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RIGIDITY OF PROJECTIVE CONJUGACY FOR QUASIPERIODIC FLOWS OF KOCH TYPE

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Abstract. For quasiperiodic flows of Koch type, we exploit an algebraic rigidity of an equivalence relation on flows, called projective conjugacy, to algebraically characterize the deviations from completeness of an absolute invariant of projective conjugacy, called the multiplier group, which describes the generalized symmetries of the flow. We then describe three ways by which two quasiperiodic flows with the same Koch field are projectively conjugate when their multiplier groups are identical. The first way involves a quantity introduced here, called the G-paragon class number of the multiplier group. The second involves the generalized Bowen–Franks groups and the class number of an order. The third involves conjugacy of the actions of the multiplier group by commuting toral automorphisms, for which one of these actions is irreducible, and a condition introduced here, called PCF, on the common real eigenvectors of the irreducible action. Additionally, we describe two ways by which similiar actions of the multiplier group can fail to be conjugate.

1. Introduction. Invariants and rigidity continue to play an essential role in classifying dynamical systems according to a prescribed equivalence relation (see [12], [14], [18], [19], and [26]). An invariant introduced recently in [6] is the multiplier group which describes the generalized symmetries that a smooth (i.e., C^∞) flow possesses. The prescribed equivalence relation under consideration is that of projective conjugacy, described in Section 2, which is stronger than smooth conjugacy. The multiplier group is an absolute invariant of projective conjugacy in that two smooth flows have the same (i.e., identical) multiplier group if they are projectively conjugate. Deviations of the multiplier group from completeness as an invariant are measured by the number of projective conjugacy classes of smooth flows having the same multiplier group. Measuring these deviations directly for arbitrary smooth flows is unwieldy because computing the multiplier group is difficult.

An algebraic rigidity of projective conjugacy, generalized symmetries, and the multiplier groups for quasiperiodic flows whose frequencies are of

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Koch type (a type introduced in [16] and named such in [20]) links these flows with algebraic number theory. This rigidity provides indirect means for identifying their multiplier groups as well as counting their concomitant projective conjugacy classes in terms of algebraic objects in a real algebraic number field. The Koch type belongs to the classical type of quasiperiodic phenomena whose frequencies satisfy a Diophantine inequality. This classical type appears in KAM theory (see [30]), in estimating the correlation dimension of quasiperiodic attractors (see [23] and [29]), and in characterizing hypoelliptic vector fields on the n-torus (see [8]). The Koch type appears in applications of renormalization group theory to KAM tori (see [16] and [20]). A quasiperiodic flow on the n-torus is of Koch type if and only if it is projectively conjugate to a minimal linear flow whose generating vector field has components that form a $\mathbb{Q}$-basis for a real algebraic number field K of degree n (see Section 2). A quasiperiodic flow on the n-torus of Koch type is F-algebraic (a type introduced in [4]) if the real algebraic number field K of degree n associated to it is F.

The equivalence classes of complete $\mathbb{Z}$-modules, or lattices, in F provide an algebraic parallel to dynamical classification problems for F-algebraic quasiperiodic flows and other dynamical systems that have links with algebraic number theory. Classes of ideals in a given order of F are equivalence classes of complete $\mathbb{Z}$-modules in F that have been applied in [1] and [21] to the topological conjugacy classification problem of torus automorphisms, or algebraic $\mathbb{Z}$-actions, where the link of these discrete dynamical systems with algebraic number theory is the Latimer–MacDuffee–Taussky theorem (see [17], [32], and [33]). It is shown in Section 2 that there is a bijection between the equivalence classes of complete $\mathbb{Z}$-modules in F and the projective conjugacy classes of F-algebraic quasiperiodic flows. Using this link, the multiplier group of an F-algebraic quasiperiodic flow is shown in Section 3 to be the group of units of an order in F. From this follows an algebraic characterization of the deviations from completeness of the multiplier group as an invariant of projective conjugacy for F-algebraic quasiperiodic flows. The deviations are the existence of inequivalent complete $\mathbb{Z}$-modules in F that have the same order (the order has class number bigger than 1), and the existence of distinct orders that have same group of units. Both obstructions to completeness are accounted for by the G-paragon class number in F being bigger than 1 (see Theorem 3.6), where the G-paragon class number in F is the number of equivalence classes of complete $\mathbb{Z}$-modules in F whose orders have the same group of units G. The multiplier group fails to be a complete invariant of projective conjugacy for quasiperiodic flows of Koch type because, as is well known, there are real algebraic number fields that have orders whose class numbers are bigger than one, and that have distinct orders with the same group of units.

Despite the failure of the multiplier group to be a complete invariant, there are ways by which two F-algebraic quasiperiodic flows ϕ, ψ with the same multiplier group can be shown to be projectively conjugate. The first way requires that the G-paragon class number of the common multiplier group be 1 (see Lemma 3.5). The second way makes use of the generalized Bowen–Franks groups introduced in [21] as invariants of topological conjugacy for toral automorphisms. The relations of these groups with the multiplier group, projective conjugacy, class numbers of orders, and G-paragon class numbers in F are given in Section 4. The second way requires that the generalized Bowen–Franks groups for the complete $\mathbb{Z}$-modules $\mathfrak{m}_\phi, \mathfrak{m}_\phi$ in F, corresponding to ϕ, ψ, are isomorphic and the class number of the order of $\mathfrak{m}_\phi$ in F is 1 (see Corollary 4.3).

The third way makes use of actions J_ϕ, J_ψ of the common multiplier group determined by commuting toral automorphisms. Such actions and their subactions have been studied elsewhere (see [11], [12], [13], [15], and [27]). Pertaining to the projective conjugacy problem for F-algebraic quasiperiodic flows, properties of these actions are given in Sections 5 and 6. The third way requires that J_ϕ and J_ψ are conjugate and that J_ϕ satisfies a condition called PCF, i.e., J_ϕ is an irreducible action and the components of the common real eigenvectors of J_ϕ are frequencies of projectively conjugate F-algebraic quasiperiodic flows (see Theorem 6.6). The actions J_ϕ and J_ψ are always similar when they are determined by the same multiplier group (see Theorem 5.1), but they may fail to be conjugate. In Section 7, two sets of sufficient conditions are given for the existence of similar nonconjugate actions. One set is based on the G-paragon class numbers in F and condition PCF (see Theorem 7.1). The second set is based on the class number of an order and the generalized Bowen–Franks groups and extends a result (Theorem 4.5 on p. 733 in [12]) about the existence of nonconjugate centralizers (see Theorem 7.4).

2. Preliminaries. The notion of projective conjugacy for smooth flows combines the equivalence relations of smooth conjugacy and scale equivalency. On a smooth manifold P without boundary, a smooth flow $\Phi : \mathbb{R}\times P \to P$ is generated by the vector field

$$X_\Phi = \frac{d}{dt}\Phi(t,p)\bigg|_{t=0}.$$

Recall that two flows Φ and Ψ on P are *smoothly conjugate* if there is an $R \in \mathrm{Diff}(P)$, the group of smooth diffeomorphisms of P, such that $R\Phi(t,p) = \Psi(t,R(p))$ for all $(t,p) \in \mathbb{R}\times P$; that is, Φ and Ψ are smoothly conjugate if $R_*X_\Phi = X_\Psi$, where $R_*X = \mathbf{T}RXR^{-1}$ is the push forward of X by R and $\mathbf{T}R$ is the tangent map of R. Two flows Φ and Ψ on P are *scale equivalent* if there is $\vartheta \in \mathbb{R}^\times \equiv \mathbb{R} \setminus \{0\}$ such that $\varPhi(t,p) = \varPsi(\vartheta t, p)$ for all $(t,p) \in \mathbb{R} \times P$, that is, $X_\varPhi = \vartheta X_\varPsi$.

Definition 2.1. Two flows $\varPhi$ and $\varPsi$ on P are *projectively conjugate* if there are $R \in \mathrm{Diff}(P)$ and $\vartheta \in \mathbb{R}^\times$ such that $R\varPhi(t,p) = \varPsi(\vartheta t, R(p))$ for all $(t,p) \in \mathbb{R} \times P$, that is, $R_* X_\varPhi = \vartheta X_\varPsi$.

Projective conjugacy is an equivalence relation on the set of smooth flows.

The multiplier group of a flow is a subgroup of $\mathbb{R}^\times$ that represents the generalized symmetries that the flow possesses. The *generalized symmetry group* of a flow $\varPhi$ on P is $S_\varPhi = \{R \in \mathrm{Diff}(P) :$ there exists α in $\mathbb{R}^\times$ such that $R_* X_\varPhi = \alpha X_\varPhi\}$. The group $S_\varPhi$ is the group-theoretic normalizer within $\mathrm{Diff}(P)$ of $F_\varPhi = \{\varPhi_t : t \in \mathbb{R}\}$, the one-parameter subgroup of diffeomorphisms generated by $\varPhi$ where $\varPhi_t(p) = \varPhi(t,p)$ (Theorem 2.5 on p. 187 in [6]). A representation of $S_\varPhi$ is the homomorphism $\varrho_\varPhi : S_\varPhi \to \mathbb{R}^\times$ that takes each generalized symmetry $R \in S_\varPhi$ to its unique multiplier $\alpha = \varrho_\varPhi(R)$ appearing in $R_* X_\varPhi = \alpha X_\varPhi$. The *multiplier group* of $\varPhi$ is

$$M_\varPhi = \varrho_\varPhi(S_\varPhi).$$

Theorem 2.2. *If $\varPhi$ and $\varPsi$ are projectively conjugate, then $M_\varPhi = M_\varPsi$.*

Proof. Suppose there is $R \in \mathrm{Diff}(P)$ and $\vartheta \in \mathbb{R}^\times$ such that $R_* X_\varPhi = \vartheta X_\varPsi$. This means that $X_\varPhi$ is smoothly conjugate to $R_* X_\varPhi$, and that $R_* X_\varPhi$ is scale equivalent to $X_\varPsi$. Since the multiplier group is an absolute invariant of smooth conjugacy (Theorem 4.2 in [6]), and since the multiplier group is an absolute invariant of scale equivalency (by a straightforward argument), it follows that the multiplier group is an absolute invariant of projective conjugacy. ■

The projective conjugacy equation $R_* X_\varPhi = \vartheta X_\varPsi$ exhibits an algebraic rigidity in the smooth conjugacy R when $\varPhi$ and $\varPsi$ are quasiperiodic flows, and yields an absolute invariant for those quasiperiodic flows whose frequency vectors are of a certain type. Recall that on the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ equipped with global coordinates $(\theta_1, \ldots, \theta_n)$, a flow $\varPhi$ is *quasiperiodic* if it is smoothly conjugate to a *minimal linear flow* ξ, i.e., a flow ξ on T^n for which

$$X_\xi = \sum_{i=1}^n a_i \frac{\partial}{\partial \theta_i}$$

is a constant vector field whose components, or frequencies, $a_1, \ldots, a_n$ are linearly independent over $\mathbb{Q}$. A minimal linear flow ξ is said to be associated to a quasiperiodic flow $\varPhi$ if ξ is smoothly conjugate to $\varPhi$. For S a commutative ring with identity, let $\mathrm{GL}(n,S)$ denote the group of $n \times n$ matrices with entries in S whose determinants are units in S.

