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ON BILINEAR BIQUANDLES

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Abstract. We define a type of biquandle which is a generalization of symplectic quandles. We use the extra structure of these bilinear biquandles to define new knot and link invariants and give some examples.

1. Introduction. A *biquandle* is an algebraic structure consisting of a set B with four binary operations $(a,b)\mapsto a^b$, a_b, $a^{\overline{b}}$ and $a_{\overline{b}}$ satisfying axioms derived from the oriented Reidemeister moves, where generators of the algebra are identified with semiarcs in an oriented link diagram. A biquandle may also be understood as a solution $S : B\times B \to B\times B$ to the *set-theoretic Yang–Baxter equation*

$$(S\times \mathrm{Id})(\mathrm{Id}\times S)(S\times \mathrm{Id}) = (\mathrm{Id}\times S)(S\times \mathrm{Id})(\mathrm{Id}\times S)$$

which satisfies some additional criteria corresponding to the first and second Reidemeister moves. Such a map S is called a *switch*, and a biquandle is an invertible switch with components $S(a,b) = (b_a, a^b)$ satisfying $S^{-1}(a,b) = (b^{\overline{a}}, a_{\overline{b}})$ and the extra conditions required by the reverse type II and type I moves.

This relationship between the biquandle axioms and the Reidemeister moves makes biquandles a natural source of knot and link invariants. For example, the biquandle counting invariant $|\mathrm{Hom}(B(L),T)|$ is the cardinality of the set of biquandle homomorphisms from the knot biquandle of a link L into a finite target biquandle T. One can think of each homomorphism as a "coloring" of the link diagram by T, assigning an element of T to every semiarc in a diagram of L such that the biquandle operations are satisfied at every crossing; we can then see that this family of invariants is a generalization of Fox's n-coloring invariants. Indeed, biquandles generalize quandles, which in turn generalize knot groups. Biquandles have been studied in recent papers such as [4], [6], [2] and more.

Many of the examples of biquandles in the current literature are natural generalizations of types of quandle structures—Alexander biquandles

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generalize Alexander quandles, Silver–Williams switches generalize Joyce's homogeneous quandles, etc. In this paper we generalize the symplectic quandles studied in [8] (also known as *quandles of transvections*) to define what we call bilinear biquandles.

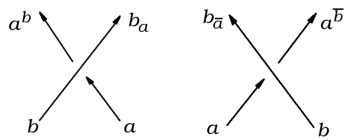


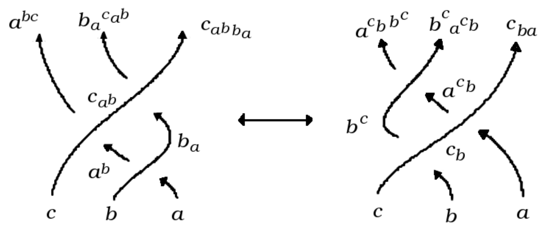
Fig. 1. Biquandle operations at crossings

The paper is organized as follows. In Section 2 we list the biquandle axioms and give examples of biquandles. In Section 3 we define bilinear biquandles, obtain some results about their structure and give an example of bilinear biquandle which is not a quandle. In Section 4 we generalize the symplectic quandle polynomial invariants defined in [8] to the biquandle case and give some examples of classical and virtual links which have the same value for the biquandle counting invariant but are distinguished by the bilinear biquandle invariant. In Section 5, we list all bilinear biquandle structures with cardinality up to 27 as determined by our computer search. In Section 6, we end with some questions for further research.

2. Biquandles and symplectic quandles. Let B be a set. A *biquandle structure* on B consists of four binary operations $(a,b) \mapsto a^b$, a_b, $a^{\overline{b}}$ and $a_{\overline{b}}$ such that

