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MEASURE-VALUED SOLUTIONS OF A HETEROGENEOUS CAHN–HILLIARD SYSTEM IN ELASTIC SOLIDS

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Abstract. The paper is concerned with the existence of measure-valued solutions to the Cahn–Hilliard system coupled with elasticity. The system under consideration is anisotropic and heterogeneous in the sense of admitting the elasticity and gradient energy tensors dependent on the order parameter. Such dependences introduce additional nonlinearities to the model for which the existence of weak solutions is not known so far.

1. Introduction. In recent years the Cahn–Hilliard problem coupled with elasticity has been the subject of extensive mathematical studies; we refer e.g. to Miranville (2003) and Bartkowiak and Pawłow (2005) for up-to-date references. The problem describes the phase separation process in a binary, deformable alloy quenched below a certain critical temperature. It is known from the materials science literature that the elastic effects strongly influence the microstructure evolution of the phase separation and that among important factors are the material anisotropies and heterogeneities. In view of that it is of importance to study the Cahn–Hilliard models accounting for such effects.

Applying a thermodynamical theory based on a microforce balance M. E. Gurtin (1996) introduced a generalized Cahn–Hilliard system coupled with elasticity. This system generalizes the classical Cahn–Hilliard equation by admitting its more general structure, chemical anisotropy and heterogeneity and the coupling with an elasticity system, in general anisotropic and heterogeneous as well. The anisotropies are represented by the matrix forms of the material coefficients whereas the heterogeneities by the coefficient dependence on the order parameter, a quantity describing the microstructure. In the case of binary phase separation the order parameter is related to the volumetric fraction of a phase.

From the mathematical point of view problems with anisotropic and heterogeneous effects lead to additional nonlinearities in the equations and make the analysis much more complicated. The main difficulty is in passing

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to the limit within nonlinearities. So far the Cahn–Hilliard system coupled with anisotropic, heterogeneous elasticity has been studied by Garcke (2000, 2003a, 2003b), and Bonetti et al. (2002). The existence results obtained in these papers are restricted to the quasi-stationary approximation of the elasticity system. We remark that the mathematical arguments applicable for the quasi-stationary elliptic elasticity do not extend to the nonstationary hyperbolic case.

The effects of chemical anisotropy and heterogeneity, to the best of the authors' knowledge, have not been much addressed so far. In this respect we mention the paper by Bonetti et al. (2002) where the order parameter dependence of the gradient coefficient representing surface tensions has been accounted for under certain structural simplifications. We also mention the paper by Bartkowiak and Pawłow (2005) which is concerned with the existence of weak solutions to the Cahn–Hilliard–Gurtin system coupled with nonstationary elasticity. We point out, however, that because of technical obstacles in passing to the limit within nonlinearities, the authors of the latter paper restricted themselves to a homogeneous problem with coefficients independent of the order parameter.

The present paper extends the results of Bartkowiak and Pawłow (2005) to the heterogeneous case by using a concept of measure-valued solutions (see Málek et al. (1996)). We use a special kind of measure-valued solution which was studied previously by Neustupa (1993) for barotropic flows, and by Kröner and Zajączkowski (1996) for Euler equations of compressible fluids.

The general Cahn–Hilliard–Gurtin system coupled with elasticity, with some prescribed initial and boundary conditions, can be represented in the form of the following system of problems for the displacement $\mathbf{u}$, order parameter χ and the chemical potential μ:

$$\begin{aligned} &\mathbf{u}_{tt} - \nabla\cdot W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi) = \mathbf{b} && \text{in } \Omega^T = \Omega\times(0,T),\\ &\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \mathbf{u}_t|_{t=0} = \mathbf{u}_1 && \text{in } \Omega,\\ &\mathbf{u} = \mathbf{0} && \text{on } S^T = S\times(0,T), \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\chi_t - \nabla\cdot(\mathbf{M}(\chi)\nabla w + \mathbf{h}\chi_t) = 0 && \text{in } \Omega^T,\\ &\chi|_{t=0} = \chi_0 && \text{in } \Omega,\\ &\mathbf{n}\cdot(\mathbf{M}(\chi)\nabla w + \mathbf{h}\chi_t) = 0 && \text{on } S^T, \end{aligned} \tag{1.2}$$

$$\begin{aligned} &\mu - \mathbf{g}\cdot\nabla\mu + \nabla\cdot(\boldsymbol{\Gamma}(\chi)\nabla\chi) - \tfrac12\nabla\chi\cdot\boldsymbol{\Gamma}'(\chi)\nabla\chi \\ &\qquad - \Psi'(\chi) - W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi) - \beta\chi_t = 0 && \text{in } \Omega^T,\\ &\mathbf{n}\cdot(\boldsymbol{\Gamma}(\chi)\nabla\chi) = 0 && \text{on } S^T, \end{aligned} \tag{1.3}$$

where the function $W(\boldsymbol{\varepsilon}(\mathbf{u}),\chi)$ represents the elastic energy, defined by

$$W(\boldsymbol{\varepsilon}(\mathbf{u}),\chi)=\tfrac{1}{2}(\boldsymbol{\varepsilon}(\mathbf{u})-\overline{\boldsymbol{\varepsilon}}(\chi))\cdot\mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u})-\overline{\boldsymbol{\varepsilon}}(\chi)).\tag{1.4}$$