THEOREM 2.3. *Suppose ξ and η are minimal linear flows associated to quasiperiodic flows Φ and Ψ on T^n. Then Φ and Ψ are projectively conjugate if and only if there exist $C \in \mathrm{GL}(n,\mathbb{Z})$ and $\vartheta \in \mathbb{R}^\times$ such that $CX_\xi = \vartheta X_\eta$.*

Proof. By the hypothesis, there are $R, Q \in \mathrm{Diff}(T^n)$ such that $R_* X_\Phi = X_\xi$ and $Q_* X_\Psi = X_\eta$ with ξ quasiperiodic, and with X_ξ and X_η both constant.

Suppose that Φ and Ψ are projectively conjugate. Then there are $V \in \mathrm{Diff}(T^n)$ and $\vartheta \in \mathbb{R}^\times$ such that $V_* X_\Phi = \vartheta X_\Psi$. Hence

$$(QVR^{-1})_* X_\xi = (QV)_* X_\Phi = Q_*(\vartheta X_\Psi) = \vartheta X_\eta.$$

Since X_ξ and ϑX_η are constant vector fields and ξ is quasiperiodic, it follows (by Theorem 2.3 in [5]) that $\mathbf{T}(QVR^{-1})$ is a constant matrix C belonging to $\mathrm{GL}(n,\mathbb{Z})$. Thus $CX_\xi = \vartheta X_\eta$.

Now suppose that there are $C \in \mathrm{GL}(n,\mathbb{Z})$ and $\vartheta \in \mathbb{R}^\times$ such that $CX_\xi = \vartheta X_\eta$. Let $V \in \mathrm{Diff}(T^n)$ be induced by C, i.e., $\mathbf{T}V = C$. Then $V_* X_\xi = \vartheta X_\eta$ because X_ξ is constant. Hence

$$(Q^{-1}VR)_* X_\Phi = (Q^{-1}V)_* X_\xi = Q_*^{-1}(\vartheta X_\eta) = \vartheta X_\Psi.$$

Therefore, Φ and Ψ are projectively conjugate. ∎

The set of vectors in $\mathbb{R}^n$ of Koch type plays a significant role in $BX_\xi = \vartheta X_\eta$. The definition of this set of vectors as given in [20] (see Definition 6.1 therein) is recalled here. Let $\omega_2, \ldots, \omega_n$ be real algebraic numbers (i.e., real roots of polynomials over $\mathbb{Q}$) that are independent over $\mathbb{Q}$, and set $\omega = (1, \omega_2, \ldots, \omega_n)$ in $\mathbb{R}^n$. For any $c \in \mathbb{R}^\times$, the vector $c\omega$ is *of Koch type* if

$$\mathbb{Q}(\omega) \equiv \mathbb{Q}(1, \omega_2, \ldots, \omega_n),$$

the algebraic extension of $\mathbb{Q}$ by $\omega_2, \ldots, \omega_n$, is of degree n, i.e., the dimension of $\mathbb{Q}(\omega)$ as a vector space over $\mathbb{Q}$ is n. Let KT_n denote the set of all vectors in $\mathbb{R}^n$ of Koch type. Each vector in KT_n satisfies a Diophantine inequality (Corollary 4.2 in [16]).

DEFINITION 2.4. A quasiperiodic flow Φ on T^n is *of Koch type* if it is smoothly conjugate to a minimal linear flow ξ for which $X_\xi \in KT_n$.

The map $c\omega \mapsto \mathbb{Q}(\omega)$ attaches to each vector in KT_n a real algebraic number field of degree n. A straightforward algebraic argument gives the proof of the following.

LEMMA 2.5. *Suppose $c\omega \in KT_n$, $\kappa = (1, \kappa_2, \ldots, \kappa_n) \in \mathbb{R}^n$, and $d \in \mathbb{R}^\times$. If there are $C \in \mathrm{GL}(n,\mathbb{Z})$ and $\vartheta \in \mathbb{R}^\times$ such that $C(c\omega) = \vartheta(d\kappa)$, then $d\kappa \in KT_n$ and $\mathbb{Q}(\omega) = \mathbb{Q}(\kappa)$.*

A minimal linear flow ξ with $X_\xi = c\omega \in KT_n$ that is associated to a quasiperiodic flow Φ of Koch type attaches to Φ the real algebraic number field $\mathbb{Q}(\omega)$ of degree n. If η is another minimal linear flow associated to Φ, then by Theorem 2.3 (with $\Phi = \Psi$) there are $C \in \mathrm{GL}(n, \mathbb{Z})$ and $\vartheta \in \mathbb{R}^\times$ such that $C(c\omega) = \vartheta X_\eta$. It follows by Lemma 2.5 that $X_\eta = d\kappa \in KT_n$ (where $d = \kappa_1$ and $\kappa = (1/d)X_\eta$) and that $\mathbb{Q}(\omega) = \mathbb{Q}(\kappa)$. Thus there is a uniquely defined real algebraic number field

$$K_\Phi = \mathbb{Q}(\omega)$$

attached to each quasiperiodic flow Φ of Koch type.

DEFINITION 2.6. The *Koch field* of a quasiperiodic flow Φ of Koch type is the unique real algebraic number field K_Φ attached to Φ.

THEOREM 2.7. *Suppose Φ and Ψ are quasiperiodic flows of Koch type. If Φ and Ψ are projectively conjugate, then $K_\Phi = K_\Psi$.*

Proof. Let ξ and η be minimal linear flows associated respectively to Φ and Ψ with $X_\xi = c\omega \in KT_n$ and $X_\eta = d\kappa \in KT_n$. If Φ and Ψ are projectively conjugate, then by Theorem 2.3 there are $C \in \mathrm{GL}(n, \mathbb{Z})$ and $\vartheta \in \mathbb{R}^\times$ such that $C(c\omega) = \vartheta(d\kappa)$. Thus $K_\Phi = \mathbb{Q}(\omega) = \mathbb{Q}(\kappa) = K_\Psi$ by Lemma 2.5. ∎

As an absolute invariant of projective conjugacy, the Koch field partitions the collection of quasiperiodic flows of Koch type.

DEFINITION 2.8. For a real algebraic number field F of degree n, a quasiperiodic flow Φ on T^n is *F-algebraic* if Φ is projectively conjugate to a minimal linear flow ϕ for which the components of X_ϕ form a $\mathbb{Q}$-basis for F.

Let $\mathfrak{X}_F$ denote the set of all F-algebraic quasiperiodic flows on T^n. It is easy to show that $\Phi \in \mathfrak{X}_F$ if and only if Φ is of Koch type and $K_\Phi = F$.

The equivalence classes of complete $\mathbb{Z}$-modules in F link bijectively with projective conjugacy classes in $\mathfrak{X}_F$. Following [24], a *complete $\mathbb{Z}$-module* or *lattice* (module for short) $\mathfrak{m}$ of F is a finitely generated subgroup of the additive group F^+ (hence a free $\mathbb{Z}$-module) whose rank is the degree of F. Each module $\mathfrak{m}$ in F has a $\mathbb{Z}$-basis: there are $a_1, \ldots, a_n \in \mathfrak{m}$ such that

$$\mathfrak{m} = \mathbb{Z}[a_1, \ldots, a_n] \equiv a_1\mathbb{Z} + \cdots + a_n\mathbb{Z}.$$

Let $\mathfrak{M}_F$ denote the set of all modules in F. If $\mathfrak{m} \in \mathfrak{M}_F$ and $\vartheta \in F^\times$ (the nonzero elements in F), then $\vartheta\mathfrak{m} \equiv \{\vartheta z : z \in \mathfrak{m}\}$ is in $\mathfrak{M}_F$. Two modules $\mathfrak{m}$ and $\mathfrak{n}$ in $\mathfrak{M}_F$ are called *equivalent* if there is $\vartheta \in F^\times$ such that $\vartheta\mathfrak{m} = \mathfrak{n}$. This equivalence of complete modules is an equivalence relation on $\mathfrak{M}_F$. Let $[\mathfrak{m}]$ denote the module class of $\mathfrak{m}$ in $\mathfrak{M}_F$. For $\Phi \in \mathfrak{X}_F$, let $[\Phi]$ denote the projective conjugacy class of Φ, and let $\mathfrak{X}_F^0$ be the subset of $\mathfrak{X}_F$ consisting of minimal linear flows ϕ for which the components of X_ϕ form a $\mathbb{Q}$-basis for F. For each $\Phi \in \mathfrak{X}_F$ there is $\phi \in \mathfrak{X}_F^0$ such that $[\Phi] = [\phi]$. Since the components of $X_\phi = \sum_{i=1}^n p_i\partial/\partial\theta_i$ form a $\mathbb{Q}$-basis for F, they are independent over $\mathbb{Z}$, and so the module $\mathfrak{m}_\phi = \mathbb{Z}[p_1, \ldots, p_n]$ is in $\mathfrak{M}_F$. The proof of the following is routine.

LEMMA 2.9. *For $\Phi \in \mathfrak{X}_F$, the module class $[\mathfrak{m}_\phi]$ in $\mathfrak{M}_F$ is independent of the choice of $\phi \in \mathfrak{X}_F^0$ for which $[\Phi] = [\phi]$.*

By Theorem 2.3, flows $\phi, \psi \in \mathfrak{X}_F^0$ are projectively conjugate if and only if there are $C \in \mathrm{GL}(n,\mathbb{Z})$ and $\vartheta \in \mathbb{R}^\times$ such that $CX_\phi = \vartheta X_\psi$. Of necessity, the ϑ appearing in $CX_\phi = \vartheta X_\psi$ belongs to $F^\times$. Let $[\phi]_0$ denote the projective conjugacy class of ϕ within $\mathfrak{X}_F^0$. Let $[\mathfrak{X}_F^0]$ denote the set of projective conjugacy classes in $\mathfrak{X}_F^0$, and $[\mathfrak{M}_F]$ denote the set of module classes in $\mathfrak{M}_F$. The map

$$\Omega : [\mathfrak{X}_F^0] \to [\mathfrak{M}_F], \qquad \Omega[\phi]_0 = [\mathfrak{m}_\phi],$$

is well-defined by Lemma 2.9. The proof of the following is also routine.