1. for all $a, b \in B$ we have
$$a = a^{b\overline{b_a}}, \qquad b = b_{\overline{a a^b}}, \qquad a = a^{\overline{b} b_{\overline{a}}}, \qquad b = b_{\overline{a} a^{\overline{b}}},$$
2. for all $a, b \in B$ there exist $x, y \in B$ such that
$$x = a^{b_{\overline{x}}}, \qquad a = x^{\overline{b}}, \qquad b = b_{\overline{x} a}, \qquad y = a^{\overline{b_y}}, \qquad a = y^b, \qquad b = b_{y\overline{a}},$$
3. for all $a, b, c \in B$ we have
$$a^{bc} = a^{c_b b^c}, \qquad c_{ba} = c_{a_b b_a}, \qquad (b_a)^{c_{a^b}} = (b^c)_{a^{c_b}},$$
$$a^{\overline{b}\overline{c}} = a^{\overline{c_{\overline{b}}}\,\overline{b^{\overline{c}}}}, \qquad c_{\overline{b}\overline{a}} = c_{\overline{a_{\overline{b}}}\,\overline{b_{\overline{a}}}}, \qquad (b_{\overline{a}})^{\overline{c_{\overline{a^{\overline{b}}}}}} = (b^{\overline{c}})_{\overline{a^{\overline{c_{\overline{b}}}}}},$$
4. for every $a \in B$ there exist $x, y \in B$ such that
$$x = a_x, \qquad a = x^a, \qquad y = a^{\overline{y}}, \qquad a = y_{\overline{a}}.$$

These axioms are obtained from the oriented Reidemeister moves by thinking of each semiarc (portion of the knot diagram between over/under crossing points) as a biquandle element; the biquandle elements on the outside of each pictured diagram portion must then agree before and after the move. For example, the oriented type III move with all positive crossings is the following:



While there are eight oriented type III moves, in the presence of the oriented type II moves we need only the two type III moves with all positive and all negative crossings. See [6] for more.

Example 1. A *quandle* is a set Q with two binary operations $\triangleright, \triangleright^{-1} : Q \times Q \to Q$ such that

(i) for all $a \in Q$, $a \triangleright a = a$,
(ii) for all $a, b \in Q$ we have $(a \triangleright b) \triangleright^{-1} b = a = (a \triangleright^{-1} b) \triangleright b$,
(iii) for all $a, b, c \in Q$ we have $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

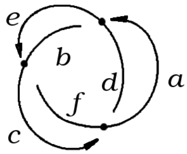
Every quandle is a biquandle with $a^b = a \triangleright b$, $a^{\overline{b}} = a \triangleright^{-1} b$, and $a_b = a_{\overline{b}} = a$.

Example 2. As an example of a non-quandle biquandle, let $B = \mathbb{Z}_n$ and let $s, t \in \mathbb{Z}_n$ be invertible elements. Then B is a biquandle with

$$a^b = ta + (1 - st)b, \quad a^{\overline{b}} = t^{-1}a + (1 - s^{-1}t^{-1})b, \quad a_b = sa, \quad a_{\overline{b}} = s^{-1}a.$$

Example 3. More generally, let M be any module over $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$. Then M is a biquandle under the operations defined in Example 2, called an *Alexander biquandle*. See [6] and [7] for more.

Example 4. Let D be an oriented link diagram, i.e. a planar 4-valent graph with two inward-oriented and two outward-oriented edges incident on every vertex, with vertices decorated to indicate crossing information. Then the *knot biquandle* $B(L)$ of the link represented by L has a presentation with one generator for each edge and relations at each crossing as depicted in Figure 1. The elements of the knot biquandle are equivalence classes of biquandle words in these generators under the equivalence relation generated by the biquandle axioms and the crossing relations. For example, the trefoil knot below has the listed knot biquandle:



$$\langle a,b,c,d,e,f \mid a^d = b,\ b_e = c,\ c^f = d,\ d_a = e,\ e^b = f,\ f_c = a\rangle.$$

If we drop the planarity requirement, such a D defines a *virtual link*, and we obtain a knot biquandle by the same procedure. Crossings arising from non-planarity are depicted as circled intersections; see [5] for more about virtual knots and links.

If $B = \{x_1, \dots, x_n\}$ is a finite biquandle, we can represent B symbolically with a $2n \times 2n$ block matrix, where the four $n \times n$ blocks encode the four biquandle operations. That is, we let

$$M_B = \left[\begin{array}{c|c} M^1 & M^2 \\ \hline M^3 & M^4 \end{array}\right], \qquad M^l_{ij} = k \quad \text{where} \quad x_k = \begin{cases} (x_i)^{\overline{(x_j)}}, & l = 1, \\ (x_i)^{(x_j)}, & l = 2, \\ (x_i)_{\overline{(x_j)}}, & l = 3, \\ (x_i)_{(x_j)}, & l = 4. \end{cases}$$

For example, the Alexander biquandle $B = \mathbb{Z}_3$ with $s = 2$, $t = 1$ has biquandle matrix

$$M_B = \left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 3 & 2 & 1 \\ 1 & 3 & 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 2 & 1 & 3 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{array}\right].$$

Biquandle matrices can be used to compute the biquandle counting invariant $|\mathrm{Hom}(B(K), T)|$ and the Yang–Baxter 2-cocycle invariants of a knot or link in an algebra-agnostic way; see [9] and [3] for more.