The corresponding derivatives

$$W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi)=\mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u})-\overline{\boldsymbol{\varepsilon}}(\chi))$$

and

$$W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}),\chi)=-\overline{\boldsymbol{\varepsilon}}'(\chi)\cdot\mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u})-\overline{\boldsymbol{\varepsilon}}(\chi))+\tfrac{1}{2}(\boldsymbol{\varepsilon}(\mathbf{u})-\overline{\boldsymbol{\varepsilon}}(\chi))\cdot\mathbf{A}'(\chi)(\boldsymbol{\varepsilon}(\mathbf{u})-\overline{\boldsymbol{\varepsilon}}(\chi))$$

represent respectively the stress tensor and the elastic contribution to the chemical potential.

Above, $\Omega\subset\mathbb{R}^n,\ n=2$ or 3, is a bounded domain with smooth boundary S, occupied by a solid body in a reference configuration, with constant mass density $\varrho=1$; $\mathbf{n}$ denotes the outward unit normal to S; $T>0$ is an arbitrary fixed time.

The unknown variables are the displacement field $\mathbf{u}:\Omega^T\to\mathbb{R}^n$, the scalar order parameter $\chi:\Omega^T\to\mathbb{R}$, and the chemical potential difference between the components (briefly referred to as the chemical potential) $\mu:\Omega^T\to\mathbb{R}$. In the case of a binary a-b alloy the order parameter is related to the volumetric fraction of one of the two phases, characterized by different crystalline structures of the components, for example $\chi=-1$ corresponds to phase a and $\chi=1$ to phase b. The second order symmetric tensor

$$\boldsymbol{\varepsilon}(\mathbf{u})=\tfrac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)$$

denotes the linearized strain (for simplicity we write $\boldsymbol{\varepsilon}$ instead of $\boldsymbol{\varepsilon}(\mathbf{u})$), and $\mathbf{b}:\Omega^T\to\mathbb{R}^n$ is the external body force.

The free energy density underlying system (1.3)–(1.5) has the Landau–Ginzburg–Cahn–Hilliard form accounting for the elastic effects,

$$f(\boldsymbol{\varepsilon},\chi,\nabla\chi)=W(\boldsymbol{\varepsilon},\chi)+\Psi(\chi)+\tfrac{1}{2}\nabla\chi\cdot\boldsymbol{\Gamma}(\chi)\nabla\chi,\tag{1.5}$$

where $W(\boldsymbol{\varepsilon},\chi)$ is the homogeneous elastic energy, $\Psi(\chi)$ is the exchange energy, and the last term with the positive definite tensor $\boldsymbol{\Gamma}(\chi)=(\Gamma_{ij}(\chi))$ is the gradient energy.

The standard form of the elastic energy $W(\boldsymbol{\varepsilon},\chi)$ is given by (1.4) where $\mathbf{A}(\chi)=(A_{ijkl}(\chi))$ is the fourth order elasticity tensor depending on the order parameter, and $\overline{\boldsymbol{\varepsilon}}(\chi)=(\overline{\varepsilon}_{ij}(\chi))$ is the symmetric eigenstrain tensor, i.e. a stress free strain at concentration χ. The exchange energy $\Psi(\chi)$ characterizes the energetic favorability of the individual phases a and b. The standard form is a double-well potential with equal minima at $\chi=-1$ and $\chi=1$:

$$\Psi(\chi)=\tfrac{1}{2}(1-\chi^2)^2.\tag{1.6}$$

Furthermore, $\mathbf{M}(\chi)=(M_{ij}(\chi))$ is the mobility matrix, $\beta\geq 0$ is the diffusional viscosity, and the vectors $\mathbf{g}=(g_i)$, $\mathbf{h}=(h_i)$ represent the cross-coupling effects; for usual isotropic materials $\mathbf{g}=\mathbf{0}$ and $\mathbf{h}=\mathbf{0}$.