THEOREM 2.10. *The map $\Omega : [\mathfrak{X}_F^0] \to [\mathfrak{M}_F]$ is a bijection.*

When $\Phi \in \mathfrak{X}_F$ is projectively conjugate to $\phi \in \mathfrak{X}_F^0$, the inclusion $[\Phi] \supset [\phi]_0$ holds, which induces a bijection $[\Phi] \leftrightarrow [\phi]_0$ between $[\mathfrak{X}_F]$ and $[\mathfrak{X}_F^0]$. The map Ω therefore extends to a bijection $[\mathfrak{X}_F] \leftrightarrow [\mathfrak{M}_F]$.

3. Algebraic characterization of deviations. The deviations of the multiplier group from completeness are characterized in terms of the group of units of the orders of modules. Let $\mathfrak{o}_F$ denote the *ring of integers* in F. Recall that an *order* in F is a subring of $\mathfrak{o}_F$ that contains 1 and, as a $\mathbb{Z}$-module, has finite index in $\mathfrak{o}_F$; and that every order in F is a module in F. The *group of units* of an order $\mathfrak{D}$ in F is the set

$$\mathfrak{D}^\times = \{\beta \in \mathfrak{D} \setminus \{0\} : \beta^{-1} \in \mathfrak{D}\}.$$

The *order of a module* $\mathfrak{m} \in \mathfrak{M}_F$ is the order

$$\mathfrak{O}(\mathfrak{m}) = \{\beta \in F : \beta\mathfrak{m} \subset \mathfrak{m}\}$$

in F. Recall that the *norm* of an element α of F, denoted by $\mathrm{norm}_{F/\mathbb{Q}}(\alpha)$, is the determinant of the matrix $B = (b_{ij}) \in \mathrm{M}(n,\mathbb{Q})$ such that $\sum_{j=1}^n b_{ij}a_j = \alpha a_i$ for all $i = 1,\dots,n$, where $a_1,\dots,a_n$ is a $\mathbb{Q}$-basis for F. (Here, $\mathrm{M}(n,S)$ denotes the set of $n \times n$ matrices with entries in a commutative ring S with identity.) For $\phi \in \mathfrak{X}_F^0$, it is known that M_ϕ is a subgroup of $\mathfrak{o}_F^\times$ (see Theorem 3.4, p. 49 in [4]).

LEMMA 3.1. *Suppose $\phi \in \mathfrak{X}_F^0$, and let $\alpha \in \mathfrak{o}_F^\times$. Then $\alpha \in M_\phi$ if and only if there is $B \in \mathrm{GL}(n,\mathbb{Z})$ such that $BX_\phi = \alpha X_\phi$.*

Proof. Suppose there is $B \in \mathrm{GL}(n,\mathbb{Z})$ such that $BX_\phi = \alpha X_\phi$. The $R \in \mathrm{Diff}(T^n)$ induced by B satisfies $R_*X_\phi = \alpha X_\phi$, and so $\alpha \in M_\phi$. On the other hand, if $\alpha \in M_\phi$, then by Corollary 4.5 in [3], there is $B \in \mathrm{GL}(n,\mathbb{Z})$ such that $BX_\phi = \alpha X_\phi$. ∎

LEMMA 3.2. *Let $\phi, \psi \in \mathfrak{X}_F^0$. If $[\phi]_0 = [\psi]_0$, then $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$.*

Proof. If $[\phi]_0 = [\psi]_0$ then $[\mathfrak{m}_\phi] = [\mathfrak{m}_\psi]$ by Theorem 2.10, from which $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$ readily follows. ∎

THEOREM 3.3. *If $\phi \in \mathfrak{X}_F^0$, then $M_\phi = \mathfrak{O}(\mathfrak{m}_\phi)^\times$ and M_ϕ is a finite index subgroup of $\mathfrak{o}_F^\times$.*

Proof. Let $\phi \in \mathfrak{X}_F^0$. Then $\Omega[\phi]_0 = [\mathfrak{m}_\phi]$ by Theorem 2.10. The order $\mathfrak{O}(\mathfrak{m}_\phi)$ is independent of the choice of representative of $[\mathfrak{m}_\phi]$ by Lemma 3.2. By an elementary argument in algebraic number theory, $\alpha \in \mathfrak{O}(\mathfrak{m}_\phi)^\times$ if and only if $\alpha \in \mathfrak{O}(\mathfrak{m}_\phi)$ and $|\mathrm{norm}_{F/\mathbb{Q}}(\alpha)| = 1$. The two conditions $\alpha \in \mathfrak{O}(\mathfrak{m}_\phi)$ and $|\mathrm{norm}_{F/\mathbb{Q}}(\alpha)| = 1$ are equivalent to the existence of $B \in \mathrm{GL}(n,\mathbb{Z})$ such that $BX_\phi = \alpha X_\phi$ because $\alpha\mathfrak{m}_\phi \subset \mathfrak{m}_\phi$ is the same as $BX_\phi = \alpha X_\phi$ for $B \in \mathrm{M}(n,\mathbb{Z})$ and because $\mathrm{norm}_{F/\mathbb{Q}}(\alpha)$ is the same as $\det(B)$. By Lemma 3.1, there exists $B \in \mathrm{GL}(n,\mathbb{Z})$ such that $BX_\phi = \alpha X_\phi$ if and only if $\alpha \in M_\phi$.

Both $\mathfrak{o}_F^\times$ and $\mathfrak{O}(\mathfrak{m}_\phi)^\times$ have the same Dirichlet rank (the same number of free generators) by Dirichlet's unit theorem (see p. 334 in [25]). Since $\mathfrak{O}(\mathfrak{m}_\phi)^\times \subset \mathfrak{o}_F^\times$, it follows that $\mathfrak{O}(\mathfrak{m}_\phi)^\times$ is a finite index subgroup of $\mathfrak{o}_F^\times$. ∎

It is an immediate consequence of Theorems 2.10 and 3.3 that the number of classes in $[\mathfrak{X}_F^0]$ with a prescribed multiplier group is found by counting the number of classes in $[\mathfrak{M}_F]$ whose orders have that multiplier group as its group of units. For a real algebraic number field F, let

$$U_F = \{\mathfrak{O}(\mathfrak{m})^\times : \mathfrak{m} \in \mathfrak{M}_F\}.$$

For each $G \in U_F$, define a *G-paragon* in F to be an $\mathfrak{m} \in \mathfrak{M}_F$ for which $\mathfrak{O}(\mathfrak{m})^\times = G$. For $G \in U_F$, let

$$\mathcal{M}_F(G) = \{\mathfrak{m} \in \mathfrak{M}_F : \mathfrak{O}(\mathfrak{m})^\times = G\},$$

the set of G-paragons in F.

DEFINITION 3.4. For $G \in U_F$, the *G-paragon class number* of F is the number of module classes in $\mathcal{M}_F(G)$, and is denoted by $\lambda_F(G)$.

LEMMA 3.5. *Suppose $\phi, \psi \in \mathfrak{X}_F^0$. If $M_\phi = M_\psi$ and $\lambda_F(M_\phi) = 1$, then $[\phi]_0 = [\psi]_0$.*

Proof. Let $\mathfrak{D} = \mathfrak{O}(\mathfrak{m}_\phi)$. Then $M_\phi = \mathfrak{D}^\times$ by Theorem 3.3, and $\lambda_F(M_\phi) = 1$ means that $[\mathfrak{m}_\phi] = \mathcal{M}_F(\mathfrak{D}^\times)$. Since $M_\phi = M_\psi$, we see that $\mathfrak{m}_\psi \in \mathcal{M}_F(\mathfrak{D}^\times)$, and so $[\mathfrak{m}_\phi] = [\mathfrak{m}_\psi]$. Hence, $[\phi]_0 = [\psi]_0$ by Theorem 2.10. ∎

THEOREM 3.6. *The multiplier group is a complete invariant for $[\mathfrak{X}_F^0]$ if and only if $\lambda_F(\mathfrak{D}^\times) = 1$ for all orders $\mathfrak{D}$ in F.*

Proof. Let $\mathfrak{D}$ be any order in F, and $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}_F$ satisfy $\mathfrak{O}(\mathfrak{m})^\times = \mathfrak{D}^\times = \mathfrak{O}(\mathfrak{n})^\times$. Since Ω is surjective, there are ϕ and ψ in $\mathfrak{X}_F^0$ such that $\Omega[\phi] = [\mathfrak{m}]$ and $\Omega[\psi] = [\mathfrak{n}]$, i.e., $\mathfrak{m} = \mathfrak{m}_\phi$ and $\mathfrak{n} = \mathfrak{m}_\psi$. Since $\mathfrak{O}(\mathfrak{m}_\phi)^\times = \mathfrak{O}(\mathfrak{m}_\psi)^\times$, we have

$M_\phi = M_\psi$ by Theorem 3.3. If the multiplier group is a complete invariant for $[\mathfrak{X}_F^0]$, then $[\phi]_0 = [\psi]_0$. This implies by Theorem 2.10 that $[\mathfrak{m}_\phi] = [\mathfrak{m}_\psi]$. So $\lambda_F(\mathfrak{D}^\times) = 1$ for any order $\mathfrak{D}$ in F.

Now suppose that $\lambda_F(\mathfrak{D}^\times) = 1$ for all orders $\mathfrak{D}$ in F, and let $\phi, \psi \in \mathfrak{X}_F^0$ satisfy $M_\phi = M_\psi$. Since $M_\phi \in U_F$, it follows from Lemma 3.5 that $[\phi]_0 = [\psi]_0$. Thus the multiplier group is a complete invariant for $[\mathfrak{X}_F^0]$. ■

Two algebraic obstructions characterize the deviations $\lambda_F(\mathfrak{D}^\times) > 1$ for orders $\mathfrak{D}$ in F. These are the existence of inequivalent modules in $\mathfrak{M}_F$ with the same order, and the existence of distinct orders in F whose groups of units are the same. For an order $\mathfrak{D}$ in F, let

$$\mathfrak{M}_F(\mathfrak{D}) = \{\mathfrak{m} \in \mathfrak{M}_F : \mathfrak{O}(\mathfrak{m}) = \mathfrak{D}\}.$$

Recall that the *class number of* $\mathfrak{D}$ *in* F, denoted by $h_F(\mathfrak{D})$, is the number of module classes in $\mathfrak{M}_F(\mathfrak{D})$. As is well known (see Theorem 1.9, p. 13 in [24]), the quantity $h_F(\mathfrak{D})$ is finite for each order $\mathfrak{D}$ in F. Since $\mathfrak{M}_F(\mathfrak{D}) \subset \mathcal{M}_F(\mathfrak{D}^\times)$, it follows that

$$h_F(\mathfrak{D}) \leq \lambda_F(\mathfrak{D}^\times)$$

for all orders $\mathfrak{D}$ in F, i.e., the class number of $\mathfrak{D}$ provides a lower bound on the $\mathfrak{D}^\times$-paragon class number of F.