EXAMPLE 5. As a final example of a non-quandle biquandle structure, let R be a ring with identity and M an R-module. Then the operations

$$y_x = Ax + By, \qquad x^y = Cx + Dy$$

with $A, B \in R$ and $C = A^{-1}B^{-1}A(1-A)$ and $D = 1 - A^{-1}B^{-1}AB$ define an *invertible switch* $S : M \times M \to M \times M$ by $S(x,y) = (y_x, x^y)$ provided $[B, (A-I)(A,B)] = 0$, where $[X,Y] = XY - YX$ and $(X,Y) = X^{-1}Y^{-1}XY$; such a switch gives a biquandle structure with barred operations $S^{-1}(x,y) =$

$(x^{\overline{y}}, y_{\overline{x}})$ provided the axioms arising from the type I and reverse type II moves are satisfied. See [1] for more.

3. Bilinear biquandles. Let R be any commutative ring and M any free module over R. Let $\langle\,,\rangle : M \times M \to R$ be an antisymmetric bilinear form. Then M is a biquandle with operations

$$\mathbf{x}^{\overline{\mathbf{y}}} = \mathbf{x} - \langle \mathbf{x}, \mathbf{y}\rangle \mathbf{y}, \quad \mathbf{x}^{\mathbf{y}} = \mathbf{x} + \langle \mathbf{x}, \mathbf{y}\rangle \mathbf{y}, \quad \mathbf{x}_{\overline{\mathbf{y}}} = \mathbf{x}, \quad \mathbf{x}_{\mathbf{y}} = \mathbf{x}.$$

Such a biquandle is in fact a quandle; quandles of this type have been called *symplectic quandles* or *quandles of transvections*. See [8] and [10].

We would like to extend this definition to define non-quandle biquandles.

Definition 1. Let M be a free module over a commutative ring R. A *bilinear biquandle* structure on M is a biquandle structure on M such that

$$\mathbf{x}^{\mathbf{y}} = \alpha \mathbf{x} + f(\mathbf{x}, \mathbf{y})\mathbf{y}, \quad \mathbf{x}^{\overline{\mathbf{y}}} = \alpha' \mathbf{x} + f'(\mathbf{x}, \mathbf{y})\mathbf{y}, \quad \mathbf{x}_{\mathbf{y}} = \beta \mathbf{x}, \quad \mathbf{x}_{\overline{\mathbf{y}}} = \beta' \mathbf{x},$$

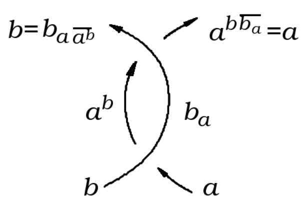
where $\alpha, \alpha', \beta, \beta' \in R$ and $f', f : M \times M \to R$ are bilinear forms.

We know already that $\alpha = \alpha' = \beta = \beta' = 1$, $f'(x,y) = -f(x,y)$, f an antisymmetric bilinear form gives us a biquandle structure, namely the symplectic quandle structure just described. We would like to know, then: what other bilinear biquandles are possible?

We start with some observations.

Proposition 1. *Let M be a bilinear biquandle. Then $\alpha' = \alpha^{-1}$ and $\beta' = \beta^{-1}$.*

Proof. Here we consider the biquandle axioms arising from the direct type II move:

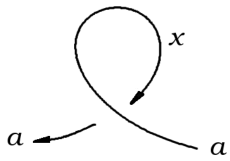


Since $b_a = \beta b$, we have $b_{a\overline{a^b}} = \beta'(\beta b) = b$, and thus $\beta\beta' = 1$.