By thermodynamical consistency the coefficient matrix

$$\mathbf{B} = \begin{bmatrix} \mathbf{M} & \mathbf{h} \\ \mathbf{g}^T & \beta \end{bmatrix} \tag{1.7}$$

has to satisfy the condition

$$\mathbf{X} \cdot \mathbf{B}\mathbf{X} \geq 0 \quad \forall \mathbf{X} = (\nabla \mu, \chi_t) \in \mathbb{R}^n \times \mathbb{R}. \tag{1.8}$$

If $\mathbf{B}$ is independent of $\mathbf{X}$ then (1.8) means the positive semi-definiteness of $\mathbf{B}$. In general, however, the quantities $\mathbf{M}, \mathbf{g}, \mathbf{h}, \beta$ may depend on $\nabla \mu, \chi_t, \boldsymbol{\varepsilon}, \chi$. Throughout this paper we shall assume that $\mathbf{M} = \mathbf{M}(\chi)$ is positive definite, and for simplicity we restrict ourselves to the special case (standard Cahn–Hilliard case)

$$\mathbf{g} = \mathbf{h} = \mathbf{0} \quad \text{and} \quad \beta = 0. \tag{1.9}$$

Later on we shall refer to the system (1.1)–(1.3) with structural simplifications (1.9) as *problem* (P_0).

In the heterogeneous case the gradient energy tensor $\mathbf{\Gamma}(\chi)$ and the elasticity tensor $\mathbf{A}(\chi)$ may be different in each of the phases, i.e., dependent on the order parameter χ. As already mentioned, the present paper is an extension of Bartkowiak and Pawłow (2005) where the existence of weak solutions has been proved for problem (P_0) and its more general version (1.1)–(1.3) only in the homogeneous case with constant tensors $\mathbf{\Gamma}$ and $\mathbf{A}$. The obstacle we were not able to overcome to establish the existence of weak solutions in the heterogeneous case was the lack of sufficiently strong a priori estimates. In fact, the χ dependence of $\mathbf{\Gamma}(\chi)$ and $\mathbf{A}(\chi)$ introduces to the field equations the nonlinear energy-like terms

$$\tfrac{1}{2}\nabla\chi \cdot \mathbf{\Gamma}'(\chi)\nabla\chi \quad \text{and} \quad \tfrac{1}{2}(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \cdot \mathbf{A}'(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \tag{1.10}$$

which are the source of mathematical difficulties. For these terms we are able to show only $L_\infty(0,T;L_1(\Omega))$-norm energy estimates which are not sufficient to prove the existence of weak solutions by passing to the limit in approximate problems.

For this reason in the present paper we propose a weaker, measure-valued sense in which the equations of (P_0) are satisfied. The idea of measure-valued solutions is taken from the papers by Neustupa (1993) and Kröner and Zajączkowski (1996) where the notion of a measure-valued solution was applied to the Euler and Navier–Stokes equations. The underlying idea is, roughly speaking, to assume that all quantities in the weak formulation of the problem are associated with some sets $E \subset \overline{\Omega}$ (E is assumed to be a Borel set) rather than points $\mathbf{x} \in \overline{\Omega}$.

The paper is organized as follows. In Section 2 we introduce a weak formulation of problem (P_0) and its measure-valued generalization. On the basis of this generalization we define a measure-valued solution of (P_0). In

Section 3 we formulate the assumptions and the main result of the paper which asserts the existence of measure-valued solutions to problem (P_0). In Section 4 we present the proof of the existence theorem. It is based on the Faedo–Galerkin approximation of (P_0) studied in Bartkowiak and Pawłow (2005). We use the following notations:

- $\mathbf{x} \in \mathbb{R}^n$, $n = 2$ or $n = 3$, is the material point; $f_{,i} = \partial f/\partial x_i$ and $f_t = df/dt$ are the material space and time derivatives,
- $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{i,j=1,\ldots,n}$,
- $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi) = (\partial W(\boldsymbol{\varepsilon},\chi)/\partial\varepsilon_{ij})_{i,j=1,\ldots,n}$, $W_{,\chi}(\boldsymbol{\varepsilon},\chi) = \partial W(\boldsymbol{\varepsilon},\chi)/\partial\chi$,
- $\boldsymbol{\Gamma}'(\chi) = (\Gamma'_{ij}(\chi))_{i,j=1,\ldots,n}$, $\Gamma'_{ij}(\chi) = d\Gamma_{ij}(\chi)/d\chi$.

For simplicity, whenever there is no danger of confusion, we omit the arguments $(\boldsymbol{\varepsilon},\chi)$. The specification of tensor indices is omitted as well.

Vector and tensor-valued mappings are denoted by bold letters. The summation convention over repeated indices is used, and we apply the following notation:

- for vectors $\mathbf{a} = (a_i)$, $\widetilde{\mathbf{a}} = (\widetilde{a}_i)$, and tensors $\mathbf{B} = (B_{ij})$, $\widetilde{\mathbf{B}} = (\widetilde{B}_{ij})$, $\mathbf{A} = (A_{ijkl})$ we write $\mathbf{a}\cdot\widetilde{\mathbf{a}} = a_i\widetilde{a}_i$, $\mathbf{B}\cdot\widetilde{\mathbf{B}} = B_{ij}\widetilde{B}_{ij}$, $\mathbf{AB} = (A_{ijkl}B_{kl})$, $\mathbf{BA} = (B_{ij}A_{ijkl})$,
- $|\mathbf{a}| = (a_ia_i)^{1/2}$, $|\mathbf{B}| = (B_{ij}B_{ij})^{1/2}$,
- ∇ and $\nabla\cdot$ denote the gradient and the divergence operators with respect to the material point $\mathbf{x} \in \mathbb{R}^n$. For the divergence of a tensor field we use contraction over the last index, $\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{x}) = (\varepsilon_{ij,j}(\mathbf{x}))$.