The class number $h_F(\mathfrak{o}_F)$ of the maximal order $\mathfrak{o}_F$ is known as the *class number* h_F of F. Each module class in $\mathfrak{M}_F(\mathfrak{o}_F)$ is an ideal class in $\mathfrak{o}_F$ which by Theorem 3.3 corresponds to the projective conjugacy class of a *maximal* F*-algebraic quasiperiodic flow* in $\mathfrak{X}_F^0$, i.e., a $\phi \in \mathfrak{X}_F^0$ with $M_\phi = \mathfrak{o}_F^\times$.

EXAMPLE 3.7. Let $F = \mathbb{Q}(\varepsilon)$ be the quartic field where ε is a real root of the irreducible $z^4 + z^3 - 2z - 1$. The ring of integers $\mathfrak{o}_F$ is the *monogenic order* $\mathbb{Z}[\varepsilon] \equiv \mathbb{Z}[1, \varepsilon, \varepsilon^2, \varepsilon^3]$, the group of units in $\mathfrak{o}_F$ is $\mathfrak{o}_F^\times = \{\pm\varepsilon_1^k\varepsilon_2^l : k, l \in \mathbb{Z}\}$ where $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = 1 + \varepsilon$, and $h_F = 1$ (see Table 4.2 of the Appendix in [25]). The $\phi, \psi \in \mathfrak{X}_F^0$ for which

$$X_\phi = \frac{\partial}{\partial\theta_1} + \varepsilon\,\frac{\partial}{\partial\theta_2} + \varepsilon^2\,\frac{\partial}{\partial\theta_3} + \varepsilon^3\,\frac{\partial}{\partial\theta_4},$$

$$X_\psi = (1 - 3\varepsilon^2)\,\frac{\partial}{\partial\theta_1} + (4\varepsilon + 4\varepsilon^2 + \varepsilon^3)\,\frac{\partial}{\partial\theta_2} + (2\varepsilon + 3\varepsilon^2 + \varepsilon^3)\,\frac{\partial}{\partial\theta_3} + (2 + \varepsilon)\,\frac{\partial}{\partial\theta_4}$$

are maximal F-algebraic: the first because $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{o}_F$, and the second because the $\mathrm{GL}(4, \mathbb{Z})$ matrices

$$B_1 = \begin{bmatrix} -2 & 3 & -6 & 1 \\ 1 & -2 & 5 & 0 \\ 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} -1 & 3 & -6 & 1 \\ 1 & -1 & 5 & 0 \\ 1 & -1 & 4 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

satisfy $B_i X_\psi = \varepsilon_i X_\psi$ for $i = 1, 2$. The only order in F to which the units $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = 1+\varepsilon$ and their inverses $\varepsilon_1^{-1} = -2+\varepsilon^2+\varepsilon^3$, $\varepsilon_2^{-1} = 2-\varepsilon^3$ belong is $\mathbb{Z}[\varepsilon] = \mathfrak{o}_F$. This implies that $\mathcal{M}_F(\mathfrak{o}_F^\times) = \mathfrak{M}_F(\mathfrak{o}_F)$ and so $\lambda_F(\mathfrak{o}_F^\times) = h_F = 1$. Since $M_\phi = \mathfrak{o}_F^\times = M_\psi$ and $\lambda_F(M_\phi) = 1$, Lemma 3.5 implies that $[\phi]_0 = [\psi]_0$. Indeed, the $\mathrm{GL}(4, \mathbb{Z})$ matrix

$$C = \begin{bmatrix} 0 & -1 & -1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and $\vartheta = (6 - 2\varepsilon + \varepsilon^2 - \varepsilon^3)/11 \in F$ satisfy $CX_\phi = \vartheta X_\psi$.

4. Detecting the equality of orders of modules. By Lemma 3.2, equality of the orders of the modules corresponding to F-algebraic quasiperiodic flows is a necessary condition for projective conjugacy. Equality of the orders together with the class number of the order being 1 is sufficient for projective conjugacy. The generalized Bowen–Franks groups introduced in [21] provide means for detecting the equality of orders. For $\mathfrak{m} \in \mathfrak{M}_F$, the *generalized Bowen–Franks groups* are the finite abelian groups

$$BF_\alpha(\mathfrak{m}) = \mathfrak{m}/\alpha\mathfrak{m} \quad \text{for nonzero } \alpha \in \mathfrak{O}(\mathfrak{m}).$$

For each $\alpha \in \mathfrak{O}(\mathfrak{m}) \setminus \{0\}$, the module $\alpha\mathfrak{m}$ is a principal ideal in $\mathfrak{m}$, and the order $|BF_\alpha(\mathfrak{m})|$ of the group $BF_\alpha(\mathfrak{m})$ is $|\mathrm{norm}_{F/\mathbb{Q}}(\alpha)|$. Recall for an order $\mathfrak{D}$ in F that a module $\mathfrak{m} \in \mathfrak{M}_F$ is called a $\mathfrak{D}$-*ideal* if $\mathfrak{O}(\mathfrak{m}) \supset \mathfrak{D}$.

LEMMA 4.1 (Martins Rodrigues, Sousa Ramos, p. 241 in [21]). *Let $\mathfrak{D}$ be an order in F, and suppose that $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}_F$ are $\mathfrak{D}$-ideals. If $\mathfrak{O}(\mathfrak{m}) \neq \mathfrak{O}(\mathfrak{n})$, then $BF_\alpha(\mathfrak{m})$ is not isomorphic to $BF_\alpha(\mathfrak{n})$ for some nonzero $\alpha \in \mathfrak{D}$.*

For modules $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}_F$ there is always an order $\mathfrak{D}$ in F for which $\mathfrak{m}$ and $\mathfrak{n}$ are both $\mathfrak{D}$-ideals. An $\mathfrak{m} \in \mathfrak{M}_F$ is an $\mathfrak{O}(\mathfrak{m})$-ideal and an $\mathfrak{n} \in \mathfrak{M}_F$ is an $\mathfrak{O}(\mathfrak{n})$-ideal. Since the orders $\mathfrak{O}(\mathfrak{m})$ and $\mathfrak{O}(\mathfrak{n})$ contain 1, and as $\mathbb{Z}$-modules, have finite index in $\mathfrak{o}_F$, it follows that $\mathfrak{O}(\mathfrak{m}) \cap \mathfrak{O}(\mathfrak{n})$ is a subring of $\mathfrak{o}_F$ that contains 1, and as a $\mathbb{Z}$-module, has finite index in $\mathfrak{o}_F$ (see Proposition 4.9 on p. 40 in [10]). The intersection $\mathfrak{D} = \mathfrak{O}(\mathfrak{m}) \cap \mathfrak{O}(\mathfrak{n})$ is then an order in F for which $\mathfrak{m}$ and $\mathfrak{n}$ are both $\mathfrak{D}$-ideals. Furthermore, any subring $\mathfrak{D}$ of $\mathfrak{O}(\mathfrak{m}) \cap \mathfrak{O}(\mathfrak{n})$ that contains 1 and has finite index in $\mathfrak{O}(\mathfrak{m}) \cap \mathfrak{O}(\mathfrak{n})$, is an order in F for which $\mathfrak{m}$ and $\mathfrak{n}$ are both $\mathfrak{D}$-ideals.

THEOREM 4.2. *For $\phi, \psi \in \mathfrak{X}_F^0$, let $\mathfrak{D}$ be any order in F for which $\mathfrak{D} \subset \mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\psi)$. If $\mathfrak{O}(\mathfrak{m}_\phi) \neq \mathfrak{O}(\mathfrak{m}_\psi)$, then $BF_\alpha(\mathfrak{m}_\phi)$ is not isomorphic to $BF_\alpha(\mathfrak{m}_\psi)$ for some nonzero $\alpha \in \mathfrak{D} \setminus \mathfrak{D}^\times$.*

Proof. By the preceding discussion, the $\mathfrak{m}_\phi, \mathfrak{m}_\psi \in \mathfrak{M}_F$ are $\mathfrak{D}$-ideals for any order $\mathfrak{D} \subset \mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\psi)$. By Lemma 4.1, if $\mathfrak{O}(\mathfrak{m}_\phi) \neq \mathfrak{O}(\mathfrak{m}_\psi)$, then $BF_\alpha(\mathfrak{m}_\phi)$ is not isomorphic to $BF_\alpha(\mathfrak{m}_\psi)$ for some nonzero $\alpha \in \mathfrak{D}$. There is no need to consider $\alpha \in \mathfrak{D}^\times$ since $|BF_\alpha(\mathfrak{m}_\phi)| = |\mathrm{norm}_{F/\mathbb{Q}}(\alpha)| = 1 = |BF_\alpha(\mathfrak{m}_\psi)|$. ∎

COROLLARY 4.3. *For $\phi, \psi \in \mathfrak{X}_F^0$, let $\mathfrak{D}$ be any order in F for which $\mathfrak{D} \subset \mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\psi)$. If $BF_\alpha(\mathfrak{m}_\phi)$ is isomorphic to $BF_\alpha(\mathfrak{m}_\psi)$ for all nonzero $\alpha \in \mathfrak{D} \setminus \mathfrak{D}^\times$ and $h_F(\mathfrak{O}(\mathfrak{m}_\phi)) = 1$, then $[\phi]_0 = [\psi]_0$.*

Proof. Suppose that $BF_\alpha(\mathfrak{m}_\phi)$ is isomorphic to $BF_\alpha(\mathfrak{m}_\psi)$ for all nonzero $\alpha \in \mathfrak{D} \setminus \mathfrak{D}^\times$. Then Theorem 4.2 implies that $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$. Hence $\mathfrak{m}_\phi, \mathfrak{m}_\psi \in \mathfrak{M}_F(\mathfrak{O}(\mathfrak{m}_\phi))$. Assuming $h_F(\mathfrak{O}(\mathfrak{m}_\phi)) = 1$ implies that $[\mathfrak{m}_\phi] = \mathfrak{M}_F(\mathfrak{O}(\mathfrak{m}_\phi)) = [\mathfrak{m}_\psi]$. Thus Theorem 2.10 implies that $[\phi]_0 = [\psi]_0$. ∎