Moreover, since $a^{b\overline{b_a}} = a$ for all $a, b \in M$, taking $b = 0 \in M$ we have $a^0 = \alpha a$, $0_a = 0$ and $a^{\overline{0}} = \alpha' a$, and thus $a^{0\overline{0_a}} = \alpha\alpha' a = a$ and $\alpha\alpha' = 1$. ∎

Proposition 2. *For any bilinear biquandle, we must have $f(a,a) = \beta^{-1} - \alpha$ and $f'(a,a) = \beta - \alpha^{-1}$.*

Proof. Consider the biquandle axiom derived from the Reidemeister type I move with a positive crossing:



$$a_x = x, \qquad x^a = a$$

Here $a_x = \beta a = x$, and we note that this x is unique since β is invertible; then

$$x^a = (\beta a)^a = \alpha(\beta a) + f(\beta a, a)a = (\alpha\beta + \beta f(a,a))a = a$$

and hence $1 = \alpha\beta + \beta f(a,a)$. Therefore

$$\beta f(a,a) = 1 - \alpha\beta, \quad \text{so} \quad f(a,a) = \beta^{-1} - \alpha.$$

The other case is similar. ∎

Corollary 3. *Let $A \in M_m(R)$ be the matrix of f with respect to an ordered basis $\{\mathbf{b_1}, \ldots, \mathbf{b_m}\}$ of B, so that*

$$f(\mathbf{x},\mathbf{y}) = \mathbf{x}A\mathbf{y}^t \quad \textit{where} \quad \mathbf{x} = \sum_{i=1}^{m} x_i \mathbf{b}_i.$$

Then the diagonal entries of A must satisfy $A_{ii} = \beta^{-1} - \alpha$.

Proposition 4. *In any bilinear biquandle, we must have*

$$f'(\mathbf{x},\mathbf{y}) = \omega f(\mathbf{x},\mathbf{y}) \quad \textit{where} \quad \omega = -\alpha^{-2}\beta^{-2} - \alpha^{-1}\beta + \alpha^{-2}.$$

Proof. Using the direct type II move, we have $\mathbf{x} = \mathbf{x}^{\mathbf{y}\overline{\mathbf{y}_\mathbf{x}}}$. Therefore,

$$\begin{aligned}
\mathbf{x} &= (\alpha\mathbf{x} + f(\mathbf{x},\mathbf{y})\mathbf{y})^{\overline{\beta\mathbf{y}}} \\
&= \alpha'(\alpha\mathbf{x} + f(\mathbf{x},\mathbf{y})\mathbf{y}) + f'(\alpha\mathbf{x} + f(\mathbf{x},\mathbf{y})\mathbf{y}, \beta\mathbf{y})\beta\mathbf{y} \\
&= \alpha'\alpha\mathbf{x} + \alpha' f(\mathbf{x},\mathbf{y})\mathbf{y} + f'(\alpha\mathbf{x}, \beta\mathbf{y})\beta\mathbf{y} + f'(f(\mathbf{x},\mathbf{y})\mathbf{y}, \beta\mathbf{y})\beta\mathbf{y}.
\end{aligned}$$

We know that $\alpha' = \alpha^{-1}$, so with further simplification, we get

$$\mathbf{x} = \mathbf{x} + \alpha^{-1} f(\mathbf{x},\mathbf{y})\mathbf{y} + \alpha\beta^2 f'(\mathbf{x},\mathbf{y})\mathbf{y} + \beta^2 f(\mathbf{x},\mathbf{y}) f'(\mathbf{y},\mathbf{y})\mathbf{y}.$$

We have $f'(\mathbf{y},\mathbf{y}) = \beta - \alpha^{-1}$, so

$$\mathbf{x} = \mathbf{x} + \alpha^{-1} f(\mathbf{x},\mathbf{y})\mathbf{y} + \alpha\beta^2 f'(\mathbf{x},\mathbf{y})\mathbf{y} + \beta^2 f(\mathbf{x},\mathbf{y})(\beta - \alpha^{-1})\mathbf{y}.$$

Then

$$0 = \alpha^{-1} f(\mathbf{x},\mathbf{y})\mathbf{y} + \alpha\beta^2 f'(\mathbf{x},\mathbf{y})\mathbf{y} + \beta^2 f(\mathbf{x},\mathbf{y})(\beta - \alpha^{-1})\mathbf{y},$$

and since this is true for all $\mathbf{y} \in B$, we must have

$$0 = \alpha^{-1} f(\mathbf{x},\mathbf{y}) + \alpha\beta^2 f'(\mathbf{x},\mathbf{y}) + \beta^2 f(\mathbf{x},\mathbf{y})(\beta - \alpha^{-1}).$$