We apply the standard Sobolev spaces notation $H^m(\Omega) = W_2^m(\Omega)$ for $m \in \mathbb{N}$, and:

- $\mathbf{L}_2(\Omega) = (L_2(\Omega))^n$, $\mathbf{V}_0 = \mathbf{H}_0^1(\Omega) = (H_0^1(\Omega))^n$, $n = 2$ or 3.
- $(\cdot,\cdot)_{L_2(\Omega)}$, $(\cdot,\cdot)_{\mathbf{L}_2(\Omega)}$ denote the scalar products in $L_2(\Omega)$ and $\mathbf{L}_2(\Omega)$.
- V' is the dual space of $V = H^1(\Omega)$ with duality pairing $\langle\cdot,\cdot\rangle_{V',V}$.
- $\mathbf{V}_0'$ is the dual of $\mathbf{V}_0$ with duality pairing $\langle\cdot,\cdot\rangle_{\mathbf{V}_0',\mathbf{V}_0}$.
- $C_c^\infty([0,T))$ is the space of smooth functions with compact support in $[0,T)$.

Throughout the paper c denotes a generic positive constant different in various instances.

2. Weak and measure-valued formulations. We introduce the following weak formulation of problem (P_0): Find functions $\mathbf{u} : \Omega^T \to \mathbb{R}^n$, $\chi : \Omega^T \to \mathbb{R}$ and $\mu : \Omega^T \to \mathbb{R}$ defined a.e. in Ω^T, satisfying

$$(2.1)\qquad \int_0^T \vartheta_1''(t)\int_\Omega \mathbf{u}\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx\,dt + \int_0^T \vartheta_1(t)\int_\Omega \mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi))\cdot\boldsymbol{\varepsilon}(\boldsymbol{\eta}_0(\mathbf{x}))\,dx\,dt$$

$$= \int_0^T \vartheta_1(t) \int_\Omega \mathbf{b}\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx\,dt + \int_\Omega \vartheta_1(0)\mathbf{u}_1\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx - \int_\Omega \vartheta_1'(0)\mathbf{u}_0\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx$$
$$\forall \boldsymbol{\eta}_0 \in (C^\infty(\overline{\Omega}))^3,\ \vartheta_1 \in C_c^\infty([0,T)) \text{ with } \boldsymbol{\eta}_0|_S = \mathbf{0},$$

where the form of the second term on the left-hand side follows from the symmetry of $\mathbf{A}$;

$$\text{(2.2)}\qquad -\int_0^T \vartheta_2'(t)\int_\Omega \chi\xi_0(\mathbf{x})\,dx\,dt + \int_0^T \vartheta_2(t)\int_\Omega \mathbf{M}(\chi)\nabla\mu\cdot\nabla\xi_0(\mathbf{x})\,dx\,dt$$
$$= \int_\Omega \vartheta_2(0)\chi_0\xi_0(\mathbf{x})\,dx \qquad \forall \xi_0 \in C^\infty(\overline{\Omega}),\ \vartheta_2 \in C_c^\infty([0,T));$$

$$\text{(2.3)}\qquad \int_0^T \vartheta_3(t)\int_\Omega \mu\zeta_0(\mathbf{x})\,dx\,dt - \int_0^T \vartheta_3(t)\int_\Omega \boldsymbol{\Gamma}(\chi)\nabla\chi\cdot\nabla\zeta_0(\mathbf{x})\,dx\,dt$$
$$- \frac{1}{2}\int_0^T \vartheta_3(t)\int_\Omega \nabla\chi\cdot\boldsymbol{\Gamma}'(\chi)\nabla\chi\ \zeta_0(\mathbf{x})\,dx\,dt - \int_0^T \vartheta_3(t)\int_\Omega \Psi'(\chi)\zeta_0(\mathbf{x})\,dx\,dt$$
$$+ \int_0^T \vartheta_3(t)\int_\Omega \overline{\boldsymbol{\varepsilon}}'(\chi)\cdot\mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi))\zeta_0(\mathbf{x})\,dx\,dt$$
$$- \frac{1}{2}\int_0^T \vartheta_3(t)\int_\Omega (\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi))\cdot\mathbf{A}'(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi))\zeta_0(\mathbf{x})\,dx\,dt = 0$$
$$\forall \zeta_0 \in C^\infty(\overline{\Omega}),\ \vartheta_3 \in C^\infty([0,T]).$$

