm{d}}(V)$ satisfying $\widehat{\mu}(\gamma_1) = 1$, $\widehat{\mu}(\gamma_2) = 0$ and length $\mu \le n$.*

Proof. For any character $\alpha \neq 1$, the range of α is a non-trivial subgroup of $\mathbb{T}$ and so there is some $g \in G$ such that $\arg\alpha(g) \in [2\pi/3, 4\pi/3]$. Consequently, $|\gamma_1\gamma_2^{-1}(g) - 1| \ge 3/2$. Now $g = v_1 \dots v_m$ for $v_i \in V$, thus

$$\begin{aligned}|\gamma_1\gamma_2^{-1}(g)-1| &\le |\gamma_1\gamma_2^{-1}(v_1)-1|+\sum_{i=1}^{m-1}|\gamma_1\gamma_2^{-1}(v_1\dots v_{i+1})-\gamma_1\gamma_2^{-1}(v_1\dots v_i)|\\ &\le \sum_{i=0}^{m-1}|\gamma_1\gamma_2^{-1}(v_{i+1})-1|.\end{aligned}$$

It follows that $|\gamma_1(v)-\gamma_2(v)| = |\gamma_1\gamma_2^{-1}(v)-1| \ge 3/(2m)$ for some $v \in V$.

Now consider the discrete measure

$$\mu = \frac{\delta_v - \gamma_2(v)\delta_e}{\gamma_1(v)-\gamma_2(v)} \in M_{\mathrm{d}}(V).$$

It is of length at most $4m/3$ and satisfies $\widehat{\mu}(\gamma_1) = 1$, $\widehat{\mu}(\gamma_2) = 0$. ■

PROPOSITION 7.2.

(i) *There exists a constant N_k such that any set of cardinality k is $I_0(N_k, 0)$.*
(ii) *If G is connected, then for any open set U there is a constant N_k, depending on k and U, such that any set of cardinality k is $I_0(U, N_k, 0)$.*

Proof. If F is a singleton, $\{\gamma\}$, given ϕ simply put $\mu = \phi(\gamma)\delta_e$.

Otherwise, assume $F = \{\gamma_i\}_{i=1}^k$ with $k > 1$ and choose an open set V, $V = G$ in (i) and satisfying $V^{k-1} \subset U$ in (ii). The connectedness of G in (ii) ensures that $V^m = G$ for some m.

For each $i \neq j$ apply Lemma 7.1 to get $\mu_{ij} \in M_{\mathrm{d}}(V)$ with length at most n and satisfying $\widehat{\mu}_{ij}(\gamma_i) = 1$, $\widehat{\mu}_{ij}(\gamma_j) = 0$. Put $\mu_i = *_{j\neq i}\,\mu_{ij} \in M_{\mathrm{d}}^+(V^{k-1})$ so that $\widehat{\mu}_i(\gamma_i) = 1$, $\widehat{\mu}_i(\gamma) = 0$ for all $\gamma \neq \gamma_i$ in F and length $\mu_i \le n^{k-1}$.

If we are given $\phi \in B(\ell^\infty(F))$, we can take $\mu = \sum_{i=1}^{k} \phi(\gamma_i)\mu_i$ and thus $N_k = kn^{k-1}$. ■

Corollary 7.3. *Suppose F has cardinality k.*

(i) *F is $RI_0(2N_{2k}, 0)$.*
(ii) *If G is connected and U is an open set, then F is $RI_0(U, 2N_{2k}(U), 0)$.*

Proof. Given Hermitian $\phi \in B(\ell^\infty(F))$, extend ϕ to $F^{-1} \setminus F$ by

$$\phi(\gamma^{-1}) = \overline{\phi(\gamma)}.$$

The set $F \cup F^{-1}$ is $I_0(U, N_{2k}, 0)$, hence there exists $\mu \in M_{\mathrm{d}}(U)$ such that length $\mu \le N_{2k}$ and $\widehat{\mu}(\gamma) = \phi(\gamma)$ for all $\gamma \in F \cup F^{-1}$. Put $\nu = (\mu + \overline{\mu})/2$ to obtain a real, discrete measure, supported on U, of length at most $2N_{2k}$ and interpolating ϕ on F. ■

7.2. *FZI_0 properties.* Lastly, we prove that all finite sets not containing 1 are FZI_0 sets with constants depending only on the cardinality of the set. The corresponding question about $FZI_0(U)$ sets is more subtle as singletons are not $FZI_0(U)$ if U is "too small".