Hence,

$$-\alpha\beta^2 f'(\mathbf{x},\mathbf{y}) = (\alpha^{-1} + \beta^2(\beta - \alpha^{-1}))f(\mathbf{x},\mathbf{y}).$$

So,

$$\begin{aligned} f'(\mathbf{x},\mathbf{y}) &= (-\alpha^{-1}\beta^{-2})(\alpha^{-1} + \beta^3 - \beta^2\alpha^{-1})f(\mathbf{x},\mathbf{y}) \\ &= (-\alpha^{-2}\beta^{-2} - \alpha^{-1}\beta + \alpha^{-2})f(\mathbf{x},\mathbf{y}). \ \blacksquare \end{aligned}$$

The type III Reidemeister move axioms impose conditions on the bilinear form $f(\mathbf{x},\mathbf{y}) = \mathbf{x}A\mathbf{y}^t$:

Proposition 5. *In any bilinear biquandle B, we must have*

$$\alpha(1-\beta^2)f(\mathbf{x},\mathbf{y}) = 0, \qquad \beta(1-\beta^2)f(\mathbf{x},\mathbf{y}) = 0.$$

In particular, the entries A_{ij} of the matrix A such that $f(\mathbf{x},\mathbf{y}) = \mathbf{x}A\mathbf{y}^t$ with respect to a basis of B must satisfy

$$\alpha(1-\beta^2)A_{ij} = \beta(1-\beta^2)A_{ij} = 0$$

for all i, j.

Proof. The middle strand in the general case gives us the equation $(b_a)^{c_{a^b}} = b^c{}_{a^{c_b}}$. Then

$$(b_a)^{c_{a^b}} = (\beta b)^{(\beta c)} = \alpha(\beta b) + f(\beta b, \beta c)\beta c = \alpha\beta b + \beta^3 f(b,c)c$$

while

$$(b^c)_{a^{c_b}} = \beta(\alpha b + f(b,c)c) = \alpha\beta b + \beta f(b,c)c,$$

so we must have $\beta f(b,c) = \beta^3 f(b,c)$ for all $b, c \in B$.

Now, we note that $\mathbf{x}^0 = \alpha\mathbf{x} + f(\mathbf{x},0)0 = \alpha\mathbf{x}$. The undercrossing strand gives us the equation $a^{bc} = a^{b_c c^b}$, so the special case $c = 0$ says

$$a^{b0} = \alpha(a^b) = \alpha(\alpha a + f(a,b)b) = \alpha^2 a + \alpha f(a,b)b$$

while

$$a^{b_0 0^b} = a^{(\beta b)0} = \alpha(\alpha a + f(a,\beta b)\beta b) = \alpha^2 a + \alpha\beta^2 f(a,b)b,$$

and hence $\alpha f(a,b) = \alpha\beta^2 f(a,b)$ for all $a, b \in B$. $\blacksquare$

These observations constrain the possible bilinear biquandle structures on $(\mathbb{Z}_n)^m$ enough to make it practical to find all such biquandle structures for small values of n and m by computer search. Specifically, given invertible $\alpha, \beta \in \mathbb{Z}_n$, we compute the corresponding list of all $x \in \mathbb{Z}_n$ satisfying $\alpha(1-\beta^2)x = \beta(1-\beta^2)x = 0$; these are candidates for entries in an $m \times m$ matrix A, with diagonal entries $\beta^{-1} - \alpha$. We then compute the biquandle operation matrix for each triple (α, β, A) over the set $(\mathbb{Z}_n)^m$ and test the resulting operation matrix for the biquandle axioms, rejecting any triples which fail to satisfy all of the axioms. `Maple` code implementing this procedure for $(\mathbb{Z}_n)^2$ and $(\mathbb{Z}_n)^3$ is available in the file `bilinear-biquandles.txt` downloadable from `www.esotericka.org/quandles`. The results for $(\mathbb{Z}_2)^2$, $(\mathbb{Z}_3)^2$, $(\mathbb{Z}_4)^2$, $(\mathbb{Z}_5)^2$, $(\mathbb{Z}_2)^3$, $(\mathbb{Z}_3)^3$ and $(\mathbb{Z}_2)^4$ are collected in Section 5.