EXAMPLE 4.4. The class number of the pure cubic number field $F = \mathbb{Q}(12^{1/3})$ is $h_F = 1$, its ring of integers is $\mathfrak{o}_F = \mathbb{Z}[1, 12^{1/3}, 18^{1/3}]$, and the group of units in $\mathfrak{o}_F$ is $\mathfrak{o}_F^\times = \{\pm\varepsilon_F^k : k \in \mathbb{Z}\}$ where $\varepsilon_F = 55 + 24 \cdot 12^{1/3} + 21 \cdot 18^{1/3}$ is the fundamental unit (see p. 63 in [31]). The $\phi, \psi \in \mathfrak{X}_F^0$ determined by

$$X_\phi = \frac{\partial}{\partial\theta_1} + 12^{1/3}\,\frac{\partial}{\partial\theta_2} + 18^{1/3}\,\frac{\partial}{\partial\theta_3},$$
$$X_\psi = (6 + 2 \cdot 18^{1/3})\,\frac{\partial}{\partial\theta_1} + (3 \cdot 12^{1/3} - 2 \cdot 18^{1/3})\,\frac{\partial}{\partial\theta_2} + (12^{1/3} + 18^{1/3})\,\frac{\partial}{\partial\theta_3}$$

are maximal F-algebraic quasiperiodic flows: the first because $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{o}_F$, and the second because the $\mathrm{GL}(3, \mathbb{Z})$ matrix

$$B = \begin{bmatrix} 55 & 81 & 63 \\ 21 & 31 & 24 \\ 69 & 102 & 79 \end{bmatrix}$$

satisfies $BX_\psi = \varepsilon_F X_\psi$. The module $\mathfrak{m}_\psi$ is not an order in F. However, it is easy to show that $\mathfrak{O}(\mathfrak{m}_\psi)$ contains the order

$$\mathfrak{D} = \mathbb{Z}[1, 45 \cdot 12^{1/3}, 42 \cdot 12^{1/3} + 3 \cdot 18^{1/3}].$$

The fundamental unit ε_F and its inverse ε_F^{-1} belong to the order $\mathfrak{D}$, which is distinct from $\mathfrak{o}_F$. This means that $\lambda_F(\mathfrak{o}_F^\times) \geq 2$, so Lemma 3.5 cannot be applied to get $[\phi]_0 = [\psi]_0$. Straightforward computations show that $BF_\alpha(\mathfrak{m}_\phi)$ and $BF_\alpha(\mathfrak{m}_\psi)$ are isomorphic for all $\alpha \in \mathfrak{D} \setminus \mathfrak{D}^\times$. Since $h_F = 1$, Corollary 4.3 implies that $[\phi]_0 = [\psi]_0$. It can also be shown directly that $\mathfrak{O}(\mathfrak{m}_\psi) = \mathfrak{o}_F$, from which $[\phi]_0 = [\psi]_0$ follows. Indeed, the $\mathrm{GL}(2, \mathbb{Z})$ matrix

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and $\vartheta = (-6 + 3 \cdot 12^{1/3} + 2 \cdot 18^{1/3})/30 \in F^\times$ satisfy $CX_\phi = \vartheta X_\psi$.

The generalized Bowen–Franks groups provide sufficient conditions for the class number of an order to be equal to the G-paragon class number for the group of units of that order.

THEOREM 4.5. *Let $\mathfrak{D}$ be an order in F. If for all ϕ, ψ in $\mathfrak{X}_F^0$ with $M_\phi = \mathfrak{D}^\times = M_\psi$ the groups $BF_\alpha(\mathfrak{m}_\phi)$ and $BF_\alpha(\mathfrak{m}_\psi)$ are isomorphic for all nonzero $\alpha \in (\mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\psi)) \setminus (\mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\psi))^\times$, then $h_F(\mathfrak{D}) = \lambda_F(\mathfrak{D}^\times)$,*

Proof. Suppose for all $\phi, \psi \in \mathfrak{X}_F^0$ with $M_\phi = \mathfrak{D}^\times = M_\psi$ that $BF_\alpha(\mathfrak{m}_\phi)$ is isomorphic to $BF_\alpha(\mathfrak{m}_\psi)$ for all $\alpha \in (\mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\phi)) \setminus (\mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\phi))^\times$. Then Theorem 4.2 implies that $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$ for all $\phi, \psi \in \mathfrak{X}_F^0$ with $M_\phi = \mathfrak{D}^\times = M_\phi$. Since $\mathfrak{D} \in \mathfrak{M}_F$, there exists a $\phi \in \mathfrak{X}_F^0$ such that $\Omega[\phi]_0 = [\mathfrak{D}]$. Hence $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{D}$, i.e., $\mathfrak{m}_\phi \in \mathfrak{M}_F(\mathfrak{D})$, and $M_\phi = \mathfrak{D}^\times$ by Theorem 3.3. It follows that $\mathfrak{D} = \mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$ for arbitrary $\psi \in \mathfrak{X}_F^0$ with $M_\psi = \mathfrak{D}^\times$. This means that $\mathfrak{M}_F(\mathfrak{D}) = \mathcal{M}_F(\mathfrak{D}^\times)$. The finiteness of $h_F(\mathfrak{D})$ therefore implies that $h_F(\mathfrak{D}) = \lambda_F(\mathfrak{D}^\times)$. ∎

For $n = 2, 3$, the converse of Lemma 4.1 is valid (see Corollary 2 on p. 242 in [21]). This means that for $n = 2, 3$, the converses of Theorems 4.2 and 4.5 are also valid.

5. The action of the multiplier group. The multiplier group of an F-algebraic quasiperiodic flow induces an action on T^n by toral automorphisms which provides a necessary condition for equality of multiplier groups of F-algebraic quasiperiodic flows. By Lemma 3.1, there is associated to each M_ϕ for $\phi \in \mathfrak{X}_F^0$ the set

$$\Pi_\phi = \{B \in \mathrm{GL}(n, \mathbb{Z}) : BX_\phi = \alpha X_\phi \text{ for } \alpha \in M_\phi\}.$$

This is a group that is isomorphic to M_ϕ, and hence abelian: there is an isomorphism $\nu_\phi : \Pi_\phi \to M_\phi$ given by $\nu_\phi(B) = \varrho_\phi(R)$ for any $R \in S_\phi$ such that $\mathbf{T}R = B$ (see Theorem 5.3 in [3]). The group Π_ϕ defines an *action of M_ϕ on T^n*:

$$J_\phi : M_\phi \times T^n \to T^n \quad \text{by} \quad J_\phi(\alpha)\theta = B\theta \quad \text{where} \quad B = \nu_\phi^{-1}(\alpha) \text{ and } \theta \in T^n.$$

Since the Dirichlet rank of M_ϕ is the same for all $\phi \in \mathfrak{X}_F^0$, the group Π_ϕ is isomorphic to Π_ψ for all $\psi \in \mathfrak{X}_F^0$. The actions J_ϕ and J_ψ are said to be *similar* (resp. *conjugate*) if the groups Π_ϕ and Π_ψ are similar (resp. conjugate) subgroups of $\mathrm{GL}(n, \mathbb{Z})$. Recall that two subgroups K_1 and K_2 of $\mathrm{GL}(n, \mathbb{Z})$ are similar (resp. conjugate) if and only if there is $C \in \mathrm{GL}(n, \mathbb{Q})$

(resp. $C \in \mathrm{GL}(n,\mathbb{Z})$) such that $\Delta_C(\Pi_\psi) = \Pi_\phi$, where Δ_C is the inner automorphism determined by C, i.e., $\Delta_C(B) = C^{-1}BC$.

THEOREM 5.1. *For $\phi, \psi \in \mathfrak{X}_F^0$, if $M_\phi = M_\psi$, then J_ϕ and J_ψ are similar.*

Proof. The components of X_ϕ and X_ψ form $\mathbb{Q}$-bases for F, and so there is $C \in \mathrm{GL}(n,\mathbb{Q})$ such that $CX_\phi = X_\psi$. Let $B \in \Pi_\phi$. Then $BX_\phi = \nu_\phi(B)X_\phi$, and so

$$CBC^{-1}X_\psi = \nu_\phi(B)X_\psi.$$

Assuming $M_\phi = M_\psi$ implies that $\nu_\phi(B) \in M_\psi$. The components of X_ψ are a $\mathbb{Z}$-basis for $\mathfrak{m}_\psi$. Since $M_\psi = \mathfrak{O}(\mathfrak{m}_\psi)^\times$ by Theorem 3.3, we have $\nu_\phi(B)\mathfrak{m}_\psi \subset \mathfrak{m}_\psi$, and so the components of $\nu_\phi(B)X_\psi$ are linear combinations over $\mathbb{Z}$ of the components of X_ψ. Uniqueness of linear combinations over $\mathbb{Z}$ implies that $CBC^{-1} \in \mathrm{M}(n,\mathbb{Z})$. As $\det(B) = \pm 1$, we then have $CBC^{-1} \in \mathrm{GL}(n,\mathbb{Z})$. Since $\nu_\phi(B) \in M_\psi$, it follows that $CBC^{-1} \in \Pi_\psi$. Thus $\Delta_{C^{-1}}(\Pi_\phi) \subset \Pi_\psi$. Similar reasoning leads to $\Delta_{C^{-1}}(\Pi_\phi) \supset \Pi_\psi$, and so Π_ϕ and Π_ψ are similar subgroups of $\mathrm{GL}(n,\mathbb{Z})$. ∎

COROLLARY 5.2. *For $\phi, \psi \in \mathfrak{X}_F^0$, if $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$, then J_ϕ and J_ψ are similar.*

Proof. Assuming $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$ implies by Theorem 3.3 that $M_\phi = M_\psi$. Now apply Theorem 5.1. ∎

However, similarity of the actions of multiplier groups fails to identify the projective conjugacy classes in $\mathfrak{X}_F^0$.

COROLLARY 5.3. *Let $\phi \in \mathfrak{X}_F^0$. If $\lambda_F(M_\phi) > 1$, then there exists $\psi \in \mathfrak{X}_F^0$ such that J_ϕ and J_ψ are similar but $[\phi]_0 \neq [\psi]_0$.*

Proof. When $\lambda_F(M_\phi) > 1$ there is by Theorems 2.10 and 3.3 a $\psi \in \mathfrak{X}_F^0$ such that $M_\phi = M_\psi$ but $[\phi]_0 \neq [\psi]_0$. By Theorem 5.1, the actions J_ϕ and J_ψ are similar. ∎

On the other hand, conjugacy of the actions of multiplier groups by commuting toral automorphisms actually provides a relative invariant for $[\mathfrak{X}_F^0]$. (Compare the following with Lemma 1 on p. 1636 in [1].)