We now introduce a measure-valued generalization of (2.1)–(2.3). To this end we apply the ideas and notations from Neustupa (1993) adapted to our setting. Assume for a while that we already know a weak solution to (2.1)–(2.3) which provides functions $\mathbf{u}$, χ, μ and their spatial gradients $\nabla\mathbf{u}$, $\nabla\chi$, $\nabla\mu$. The idea underlying a measure-valued generalization of (2.1)–(2.3) is to assume that the quantities ($\mathbf{u}$, $\boldsymbol{\varepsilon}(\mathbf{u})$, χ, $\nabla\chi$, μ, $\nabla\mu$) are connected with some sets in $\overline{\Omega}$ rather than points. Then at time $t \in (0,T)$ the quantities $(\mathbf{u}, \boldsymbol{\varepsilon}(\mathbf{u}), \chi, \nabla\chi, \mu, \nabla\mu)$ assigned to a set $E \subset \overline{\Omega}$ represent collections of points from the space

$$\mathbb{R}^M \equiv \mathbb{R}^n \times \mathbb{R}^{n^2} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n.$$

Denote a point in $\mathbb{R}^M$ by

$$\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\mathbf{u}}, \boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})}, \lambda_\chi, \boldsymbol{\lambda}_{\nabla\chi}, \lambda_\mu, \boldsymbol{\lambda}_{\nabla\mu}) \in \mathbb{R}^M.$$

Let $E \subset \overline{\Omega}$ and $I \subset \mathbb{R}^M$ be Borel sets. The collection of unknowns associated with $E \subset \overline{\Omega}$ can be characterized by a measure $\nu_{t,E}$ on $\mathbb{R}^M$ such that if I is a Borel set in $\mathbb{R}^M$ then $\nu_{t,E}(I)$ is the Lebesgue measure of the subset E' of

E consisting of the points $\mathbf{x}$ such that

$$(\mathbf{u}(\mathbf{x},t),\boldsymbol{\varepsilon}(\mathbf{u})(\mathbf{x},t),\chi(\mathbf{x},t),\nabla\chi(\mathbf{x},t),\mu(\mathbf{x},t),\nabla\mu(\mathbf{x},t))\in I.$$

Denoting by $\delta_{[\mathbf{x},t]}$ the Dirac measure in $\mathbb{R}^M$ with support at $(\mathbf{u}(\mathbf{x},t),$ $\boldsymbol{\varepsilon}(\mathbf{u})(\mathbf{x},t),\chi(\mathbf{x},t),\nabla\chi(\mathbf{x},t),\mu(\mathbf{x},t),\nabla\mu(\mathbf{x},t))$, and by $\varkappa_{E'}$ the characteristic function of E', we write

$$\varkappa_{E'}(\mathbf{x})=\int_I d\delta_{[\mathbf{x},t]}(\boldsymbol{\lambda})\quad\text{ for a.e. }\mathbf{x}\in\Omega,$$

and

$$\nu_{t,E}(I)=\int_E \varkappa_{E'}(\mathbf{x})\,dx=\int_E\int_I d\delta_{[\mathbf{x},t]}(\boldsymbol{\lambda})\,dx. \tag{2.4}$$

The measure $\nu_{t,E}(I)$ can be viewed as a value of a set function defined on subsets of $\overline{\Omega}\times\mathbb{R}^M$ of type $E\times I$ with E and I being Borel sets in $\overline{\Omega}$ and $\mathbb{R}^M$, respectively. Such a function has a unique extension to a nonnegative regular measure ν_t on $\overline{\Omega}\times\mathbb{R}^M$ such that $\nu_t(E\times I)=\nu_{t,E}(I)$ for all Borel sets E in $\overline{\Omega}$ and I in $\mathbb{R}^M$.

Making use of the introduced measure $\nu_t(\mathbf{x},\boldsymbol{\lambda})$ we now reformulate (2.1)–(2.3). To this end we note that the integrands in (2.1)–(2.3) contain terms of the type

$$\int_\Omega f(\mathbf{u},\boldsymbol{\varepsilon}(\mathbf{u}),\chi,\nabla\chi,\mu,\nabla\mu)\xi(\mathbf{x})\,dx. \tag{2.5}$$

The function $f(\mathbf{u},\boldsymbol{\varepsilon}(\mathbf{u}),\chi,\nabla\chi,\mu,\nabla\mu)$ with $(\mathbf{u},\boldsymbol{\varepsilon}(\mathbf{u}),\chi,\nabla\chi,\mu,\nabla\mu)$ evaluated at $(\mathbf{x},t)$ can be expressed in the form

$$\begin{aligned}f(\mathbf{u}(\mathbf{x},t),\boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x},t)),\chi(\mathbf{x},t),\nabla\chi(\mathbf{x},t),\mu(\mathbf{x},t),\nabla\mu(\mathbf{x},t))&\\ =\int_{\mathbb{R}^M} f(\boldsymbol{\lambda}_{\mathbf{u}},\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})},\lambda_\chi,\boldsymbol{\lambda}_{\nabla\chi},\lambda_\mu,\boldsymbol{\lambda}_{\nabla\mu})\,d\delta_{[\mathbf{x},t]}(\boldsymbol{\lambda}).&\end{aligned}$$