EXAMPLE 6. Let $T = (\mathbb{Z}_4)^2$ and let $\alpha = \beta = 3$, $\omega = -\alpha^{-2}\beta^{-2} - \alpha^{-1}\beta + \alpha^{-2} = 1$ and $A = \left[\begin{smallmatrix} 0 & 2 \\ 2 & 0 \end{smallmatrix}\right]$. Then one checks that the operations

$$\mathbf{x}^{\mathbf{y}} = \mathbf{x}^{\overline{\mathbf{y}}} = 3\mathbf{x} + \mathbf{x}\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\mathbf{y}^t\mathbf{y}, \qquad \mathbf{x}_{\mathbf{y}} = 3\mathbf{x} = \mathbf{x}_{\overline{\mathbf{y}}}$$

define a bilinear biquandle structure on T.

For example, let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Then after a bit of arithmetic we find that

$$\begin{aligned}\mathbf{x}^{\mathbf{y}\overline{\mathbf{y}}_{\mathbf{x}}} = (&x_1 + 20x_2y_1^2 + 20y_1x_1y_2 + 72y_2x_2y_1^3 + 72y_1^2x_1y_2^2, \\ &x_2 + 20y_2x_2y_1 + 20x_1y_2^2 + 72y_2^2x_2y_1^1 + 72y_1x_1y_2^3),\end{aligned}$$

which is just (x_1, x_2) in $(\mathbb{Z}_4)^2$.

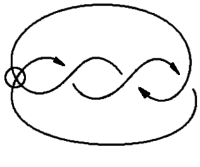
4. Link invariants from bilinear biquandles. Since a bilinear biquandle is not just a biquandle but also an R-module, we can take advantage of this extra structure to enhance the biquandle counting invariant as in [8].

DEFINITION 2. Let L be a link, $B(L)$ the knot biquandle of L, and T a finite bilinear biquandle. Define the *bilinear biquandle polynomial* of L with respect to T to be

$$\phi_{\mathrm{BB}}(L, T, q, z) = \sum_{f \in \mathrm{Hom}(B(L),T)} q^{|\mathrm{Im}(f)|} z^{|\mathrm{Span}(\mathrm{Im}(f))|}.$$

Since ϕ_{BB} is determined by the set $\mathrm{Hom}(B(L), M)$ which is an invariant of link type, so is ϕ_{BB}. In particular, ϕ_{BB} specializes to the biquandle counting invariant $|\mathrm{Hom}(B(L), M)|$ when $q = z = 1$, though in general ϕ_{BB} contains more information than the counting invariant alone.

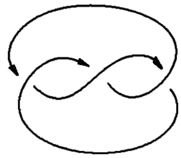
EXAMPLE 7. Let T be the bilinear biquandle defined in Example 6. Then the virtual link L pictured below is distinguished from the trefoil knot 3_1 by the associated bilinear biquandle invariant ϕ_{BB}, though both links have the same counting invariant value.



L

$$\phi_{\mathrm{BB}}(L, T, q, z) = qz + 6q^2z^2 + 3qz^2 + 6q^2z^4$$

$$|\mathrm{Hom}(B(L), T)| = 16$$



$$\phi_{\mathrm{BB}}(3_1, T, z, q) = qz + 3qz^2 + 12q^2z^4$$

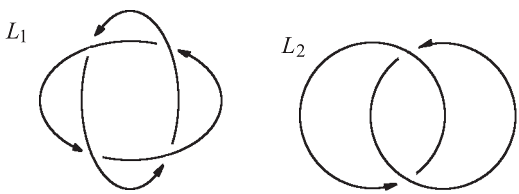
$$|\mathrm{Hom}(B(L), T)| = 16$$

Note that if we specialize $z = 1$, ignoring the module structure and using only the biquandle structure, the resulting invariant fails to distinguish the virtual links as both reduce to $q + 3q + 12q^2$.