We need another definition.

Definition 7. We will say F is *local $FZI_0(U, N, \varepsilon)$* if each singleton $\{\gamma\}$, $\gamma \in F$, is $FZI_0(U, N, \varepsilon)$.

Lemma 7.4. *Suppose U is a symmetric e-neighbourhood. For each integer $n_0 \ge 1$ there exists an integer $n_1 = n_1(n_0)$ such that if F is local $FZI_0(U, n_0, 1/2)$, then F is local $FZI_0(U, n_1, 0)$.*

Proof. Without loss of generality $F = \{\gamma\}$. If γ has order 2, then find $\mu \in M_{\mathrm{d}}^+(U)$ such that $|\widehat{\mu}(\gamma) + 1| \le 1/2$ and take $\mu_1 = (\mu + \widetilde{\mu})/(-2\Re\widehat{\mu}(\gamma))$ to interpolate -1 exactly with a length $2n_0$ measure. Given $\phi(\gamma)$ real-valued, either $\phi(\gamma)\delta_e$ or $\phi(\gamma)\mu_1$ does the required interpolation.

Otherwise, choose $\mu_1, \mu_2 \in M_{\mathrm{d}}^+(U)$ with lengths at most n_0 and such that $|\widehat{\mu}_1(\gamma) - e^{i3\pi/4}| \le 1/2$ and $|\widehat{\mu}_2(\gamma) - e^{i5\pi/4}| \le 1/2$. Then $\Re\widehat{\mu}_j(\gamma) \le -(\sqrt{2}-1)/2 \le -1/5$. The imaginary parts of $\widehat{\mu}_j(\gamma)$ are opposite in sign and both are also bounded away from zero by $-1/5$.

Given Hermitian $\phi(\gamma) = a^+ - a^- + i(b^+ - b^-)$ of norm 1, we can interpolate exactly at γ with the Fourier transform of the measure $\mu \in M_{\mathrm{d}}^+(U)$ given by

$$\mu = a^+\delta_e + a^- \frac{(\mu_1 + \widetilde{\mu}_1)/2}{|\Re\widehat{\mu}_1(\gamma)|} + b^+\left(\frac{\mu_1 - \Re\widehat{\mu}_1(\gamma)\delta_e}{\Im\widehat{\mu}_1(\gamma)}\right) + b^-\left(\frac{\mu_2 - \Re\widehat{\mu}_2(\gamma)\delta_e}{-\Im\widehat{\mu}_2(\gamma)}\right).$$

The length of μ is at most $13n_0$. ■

LEMMA 7.5. *Suppose that F has cardinality k and that V is a symmetric e-neighbourhood. Assume that F is local $FZI_0(V,n,0)$ and that either G is connected or $V=G$. Then there is an integer $n_2=n_2(n,V)$ such that for all $\gamma_1,\gamma_2\in F$, the doubleton $\{\gamma_1,\gamma_2\}$ is $FZI_0(V^2,n_2,0)$.*

Proof. As F is local $FZI_0(V,n,0)$ there is some $\mu_1\in M_{\mathrm{d}}^+(V)$ such that $\widehat{\mu}_1(\gamma_1)=-1$ and length $\mu_1\le n$. By Corollary 7.3 there is some $n_1=n_1(V)$ such that all two-element sets are $RI_0(V,n_1,0)$.

Consider the strip $S=\{z\in\mathbb{C}:|\Re z+1|\le\varepsilon\}$ where $\varepsilon n_1=1/2$.

If $\widehat{\mu}_1(\gamma_2)\notin S$, then $\widehat{\mu}_1(\gamma_1)=-1$ implies $|\Re\widehat{\mu}_1(\gamma_1)-\Re\widehat{\mu}_1(\gamma_2)|\ge\varepsilon$. Without loss of generality we can assume $\Re\widehat{\mu}_1(\gamma_1)\ge\Re\widehat{\mu}_1(\gamma_2)$. We define

$$\mu=\frac{(\mu_1+\widetilde{\mu}_1)/2-\Re\widehat{\mu}_1(\gamma_2)\delta_e}{\Re\widehat{\mu}_1(\gamma_1)-\Re\widehat{\mu}_1(\gamma_2)}$$

The choice ensures that $\mu\in M_{\mathrm{d}}^+(V)$, $\widehat{\mu}$ is $1,0$-valued on γ_1,γ_2, respectively, and the length of μ is at most $(2n+1)/\varepsilon=(2n+1)2n_1$. As F is local $FZI_0(V,n,0)$, routine arguments show that $\{\gamma_1,\gamma_2\}$ is $FZI_0(V^2,n_2,0)$ for a suitable n_2.