Hence, the integral in (2.5) is equal to

$$\int_\Omega\int_{\mathbb{R}^M} f(\boldsymbol{\lambda}_{\mathbf{u}},\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})},\lambda_\chi,\boldsymbol{\lambda}_{\nabla\chi},\lambda_\mu,\boldsymbol{\lambda}_{\nabla\mu})\xi(\mathbf{x})\,d\delta_{[\mathbf{x},t]}(\boldsymbol{\lambda})\,dx,$$

and due to (2.4), it can also be expressed as

$$\int_\Omega\int_{\mathbb{R}^M} f(\boldsymbol{\lambda}_{\mathbf{u}},\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})},\lambda_\chi,\boldsymbol{\lambda}_{\nabla\chi},\lambda_\mu,\boldsymbol{\lambda}_{\nabla\mu})\xi(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda}). \tag{2.6}$$

Consequently, we can write the integral identities (2.1)–(2.3) in the form

$$\begin{aligned}&\int_0^T\vartheta_1''(t)\int_\Omega\int_{\mathbb{R}^M}\boldsymbol{\lambda}_{\mathbf{u}}\cdot\boldsymbol{\eta}_0(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt\\ &\qquad+\int_0^T\vartheta_1(t)\int_\Omega\int_{\mathbb{R}^M}\mathbf{A}(\lambda_\chi)(\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})}-\overline{\boldsymbol{\varepsilon}}(\lambda_\chi))\cdot\boldsymbol{\varepsilon}(\boldsymbol{\eta}_0(\mathbf{x}))\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt\end{aligned} \tag{2.7}$$

$$= \int_0^T \vartheta_1(t) \int_\Omega \mathbf{b}\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx\,dt + \int_\Omega \vartheta_1(0)\mathbf{u}_1\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx$$
$$- \int_\Omega \vartheta_1'(0)\mathbf{u}_0\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx$$
$$\forall \boldsymbol{\eta}_0 \in (C^\infty(\overline{\Omega}))^3,\ \vartheta_1 \in C_c^\infty([0,T)) \text{ with } \boldsymbol{\eta}_0|_S = \mathbf{0};$$

$$-\int_0^T \vartheta_2'(t) \int_\Omega \int_{\mathbb{R}^M} \lambda_\chi \xi_0(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt \tag{2.8}$$
$$+ \int_0^T \vartheta_2(t) \int_\Omega \int_{\mathbb{R}^M} \mathbf{M}(\lambda_\chi)\boldsymbol{\lambda}_{\nabla\mu}\cdot\nabla\xi_0(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt$$
$$= \int_\Omega \vartheta_2(0)\chi_0\xi_0(\mathbf{x})\,dx \qquad \forall \xi_0 \in C^\infty(\overline{\Omega}),\ \vartheta_2 \in C_c^\infty([0,T));$$

and

$$\int_0^T \vartheta_3(t) \int_\Omega \int_{\mathbb{R}^M} \lambda_w \zeta_0(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt \tag{2.9}$$
$$- \int_0^T \vartheta_3(t) \int_\Omega \int_{\mathbb{R}^M} \boldsymbol{\Gamma}(\lambda_\chi)\boldsymbol{\lambda}_{\nabla\chi}\cdot\nabla\zeta_0(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt$$
$$- \frac{1}{2}\int_0^T \vartheta_3(t) \int_\Omega \int_{\mathbb{R}^M} \boldsymbol{\lambda}_{\nabla\chi}\cdot\boldsymbol{\Gamma}'(\lambda_\chi)\boldsymbol{\lambda}_{\nabla\chi}\zeta_0(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt$$
$$- \int_0^T \vartheta_3(t) \int_\Omega \int_{\mathbb{R}^M} \Psi'(\lambda_\chi)\zeta_0(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt$$
$$+ \int_0^T \vartheta_3(t) \int_\Omega \int_{\mathbb{R}^M} \overline{\boldsymbol{\varepsilon}}'(\lambda_\chi)\cdot\mathbf{A}(\lambda_\chi)(\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})} - \overline{\boldsymbol{\varepsilon}}(\lambda_\chi))\zeta_0(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt$$
$$- \frac{1}{2}\int_0^T \vartheta_3(t) \int_\Omega \int_{\mathbb{R}^M} (\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})} - \overline{\boldsymbol{\varepsilon}}(\lambda_\chi))$$
$$\cdot\,\mathbf{A}'(\lambda_\chi)(\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})} - \overline{\boldsymbol{\varepsilon}}(\lambda_\chi))\zeta_0(\mathbf{x})\,d\nu_t(\mathbf{x},\boldsymbol{\lambda})\,dt = 0$$
$$\forall \zeta_0 \in C^\infty(\overline{\Omega}),\ \vartheta_3 \in C^\infty([0,T]).$$