Example 8. Each monomial in a value of ϕ_{BB} corresponds to a sub-biquandle of the target biquandle, namely the image of the knot biquandle under some homomorphism. If two knots or links have different values of ϕ_{BB}, the difference in these values can yield information about the difference between the knots not apparent from the counting invariant alone. Let T be the bilinear biquandle $T = (\mathbb{Z}_4)^2$ with

$$\mathbf{x}^{\mathbf{y}} = \mathbf{x}^{\overline{\mathbf{y}}} = \mathbf{x} + \mathbf{x}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{y}^t\mathbf{y}, \qquad \mathbf{x}_{\mathbf{y}} = \mathbf{x}_{\overline{\mathbf{y}}} = 3\mathbf{x}.$$

Then the links L_1 and L_2 shown below have invariant values $\phi_{\mathrm{BB}}(L_1) = qz + 48q^3z^4 + 6q^2z^2 + 30q^2z^4 + 72q^4z^8 + 3qz^2$ and $\phi_{\mathrm{BB}}(L_2) = qz + 6q^2z^2 + 30q^2z^4 + 3qz^2$ respectively.



Thus, there are sub-biquandles of $B_1, B_2 \subset T$ with cardinality 3 and 4 respectively such that L_1 has biquandle colorings by B_1 and B_2 while L_2 does not. Moreover, the submodule spanned by B_1 has cardinality 4, while the submodule spanned by B_2 has cardinality 8.

5. Bilinear biquandles of small cardinality. In Table 1 we list the results of our computer search for bilinear biquandles of small cardinality. These results were obtained using `Maple` programs in the file `bilinear-biquandles.txt` available at `www.esotericka.org/quandles`. In light of the results of Section 3, we identify each bilinear biquandle by listing α, β and the matrix A of $f(\mathbf{x}, \mathbf{y})$ with respect to the standard basis of $(\mathbb{Z}_n)^m$. We list only those bilinear biquandles which are not symplectic quandles and which have cardinality less than or equal to 27.

Table 1. Non-quandle bilinear biquandles of cardinality ≤ 27

$(\mathbb{Z}_n)^m$	α	β	A	$(\mathbb{Z}_n)^m$	α	β	A	$(\mathbb{Z}_n)^m$	α	β	A
$(\mathbb{Z}_3)^2$	2	2	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$(\mathbb{Z}_3)^2$	2	2	$\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$	$(\mathbb{Z}_4)^2$	1	3	$\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$
$(\mathbb{Z}_4)^2$	1	3	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	$(\mathbb{Z}_4)^2$	3	1	$\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$	$(\mathbb{Z}_4)^2$	3	1	$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
$(\mathbb{Z}_4)^2$	3	3	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$(\mathbb{Z}_4)^2$	3	3	$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$	$(\mathbb{Z}_4)^2$	3	3	$\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$
$(\mathbb{Z}_5)^2$	4	4	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$(\mathbb{Z}_5)^2$	4	4	$\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$	$(\mathbb{Z}_3)^3$	2	2	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

6. Questions. In this section we collect a few questions for future research.

Our initial computer search did not give any examples of finite biquandles in which all four operations have the form

$$(\mathbf{x}, \mathbf{y}) \mapsto \alpha_i \mathbf{x} + f_i(\mathbf{x}, \mathbf{y})\mathbf{y}$$

where $f_i : B \times B \to R$ is a non-zero bilinear form for all $i = 1, 2, 3, 4$. Are there any examples of such biquandles?

We notice from Table 1 that for $n = 2, 3, 5$ the only bilinear biquandle structures on $(\mathbb{Z}_n)^m$ we seem to find are only slight variations of the symplectic quandle structure—the bilinear form is antisymmetric and $\alpha, \beta \in \{-1, 1\}$. Does this pattern hold for all prime n?

There are, of course other possible combinations of module elements and bilinear forms similar to the symplectic quandle structure that one could use in looking for finite biquandles, such as

$$(\mathbf{x}, \mathbf{y}) \mapsto f_i(\mathbf{x}, \mathbf{y})\mathbf{x} + g_i(\mathbf{x}, \mathbf{y})\mathbf{y}, \qquad i = 1, 2, 3, 4,$$

or

$$(\mathbf{x}, \mathbf{y}) \mapsto A_i \mathbf{x} + g_i(\mathbf{x}, \mathbf{y})\mathbf{y}, \qquad i = 1, 2, 3, 4,$$

where f_i, g_i are bilinear forms and $A_i \in M_m(R)$. All of these should have ϕ_{BB} type invariants associated. Which of these formats give interesting new finite biquandles?

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