THEOREM 5.4. *For $\phi, \psi \in \mathfrak{X}_F^0$, if $[\phi]_0 = [\psi]_0$, then J_ϕ and J_ψ are conjugate.*

Proof. Suppose $[\phi]_0 = [\psi]_0$. Then $M_\phi = M_\psi$ by Theorem 2.2, and there is $C \in \mathrm{GL}(n,\mathbb{Z})$ and $\vartheta \in F^\times$ such that $CX_\phi = \vartheta X_\psi$. For $B \in \Pi_\psi$,

$$C^{-1}BCX_\phi = \vartheta C^{-1}BX_\psi = \vartheta\nu_\psi(B)C^{-1}X_\psi = \nu_\psi(B)X_\phi.$$

Since $\nu_\psi(B) \in M_\psi = M_\phi$ and $C^{-1}BC \in \mathrm{GL}(n,\mathbb{Z})$, it follows that $C^{-1}BC \in \Pi_\phi$. This implies that $\Delta_C(\Pi_\psi) \subset \Pi_\phi$. Similar reasoning leads to $\Delta_C(\Pi_\psi) \supset \Pi_\phi$, and so Π_ϕ and Π_ψ are conjugate subgroups of $\mathrm{GL}(n,\mathbb{Z})$. ∎

A consequence of Theorem 5.4 is that $\phi, \psi \in \mathfrak{X}_F^0$ with $M_\phi = M_\psi$ and/or $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$ fail to be projectively conjugate precisely when the similar actions J_ϕ and J_ψ are not conjugate. The existence of similar nonconjugate actions (discussed in Section 7) has consequences on class numbers of orders and G-paragon class numbers in F.

Corollary 5.5. *Let $\phi, \psi \in \mathfrak{X}_F^0$ be such that $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$. If J_ϕ and J_ψ are not conjugate, then $h_F(\mathfrak{O}(\mathfrak{m}_\phi)) > 1$.*

Proof. Supposing $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$ and $h_F(\mathfrak{O}(\mathfrak{m}_\phi)) = 1$ implies that $[\mathfrak{m}_\phi] = [\mathfrak{m}_\psi]$. Then $[\phi]_0 = [\psi]_0$ by Theorem 2.10. Hence by Theorem 5.4, it follows that J_ϕ and J_ψ are conjugate. ∎

Corollary 5.6. *Let $\phi, \psi \in \mathfrak{X}_F^0$ be such that $M_\phi = M_\psi$. If J_ϕ and J_ψ are not conjugate, then $\lambda_F(M_\phi) > 1$.*

Proof. Supposing $M_\phi = M_\psi$ and $\lambda_F(M_\phi) = 1$ implies by Lemma 3.5 that $[\phi]_0 = [\psi]_0$. Then J_ϕ and J_ψ are conjugate by Theorem 5.4. ∎

6. Condition PCF for irreducible actions. By Theorem 5.4, a necessary condition for F-algebraic quasiperiodic flows with the same multiplier to be projectively conjugate is that their similar actions are conjugate. But this necessary condition fails to be sufficient. A way to construct a counterexample is based on a $\mathrm{GL}(n,\mathbb{Z})$ matrix with an irreducible characteristic polynomial that is similar, but not conjugate, to its inverse. Recall that $B_1, B_2 \in \mathrm{GL}(n,\mathbb{Z})$ are conjugate if there is $C \in \mathrm{GL}(n,\mathbb{Z})$ such that $\Delta_C(B_2) = B_1$, and are similar if there is $C \in \mathrm{GL}(n,\mathbb{Q})$ such that $\Delta_C(B_2) = B_1$.

Example 6.1. Let $F = \mathbb{Q}(\sqrt{11})$. The group of units in the ring of integers of F is $\mathfrak{o}_F^\times = \{\pm\varepsilon_F^k : k \in \mathbb{Z}\}$, where $\varepsilon_F = 10 + 3\sqrt{11}$ is the fundamental unit (see Appendix B in [22]). The $\phi, \psi \in \mathfrak{X}_F^0$ determined by

$$X_\phi = 10\,\frac{\partial}{\partial\theta_1} + (7 + 3\sqrt{11})\,\frac{\partial}{\partial\theta_2}, \qquad X_\psi = -10\,\frac{\partial}{\partial\theta_1} + (-7 + 3\sqrt{11})\,\frac{\partial}{\partial\theta_2}$$

are both maximal F-algebraic quasiperiodic flows: the $\mathrm{GL}(2,\mathbb{Z})$ matrices

$$B_1 = \begin{bmatrix} 3 & 10 \\ 5 & 17 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 17 & -10 \\ -5 & 3 \end{bmatrix}$$

satisfy $B_1 X_\phi = \varepsilon_F X_\phi$ and $B_2 X_\psi = \varepsilon_F X_\psi$. Thus $\Pi_\phi = \{\pm B_1^k : k \in \mathbb{Z}\}$ and $\Pi_\psi = \{\pm B_2^k : k \in \mathbb{Z}\}$. For $l \in \mathbb{Z} \setminus \{0\}$, the inclusion $l\sqrt{11}\,\mathfrak{m}_\phi \subset \mathfrak{m}_\phi$ requires that 3 divides l, and so $\mathfrak{O}(\mathfrak{m}_\phi) = \mathbb{Z}[3\sqrt{11}]$. A similar calculation shows that $\mathfrak{O}(\mathfrak{m}_\psi) = \mathbb{Z}[3\sqrt{11}]$. The characteristic polynomial of B_1 is irreducible. The matrix B_1 is similar, but not conjugate, to $B_2 = B_1^{-1}$ (see p. 1647 in [1]). If $[\phi]_0 = [\psi]_0$, then J_ϕ and J_ψ would be conjugate actions by Theorem 5.4, which would imply that B_1 and B_2 are conjugate. This contradiction means that $[\phi]_0 \neq [\psi]_0$, and so $h_F(\mathbb{Z}[3\sqrt{11}]) \geq 2$, whence $\lambda_F(\mathbb{Z}[3\sqrt{11}]^\times) \geq 2$. However, since $B_2 = B_1^{-1}$, it follows that J_ϕ and J_ψ are trivially conjugate.

There is, though, a partial converse to Theorem 5.4 (given below in Theorem 6.6) by which conjugacy of the actions does imply projective conjugacy. It is based on the real eigenvectors of the matrices in $\Pi_\phi \setminus \{I, -I\}$, where I is the identity matrix. For $\phi \in \mathfrak{X}_F^0$, the action J_ϕ is called *irreducible* if there is $B \in \Pi_\phi$ with an irreducible characteristic polynomial (cf. Proposition 3.1, p. 726 in [12]). If J_ϕ is irreducible, then there are $1 \leq k \leq n$ real eigenvectors $X_\phi, v_2, \ldots, v_k$ common to all $B \in \Pi_\phi$ (see Lemma 3.4, p. 516 in [7]). Let $\nu_\phi(B), \alpha_2, \ldots, \alpha_k$ be the corresponding eigenvalues of a $B \in \Pi_\phi$ that has an irreducible characteristic polynomial. The n components of each v_i, $i = 2, \ldots, k$, are linearly independent over $\mathbb{Q}$ and can be chosen so as to belong to $\mathfrak{o}_{F_i}$ where $F_i = \mathbb{Q}(\alpha_i)$ is a real algebraic number field of degree n over $\mathbb{Q}$ (see Propositions 1 and 8 in [34]; although these propositions are stated for $\mathrm{SL}(n, \mathbb{Z})$ matrices, their proofs readily extend to $\mathrm{GL}(n, \mathbb{Z})$ matrices). Since for each $i = 2, \ldots, k$, the components of v_i form a $\mathbb{Q}$-basis for F_i, and since v_i is an eigenvector common to all the matrices in Π_ϕ that have an irreducible characteristic polynomial, the field F_i is independent of the eigenvalue α_i corresponding to the eigenvector v_i of B.

Definition 6.2. For $\phi \in \mathfrak{X}_F^0$ suppose that J_ϕ is irreducible and k is the number of real eigenvectors common to Π_ϕ. The *quasiperiodic flows* associated to the common real eigenvectors $v_2, \ldots, v_k$ of Π_ϕ other than X_ϕ are the $\phi_i \in \mathfrak{X}_{F_i}^0$ determined by $X_{\phi_i} = v_i$ for $i = 2, \ldots, k$.

Irreducibility of J_ϕ implies that Π_ϕ possesses group-theoretic centralizer properties. Recall that the *centralizer* of a nonempty subset K of $\mathrm{GL}(n, \mathbb{Z})$ within $\mathrm{GL}(n, \mathbb{Z})$ is $Z(K) = \{A \in \mathrm{GL}(n, \mathbb{Z}) : BA = AB \text{ for all } B \in K\}$, while for $B \in \mathrm{GL}(n, \mathbb{Z})$, it is $Z(B) = \{A \in \mathrm{GL}(n, \mathbb{Z}) : BA = AB\}$. The proof of the following is straightforward.

Theorem 6.3. *Let $\phi \in \mathfrak{X}_F^0$. If there is $B \in \Pi_\phi$ with an irreducible characteristic polynomial, then $Z(B) = \Pi_\phi$.*

Irreducibility of the action J_ϕ always implies that Π_ϕ is a *maximal abelian* subgroup of the nonabelian group $\mathrm{GL}(n, \mathbb{Z})$, i.e., for any abelian subgroup K of $\mathrm{GL}(n, \mathbb{Z})$ such that $\Pi_\phi \subset K$ it follows that $\Pi_\phi = K$. In this case, Π_ϕ being maximal abelian is equivalent to $Z(\Pi_\phi) = \Pi_\phi$ (see p. 86 in [28]).

Corollary 6.4. *Let $\phi \in \mathfrak{X}_F^0$ and suppose that J_ϕ is irreducible and the number k of common real eigenvectors for Π_ϕ is strictly bigger than* 1. *Then $\Pi_\phi = \Pi_{\phi_i}$ for all $i = 2, \ldots, k$ for the k quasiperiodic flows ϕ_i associated to the real common eigenvectors of Π_ϕ.*

Proof. Let $B \in \Pi_\phi$ have an irreducible characteristic polynomial. Since $BX_{\phi_i} = \nu_{\phi_i}(B)X_{\phi_i}$, we get $B \in \Pi_{\phi_i}$. By Theorem 6.3 it follows that $\Pi_\phi = Z(B) = \Pi_{\phi_i}$ for all $i = 2, \ldots, k$. ∎

For $\phi \in \mathfrak{X}_F^0$ with J_ϕ irreducible and k the number of common real eigenvectors for Π_ϕ, projective conjugacy of ϕ with ϕ_i for any $i = 2, \ldots, k$ does not follow from the equalities $\Pi_\phi = \Pi_{\phi_i}$. It may happen that the Koch fields $F = K_\phi$ and $F_i = K_{\phi_i}$ are different for some $i = 2, \ldots, k$, so that $[\phi]_0 \neq [\phi_i]_0$ by Theorem 2.7, even though $F, F_2, \ldots, F_k$ are isomorphic as fields because they are extensions of $\mathbb{Q}$ by roots of the same irreducible polynomial (see Corollary 1.9, p. 236 in [10]). When $F \neq F_i$ for some i, then of necessity F is not normal over $\mathbb{Q}$. (Recall that F being *normal* over $\mathbb{Q}$ means that every irreducible polynomial in $\mathbb{Q}[z]$ that has a root in F splits in the polynomial ring $F[z]$). However, even if $F = F_i$ for all $i = 2, \ldots, k$, it may be that $[\phi]_0 \neq [\phi_i]_0$ for some i, and this happens by Theorem 2.2 when $M_\phi \neq M_{\phi_i}$.