Otherwise, $\widehat{\mu}_1(\gamma_2)\in S$. We put $\mu=(\mu_1+\widetilde{\mu}_1)/2\in M_{\mathrm{d}}^+(V)$. Given Hermitian $\phi\in B(\ell^\infty\{\gamma_1,\gamma_2\})$ obtain $\nu\in M_{\mathrm{d}}^{\mathrm{r}}(V)$ of length n_1 such that $\widehat{\nu}(\gamma_i)=\phi(\gamma_i)$. Assume $v=\sum(a_k^+-a_k^-)\delta_{x_k}$ with $a_k^+,a_k^-\ge 0$. Now set

$$\omega=\sum a_k^+\delta_{x_k}+\sum a_k^-\mu*\delta_{x_k}\in M_{\mathrm{d}}^+(V^2).$$

One can easily see that the length of ω is bounded by a function of n,n_1, and

$$|\widehat{\omega}(\gamma_i)-\phi(\gamma_i)|\le|\widehat{\nu}(\gamma_i)-\phi(\gamma_i)|+\sum a_k^-|\Re\widehat{\mu}_1(\gamma_i)+1|\le n_1\varepsilon=1/2.$$

Thus $\{\gamma_1,\gamma_2\}$ is $FZI_0(V^2,n_2',1/2)$. By approximating ±1, we can find $\sigma\in M_{\mathrm{d}}^+(V)$ of length n_2' with $|\Re\widehat{\mu}_1(\gamma_1)-\Re\widehat{\mu}_1(\gamma_2)|\ge 1$. This essentially reduces the problem to the first part of the argument. ∎

PROPOSITION 7.6. *Assume F is a set of cardinality k. Suppose U is an e-neighbourhood and that there exists a symmetric e-neighbourhood V with $V^{2k}\subseteq U$ such that F is local $FZI_0(V,n,0)$. Assume that either G is connected or $V=G$. Then there is an integer $N_k=N_k(U,k,n)$ such that F is $FZI_0(U,N_k,0)$.*

Proof. Let $F=\{\gamma_i\}_{i=1}^k$. By the previous lemma, for each $i\ne j$, obtain a measure $\mu_{ij}\in M_{\mathrm{d}}^+(V^2)$, of length $\le n_2$, satisfying $\widehat{\mu}_{ij}(\gamma_i)=1$, $\widehat{\mu}_{ij}(\gamma_j)=0$. Put $\mu_i=*_{j\ne i}\,\mu_{ij}\in M_{\mathrm{d}}^+(V^{2(k-1)})$. Given Hermitian $\phi\in B(\ell^\infty(F))$, we can obtain $\nu_i\in M_{\mathrm{d}}^+(V)$ with $\widehat{\nu}_i(\gamma_i)=\phi(\gamma_i)$. Put $\omega=\sum_{i=1}^k\mu_i*\nu_i$ to obtain the appropriate interpolating measure. ∎

COROLLARY 7.7. *There is a constant N_k such that if F is any set of cardinality k, not containing the identity, then F is $FZI_0(N_k,0)$.*

Proof. It suffices to prove that singletons $\{\gamma\}$ (other than the identity character) are $FZI_0(2, \pi/4)$. To see this, first suppose γ is of order 2. Then we need only interpolate values $\phi(\gamma)$ in $[-1, 1]$, which we can do with either $\phi(\gamma)\delta_e$ or $\phi(\gamma)\delta_x$ where $\gamma(x) = -1$. Thus the order 2 elements are actually $FZI_0(1, 0)$.

If γ has order at least 4 (including infinite order), then given any $t \in \mathbb{T}$ there exists $x \in G$ such that $|\widehat{\delta}_x(\gamma) - t| \le \pi/4$. This suffices as every point in the unit ball is a convex combination of two points of modulus 1.

Otherwise, γ has order 3 and we can find either a point x or pair x, y such that $|\gamma(x) - t| \le \pi/6$ or $|\gamma(x) + \gamma(y) - t| \le \pi/6$. In either case we can obtain a length 2 measure μ satisfying $|\widehat{\mu}(\gamma) - t| \le \pi/6$. ∎

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