Let Y be the linear space of all functions on $\mathbb{R}^M$ which can be expressed in the form

$$f(\boldsymbol{\lambda}_{\mathbf{u}}, \boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})}, \lambda_\chi, \boldsymbol{\lambda}_{\nabla\chi}, \lambda_\mu, \boldsymbol{\lambda}_{\nabla\mu}) = \mathbf{d}_1\cdot\boldsymbol{\lambda}_{\mathbf{u}} + \mathbf{B}\cdot\mathbf{A}(\lambda_\chi)(\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})} - \overline{\boldsymbol{\varepsilon}}(\lambda_\chi)) + a_1\lambda_\chi + \mathbf{d}_2\cdot\mathbf{M}(\lambda_\chi)\boldsymbol{\lambda}_{\nabla\mu} \tag{2.10}$$

$$+ a_2\lambda_w + \mathbf{d}_3 \cdot \mathbf{\Gamma}(\lambda_\chi)\boldsymbol{\lambda}_{\nabla\chi}$$
$$+ a_3\Psi'(\lambda_\chi) + a_4\overline{\boldsymbol{\varepsilon}}'(\lambda_\chi) \cdot \mathbf{A}(\lambda_\chi)(\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})} - \overline{\boldsymbol{\varepsilon}}(\lambda_\chi))$$
$$+ a_5\boldsymbol{\lambda}_{\nabla\chi} \cdot \mathbf{\Gamma}'(\lambda_\chi)\boldsymbol{\lambda}_{\nabla\chi}$$
$$+ a_6(\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})} - \overline{\boldsymbol{\varepsilon}}(\lambda_\chi)) \cdot \mathbf{A}'(\lambda_\chi)(\boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})} - \overline{\boldsymbol{\varepsilon}}(\lambda_\chi)),$$

where $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 \in \mathbb{R}^n, \mathbf{B} \in \mathbb{R}^{n^2}$ is a symmetric second order tensor, and $a_1, \ldots, a_6 \in \mathbb{R}$.

We can now introduce

Definition 2.1. Assume that $\mathbf{u}_0$, $\mathbf{u}_1$, χ_0 are given functions defined a.e. in Ω, and $\mathbf{b}$ is a given function defined a.e. in Ω^T, such that $\mathbf{u}_0, \mathbf{u}_1 \in L_1(\Omega)$, $\chi_0 \in L_1(\Omega)$, $\mathbf{b} \in \mathbf{L}_1(\Omega^T)$. By a *measure-valued solution* of problem (P_0) we mean a mapping assigning to a.e. $t \in (0,T)$ a regular nonnegative Borel measure ν_t on $\overline{\Omega} \times \mathbb{R}^M$ such that

$$\int_\Omega \int_{\mathbb{R}^M} |f(\boldsymbol{\lambda}_\mathbf{u}, \boldsymbol{\lambda}_{\boldsymbol{\varepsilon}(\mathbf{u})}, \lambda_\chi, \boldsymbol{\lambda}_{\nabla\chi}, \lambda_\mu, \boldsymbol{\lambda}_{\nabla\mu})|\, d\nu_t(\mathbf{x}, \boldsymbol{\lambda}) < \infty \quad \forall f \in Y, \tag{2.11}$$

and the identities (2.7)–(2.9) are satisfied for all test functions $\boldsymbol{\eta}_0 \in (C^\infty(\overline{\Omega}))^3$, $\vartheta_1 \in C_c^\infty([0,T))$ with $\boldsymbol{\eta}_0|_S = \mathbf{0}$, $\xi_0 \in C^\infty(\overline{\Omega})$, $\vartheta_2 \in C_c^\infty([0,T))$, and $\zeta_0 \in C^\infty(\overline{\Omega})$, $\vartheta_3 \in C^\infty([0,T])$.

Definition 2.2. For a given $t \in (0,T)$ a measure-valued solution ν_t is *continuous with respect to the Lebesgue measure* m in Ω (briefly, *m-continuous*) if $m(E) = 0$ implies $\nu_t(E \times \mathbb{R}^M) = 0$ for each Borel set $E \subset \Omega$.