Definition 6.5. For $\phi \in \mathfrak{X}_F^0$, the action J_ϕ is said to satisfy *condition PCF* (projectively conjugate flows) if J_ϕ is irreducible and the k quasiperiodic flows ϕ_i associated to the common real eigenvectors of Π_ϕ other than X_ϕ are F-algebraic and satisfy $[\phi]_0 = [\phi_i]_0$ for all $i = 2, \ldots, k$.

It is easy to show that for $\phi, \psi \in \mathfrak{X}_F^0$, the action J_ϕ satisfying condition PCF implies that J_ψ satisfies condition PCF whenever J_ϕ and J_ψ are conjugate actions.

For a given real algebraic number field F, not every $\phi \in \mathfrak{X}_F^0$ has J_ϕ satisfying condition PCF. In Example 6.1, the $\phi \in \mathfrak{X}_F^0$ for $F = \mathbb{Q}(\sqrt{11})$ with $X_\phi = 10\partial/\partial\theta_1 + (7 + 3\sqrt{11})\partial/\partial\theta_2$ has J_ϕ failing to satisfy condition PCF. In this case, the eigenvectors common to all $B \in \Pi_\phi$ are X_ϕ and

$$X_{\phi_2} = 10\,\frac{\partial}{\partial\theta_1} + (7 - 3\sqrt{11})\,\frac{\partial}{\partial\theta_2}.$$

As given in Example 6.1, the flow $\psi \in \mathfrak{X}_F^0$ with $X_\psi = -10\partial/\partial\theta_1 + (-7 + 3\sqrt{11})\partial/\partial\theta_2$ satisfies $X_\psi = -X_{\phi_2}$. This means that $[\psi]_0 = [\phi_2]_0$. But since $[\psi]_0 \neq [\phi]_0$, it follows that $[\phi]_0 \neq [\phi_2]_0$, even though $\Pi_\phi = \Pi_{\phi_2}$.

Theorem 6.6. *Suppose $\phi, \psi \in \mathfrak{X}_F^0$. If J_ϕ and J_ψ are conjugate and J_ϕ satisfies condition PCF, then $[\phi]_0 = [\psi]_0$.*

Proof. Suppose there is $C \in \mathrm{GL}(n, \mathbb{Z})$ such that $\Delta_C(\Pi_\psi) = \Pi_\phi$. Let $B \in \Pi_\phi$ be a matrix with irreducible characteristic polynomial. Then there is $B' \in \Pi_\psi$ such that $BC^{-1} = C^{-1}B'$. Hence $BC^{-1}X_\psi = \nu_\psi(B')C^{-1}X_\psi$, and so $C^{-1}X_\psi$ is a real eigenvector for B. Irreducibility of the characteristic polynomial of B implies that the real eigenspaces of B are one-dimensional. If $C^{-1}X_\psi = \vartheta X_\phi$ for some $\vartheta \in \mathbb{R}^\times$, then $\vartheta \in F^\times$ since $C \in \mathrm{GL}(n, \mathbb{Z})$ and $\phi, \psi \in \mathfrak{X}_F^0$, and so $[\phi]_0 = [\psi]_0$. Otherwise, there is $\vartheta \in \mathbb{R}^\times$ such that

$C^{-1}X_\psi = \vartheta X_{\phi_i}$ for one of the quasiperiodic flows ϕ_i, $i = 2, \ldots, k$, associated to the $k-1$ common real eigenvectors of Π_ϕ other than X_ϕ. Again, since $C \in \mathrm{GL}(n, \mathbb{Z})$, and $\psi \in \mathfrak{X}_F^0$, and $\phi_i \in \mathfrak{X}_F^0$ by hypothesis, it follows that $\vartheta \in F^\times$, and so $[\psi]_0 = [\phi_i]_0$. Then $[\phi]_0 = [\phi_i]_0 = [\psi]_0$ by condition PCF. ∎

When the degree of F is 2, there is a condition on the frequencies of an F-algebraic quasiperiodic flow ϕ that implies condition PCF for J_ϕ. For each $\phi \in \mathfrak{X}_F^0$ with $X_\phi = a_1\partial/\partial\theta_1 + a_2\partial/\partial\theta_2$, the ratio a_2/a_1 of the frequencies of ϕ is a root of an irreducible quadratic monic polynomial in the polynomial ring $\mathbb{Q}[z]$ (see Theorem 2.1 in [2], where the irreducible monic polynomial is explicitly constructed). Each $B \in \Pi_\phi \setminus \{I, -I\}$ has two eigenvectors: one is X_ϕ, and the other is determined by the irreducible monic polynomial for which a_2/a_1 is a root (which eigenvector is explicitly given in the proof of Corollary 2.4 in [2], cf. the eigenvectors given in [1] for an $\mathrm{M}(2, \mathbb{Z})$ matrix).

Lemma 6.7. *Let $\phi \in \mathfrak{X}_F^0$ where F is of degree* 2. *For $X_\phi = a_1\partial/\partial\theta_1 + a_2\partial/\partial\theta_2$, let $z^2 + pz - q \in \mathbb{Q}[z]$ be an irreducible polynomial for which a_2/a_1 is a root. Then all $B \in \Pi_\phi \setminus \{I, -I\}$ have X_ϕ and $[q^{-1}a_2, -a_1]^T$ as eigenvectors.*

The action J_ϕ corresponding to an F-algebraic quasiperiodic flow ϕ is always irreducible when F is real quadratic. Each $B \in \Pi_\phi \setminus \{I, -I\}$ has an irreducible characteristic polynomial because both of its roots $\nu_\phi(B)$, $\det(B)/\nu_\phi(B)$ are in $F \setminus \mathbb{Q}$ (see Corollary 4.4, p. 10 in [3]). The flow ϕ_2 corresponding to the common real eigenvector of Π_ϕ other than X_ϕ, given in Lemma 6.7, is always F-algebraic.

Theorem 6.8. *Let $\phi \in \mathfrak{X}_F^0$ where F is of degree* 2. *For $X_\phi = a_1\partial/\partial\theta_1 + a_2\partial/\partial\theta_2$, let $z^2 + pz - q \in \mathbb{Q}[z]$ be an irreducible polynomial for which a_2/a_1 is a root. If either* (i) *$p \in \mathbb{Z}$ or* (ii) *$p/q \in \mathbb{Z}$, then J_ϕ satisfies condition PCF.*

Proof. Fix $B \in \Pi_\phi \setminus \{I, -I\}$. Then $B[1, a_2/a_1]^T = \nu_\phi(B)[1, a_2/a_1]^T$ where $[1, a_2/a_1]^T$ is a scale equivalent of X_ϕ. Assuming that a_2/a_1 is a root of an irreducible quadratic $z^2 + pz - q \in \mathbb{Q}[z]$ means by Lemma 6.7 that the other real eigenvector of B is $[q^{-1}a_2/a_1, -1]^T$. Since a_2/a_1 is a root of z^2+pz-q, either $a_2/a_1 = (-p+\sqrt{p^2+4q})/2$ or $a_2/a_1 = (-p-\sqrt{p^2+4q})/2$. In both cases a straightforward calculation shows that

$$\begin{bmatrix} 1 & 0 \\ -p & -1 \end{bmatrix} \begin{bmatrix} 1 \\ (-p \pm \sqrt{p^2+4q})/2 \end{bmatrix} = \frac{2q}{-p \pm \sqrt{p^2+4q}} \begin{bmatrix} q^{-1}(-p \pm \sqrt{p^2+4q})/2 \\ -1 \end{bmatrix}.$$

If $p \in \mathbb{Z}$, then $[\phi]_0 = [\phi_2]_0$. If $p \notin \mathbb{Z}$, but $p/q \in \mathbb{Z}$ then the above can be repeated with a_1/a_2 in place of a_2/a_1 because a_1/a_2 is a root of the irreducible quadratic $z^2 - pq^{-1}z - q^{-1}$. Thus J_ϕ satisfies condition PCF. ■

COROLLARY 6.9. *Let $\phi, \psi \in \mathfrak{X}_F^0$ where F is of degree 2. For $X_\phi = a_1\partial/\partial\theta_1 + a_2\partial/\partial\theta_2$, let $z^2 + pz - q \in \mathbb{Q}[z]$ be an irreducible polynomial for which a_2/a_1 is a root. If J_ϕ and J_ψ are conjugate, and either* (i) $p \in \mathbb{Z}$ *or* (ii) $p/q \in \mathbb{Z}$, *then* $[\phi]_0 = [\psi]_0$.

Proof. The action J_ϕ satisfies condition PCF by Theorem 6.8. Then $[\phi]_0 = [\psi]_0$ by Theorem 6.6. ■

EXAMPLE 6.10. The class number of $F = \mathbb{Q}(\sqrt{235})$ is $h_F = 6$, the ring of integers in F is $\mathfrak{o}_F = \mathbb{Z}[\sqrt{235}]$, and the group of units in $\mathfrak{o}_F$ is $\mathfrak{o}_F^\times = \{\pm\varepsilon_F^k : k \in \mathbb{Z}\}$ where $\varepsilon_F = 46 + 3\sqrt{235}$ is the fundamental unit (see Appendices B and C in [22]). The $\phi, \psi \in \mathfrak{X}_F^0$ determined by

$$X_\phi = 2\,\frac{\partial}{\partial\theta_1} + (15 + \sqrt{235})\,\frac{\partial}{\partial\theta_2}, \qquad X_\psi = 42\,\frac{\partial}{\partial\theta_1} + (37 + \sqrt{235})\,\frac{\partial}{\partial\theta_2}$$

are both maximal F-algebraic: the $\mathrm{GL}(2,\mathbb{Z})$ matrices

$$B_1 = \begin{bmatrix} 1 & 6 \\ 15 & 91 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -65 & 126 \\ -81 & 157 \end{bmatrix}$$

satisfy $B_1X_\phi = \varepsilon_F X_\phi$ and $B_2X_\psi = \varepsilon_F X_\psi$. Since $\sqrt{235}\,\mathfrak{m}_\phi \subset \mathfrak{m}_\phi$, we have $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{o}_F$. Neither Lemma 3.5 nor Corollary 4.3 can be used to show that $[\phi]_0 = [\psi]_0$ because $\lambda_F(\mathfrak{o}_F^\times) \geq h_F(\mathfrak{O}(\mathfrak{m}_\phi)) = h_F = 6$. The $\mathrm{GL}(2,\mathbb{Z})$ matrix

$$C = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

satisfies $\Delta_C(B_2^{-1}) = B_1$. This implies that $\Delta_C(\Pi_\psi) = \Pi_\phi$ since $\Pi_\phi = \{\pm B_1^k : k \in \mathbb{Z}\}$ and $\Pi_\psi = \{\pm B_2^k : k \in \mathbb{Z}\}$. Thus, J_ϕ and J_ψ are conjugate. The frequencies $a_1 = 2$ and $a_2 = 15 + \sqrt{235}$ of ϕ have a ratio $a_2/a_1 = (15 + \sqrt{235})/2$, which is a root of the irreducible $z^2 - 15z + 5/2$. Corollary 6.9 now implies that $[\phi]_0 = [\psi]_0$. Indeed, the $\mathrm{GL}(2,\mathbb{Z})$ matrices C and

$$C' = \begin{bmatrix} 1 & 0 \\ 15 & -1 \end{bmatrix}$$

(where C' is as in the proof of Theorem 6.8) and $\vartheta = (16 - \sqrt{235})/21 \in F^\times$ satisfy $CC'X_\phi = \vartheta X_\psi$.

For irreducible actions associated to real algebraic number fields F of degree 3 or more, the scope of condition PCF seems somewhat limited. One notable exception is when the degree n of F is odd and bigger than or equal to 3. In this case, an action J_ϕ for $\phi \in \mathfrak{X}_F^0$ vacuously satisfies condition PCF when there is $B \in \Pi_\phi$ with an irreducible characteristic polynomial and X_ϕ as its only real eigenvector.

7. Existence of similar nonconjugate actions. The existence of similar nonconjugate actions of multiplier groups for F-algebraic quasiperiodic flows requires by Corollaries 5.5 and 5.6 that $\lambda_F(G) > 1$ for some $G \in U_F$. A partial converse of Corollary 5.6 gives one way by which the existence of similar nonconjugate actions is proved.

THEOREM 7.1. *Let $\phi \in \mathfrak{X}_F^0$. If J_ϕ satisfies condition PCF and $\lambda_F(M_\phi) > 1$, then there is $\psi \in \mathfrak{X}_F^0$ such that J_ϕ and J_ψ are similar but not conjugate.*

Proof. Supposing $\lambda_F(M_\phi) > 1$ implies there is $\psi \in \mathfrak{X}_F^0$ such that $M_\phi = M_\psi$ but $[\phi]_0 \neq [\psi]_0$. By Theorem 5.1, the actions J_ϕ and J_ψ are similar. Assuming that J_ϕ satisfies condition PCF implies by Theorem 6.6 that J_ϕ and J_ψ are not conjugate. ■

A partial converse of Corollary 5.5 gives another way to prove the existence of similar nonconjugate actions of multiplier groups. It is stated in Theorem 7.4 along with the comments that follow it. This partial converse is based on classical results of Latimer and MacDuffee [17], Taussky [32], [33], and Wallace [34] about bijections between classes of ideals in an order in F and $\mathrm{GL}(n,\mathbb{Z})$-conjugacy classes of matrices in $\mathrm{M}(n,\mathbb{Z})$, properties of the actions of multiplier groups, and the generalized Bowen–Franks groups. Recall that $\mathrm{SL}(n,\mathbb{Z})$ is the subgroup of $\mathrm{GL}(n,\mathbb{Z})$ consisting of those matrices with determinant 1. A matrix $B \in \mathrm{SL}(n,\mathbb{Z})$ is called *real hyperbolic* if the eigenvalues of B are real, distinct, and none has absolute value 1.

THEOREM 7.2 (Katok, Katok, Schmidt, p. 733 of [12]). *Suppose that $B \in \mathrm{SL}(n,\mathbb{Z})$ is real hyperbolic with an irreducible characteristic polynomial and its distinct real eigenvalues are $\varepsilon_1,\ldots,\varepsilon_n$, that $F = \mathbb{Q}(\varepsilon)$ where ε is one of the eigenvalues of B, and that $\mathfrak{o}_F = \mathbb{Z}[\varepsilon]$. If h_F is bigger than the number r of eigenvalues $\varepsilon_1,\ldots,\varepsilon_n$ that belong to F, then there exists $B' \in \mathrm{SL}(n,\mathbb{Z})$ having the same eigenvalues as B for which $Z(B)$ and $Z(B')$ are not conjugate subgroups of $\mathrm{GL}(n,\mathbb{Z})$.*

The requirement in Theorem 7.2 that the maximal order $\mathfrak{o}_F$ be the monogenic order $\mathbb{Z}[\varepsilon]$ is not needed in the proof since the bijection of Wallace (Theorem 2, p. 180 in [34]) used in the proof applies directly to the monogenic order $\mathbb{Z}[\varepsilon]$. Replacing h_F with $h_F(\mathfrak{D})$ where $\mathfrak{D} = \mathbb{Z}[\varepsilon]$ gives the existence of nonconjugate centralizers in $\mathrm{GL}(n,\mathbb{Z})$ of matrices $B, B' \in \mathrm{SL}(n,\mathbb{Z})$ with the same eigenvalues, i.e., the same characteristic polynomial, whenever $h_F(\mathfrak{D})$ is bigger than the number of eigenvalues of B lying in $F = \mathbb{Q}(\varepsilon)$.

LEMMA 7.3. *Let $\phi \in \mathfrak{X}_F^0$. If $B \in \Pi_\phi$ has an irreducible characteristic polynomial, then $\mathbb{Z}[\nu_\phi(B)]$ is a monogenic order in F for which $\mathfrak{O}(\mathfrak{m}_\phi) \supset \mathbb{Z}[\nu_\phi(B)]$.*

Proof. The eigenvalue $\nu_\phi(B)$ of B belongs to $M_\phi \subset \mathfrak{o}_F^\times \subset F$. Since $M_\phi = \mathfrak{O}(\mathfrak{m}_\phi)^\times$ by Theorem 3.3, it follows that $\nu_\phi(B)$ belongs to $\mathfrak{O}(\mathfrak{m}_\phi)$. Irreducibility of the characteristic polynomial of B implies that $\mathbb{Z}[\nu_\phi(B)]$ is an order in F, and $\nu_\phi(B)$ belonging to $\mathfrak{O}(\mathfrak{m}_\phi)$ implies that $\mathfrak{O}(\mathfrak{m}_\phi) \supset \mathbb{Z}[\nu_\phi(B)]$. ∎

THEOREM 7.4. *For $\phi \in \mathfrak{X}_F^0$ suppose there is a real hyperbolic matrix in $B \in \Pi_\phi$ with an irreducible characteristic polynomial and determinant 1, and let $\varepsilon = \nu_\phi(B)$. If $h_F(\mathbb{Z}[\varepsilon])$ is bigger than the number of eigenvalues of B that lie in F, then there exists $\psi \in \mathfrak{X}_F^0$ such that $\mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\psi) \supset \mathbb{Z}[\varepsilon]$ while J_ϕ and J_ψ are not conjugate.*

Proof. By Lemma 7.3 the module $\mathbb{Z}[\varepsilon]$ is an order in F for which $\mathfrak{O}(\mathfrak{m}_\phi) \supset \mathbb{Z}[\varepsilon]$. If $h_F(\mathbb{Z}[\varepsilon])$ is bigger than the number of eigenvalues of B that belong to F, then by Theorem 7.2 and the comments that followed, there is $B' \in \mathrm{SL}(n, \mathbb{Z})$ which has the same eigenvalues as B, but for which $Z(B)$ and $Z(B')$ are not conjugate subgroups of $\mathrm{GL}(n, \mathbb{Z})$. Corresponding to the eigenvalue ε of B' is an eigenvector whose components can be chosen to form a $\mathbb{Z}$-basis for an ideal $\mathfrak{n}$ in $\mathbb{Z}[\varepsilon]$ (see Proposition 7, p. 180 in [34]). By Theorem 2.10, there is a $\psi \in \mathfrak{X}_F^0$ such that $\mathfrak{m}_\psi = \mathfrak{n}$. Thus X_ψ is an eigenvector of B' corresponding to ε, and so $\varepsilon \in M_\psi$ and $B' \in \Pi_\psi$. Since the characteristic polynomial of B' is the same as that of B, it follows by Lemma 7.3 that $\mathfrak{O}(\mathfrak{m}_\psi) \supset \mathbb{Z}[\varepsilon]$. This means that $\mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\psi) \supset \mathbb{Z}[\varepsilon]$. Furthermore, $\Pi_\phi = Z(B)$ and $\Pi_\psi = Z(B')$ by Theorem 6.3, and so J_ϕ and J_ψ are not conjugate. ∎

Suppose, as in the conclusion of Theorem 7.4, that there are $\phi, \psi \in \mathfrak{X}_F^0$ with $\mathfrak{O}(\mathfrak{m}_\phi) \cap \mathfrak{O}(\mathfrak{m}_\psi) \supset \mathbb{Z}[\varepsilon]$ for $\varepsilon = \nu_\phi(B)$ where $B \in \Pi_\phi$, and that J_ϕ and J_ψ are not conjugate. It follows by Theorem 5.4 that $[\phi]_0 \neq [\phi]_0$. However, the modules $\mathfrak{m}_\phi, \mathfrak{m}_\psi \in \mathfrak{M}_F$ are both $\mathbb{Z}[\varepsilon]$-ideals. Their orders $\mathfrak{O}(\mathfrak{m}_\phi)$ and $\mathfrak{O}(\mathfrak{m}_\psi)$ lie in a finite lattice of orders, ordered by inclusion, with the monogenic $\mathbb{Z}[\varepsilon]$ at the bottom and the maximal order $\mathfrak{o}_F$ at the top (see Theorem 9.28, p. 77 in [9]). Determining when $\mathfrak{O}(\mathfrak{m}_\phi)$ and $\mathfrak{O}(\mathfrak{m}_\psi)$ are the same or different is the situation addressed in [21]. When $\mathbb{Z}[\varepsilon] \neq \mathfrak{o}_F$, the generalized Bowen–Franks groups provide a sufficient condition for detecting equality of the orders: by Theorem 4.2, if $BF_\alpha(\mathfrak{m}_\phi)$ is isomorphic to $BF_\alpha(\mathfrak{m}_\psi)$ for all nonzero $\alpha \in \mathbb{Z}[\varepsilon] \setminus \mathbb{Z}[\varepsilon]^\times$, then $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$. When $\mathbb{Z}[\varepsilon] = \mathfrak{o}_F$, it follows that $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$. Whenever $\mathfrak{O}(\mathfrak{m}_\phi) = \mathfrak{O}(\mathfrak{m}_\psi)$, Corollary 5.2 implies that J_ϕ and J_ψ are similar. However, it may happen that $\mathfrak{O}(\mathfrak{m}_\phi) \neq \mathfrak{O}(\mathfrak{m}_\psi)$ but $M_\phi = M_\psi$, so Theorem 5.1 implies that J_ϕ and J_ψ are similar.

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