We recall here the following property of m-continuous measure-valued solutions. If ν_t is m-continuous for a.a. $t \in (0,T)$ then for a.a. $(\mathbf{x}, t) \in \Omega^T$ there exists a Borel measure $\varepsilon_{[\mathbf{x},t]}$ on $\mathbb{R}^M$ so that if g is a ν_t-integrable function on $\Omega \times \mathbb{R}^M$ then

$$\int_\Omega \int_{\mathbb{R}^M} g(\mathbf{x}, \boldsymbol{\lambda})\, d\nu_t(\mathbf{x}, \boldsymbol{\lambda}) = \int_\Omega \int_{\mathbb{R}^M} g(\mathbf{x}, \boldsymbol{\lambda})\, d\varepsilon_{[\mathbf{x},t]}(\boldsymbol{\lambda})\, dx \quad \text{for a.a. } t \in (0,T).$$

If in addition $\varepsilon_{[\mathbf{x},t]}$ is the Dirac measure with support at the point

$$\boldsymbol{\lambda}(\mathbf{x}, t) = (\mathbf{u}(\mathbf{x}, t), \boldsymbol{\varepsilon}(\mathbf{u})(\mathbf{x}, t), \chi(\mathbf{x}, t), \nabla\chi(\mathbf{x}, t), \mu(\mathbf{x}, t), \nabla\mu(\mathbf{x}, t)) \in \mathbb{R}^M$$

for a.a. $(\mathbf{x}, t) \in \Omega^T$, then

$$\int_\Omega \int_{\mathbb{R}^M} g(\mathbf{x}, \boldsymbol{\lambda})\, d\nu_t(\mathbf{x}, \boldsymbol{\lambda})$$
$$= \int_\Omega g(\mathbf{x}, \mathbf{u}(\mathbf{x}, t), \boldsymbol{\varepsilon}(\mathbf{u})(\mathbf{x}, t), \chi(\mathbf{x}, t), \nabla\chi(\mathbf{x}, t), \mu(\mathbf{x}, t), \nabla\mu(\mathbf{x}, t))\, dx.$$

3. The existence of measure-valued solutions to problem (P_0). System (1.1)–(1.3) (and its special case (P_0)) was studied in Bartkowiak and Pawłow (2005) by means of the Faedo–Galerkin approximation. The existence results obtained there for approximate problems covered the heterogeneous case. The restriction to the homogeneous case was needed to pass to the limit in the weak formulation of approximate problems. It was shown there that the approximate problem corresponding to (1.1)–(1.3) in the heterogeneous case, i.e. with tensors $\mathbf{\Gamma}$ and $\mathbf{A}$ depending on χ, has a solution. However, in order to pass to the limit in the weak formulations of the approximate problems it was necessary to impose the assumption that $\mathbf{\Gamma}$ and $\mathbf{A}$ were constant.

In the present paper we apply the same Faedo–Galerkin approximation to prove the existence of measure-valued solutions in the heterogeneous case. First, we recall from Bartkowiak and Pawłow (2005) the corresponding assumptions.

(A1) The domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is bounded with smooth boundary S.

The subsequent assumptions concern the ingredients of the Landau–Ginzburg free energy

$$f(\boldsymbol{\varepsilon}, \chi, \nabla\chi) : \mathcal{S}^2 \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

given by (1.5), where $\mathcal{S}^2$ denotes the set of symmetric second order tensors in $\mathbb{R}^n$.

(A2) The elasticity tensor $\mathbf{A}(\chi) = (A_{ijkl}(\chi)) : \mathcal{S}^2 \to \mathcal{S}^2$:

- (i) is a linear mapping, of class $C^{1,1}$ with respect to χ: $A_{ijkl}(\cdot) \in C^1(\mathbb{R})$ with $A'_{ijkl}(\cdot)$ Lipschitz continuous,
- (ii) satisfies the symmetry conditions $A_{ijkl}(\cdot) = A_{jikl}(\cdot) = A_{klij}(\cdot)$,
- (iii) is positive definite and bounded uniformly with respect to χ: there exist constants $0 < \underline{c}_A < \overline{c}_A$ such that

$$\underline{c}_A|\boldsymbol{\varepsilon}|^2 \le \boldsymbol{\varepsilon} \cdot \mathbf{A}(\chi)\boldsymbol{\varepsilon} \le \overline{c}_A|\boldsymbol{\varepsilon}|^2 \quad \forall \boldsymbol{\varepsilon} \in \mathcal{S}^2 \text{ and } \chi \in \mathbb{R},$$

- (iv) is such that $\mathbf{A}'(\chi) = (A'_{ijkl}(\chi)) : \mathcal{S}^2 \to \mathcal{S}^2$ is uniformly bounded with respect to χ: there exists a constant $c_{A'} > 0$ such that

$$|\mathbf{A}'(\chi)\boldsymbol{\varepsilon}| \le c_{A'}|\boldsymbol{\varepsilon}| \quad \forall \boldsymbol{\varepsilon} \in \mathcal{S}^2 \text{ and } \chi \in \mathbb{R}.$$

We remark that we do not require that $\mathbf{A}(\chi)$ is isotropic.

(A3) The eigenstrain $\overline{\boldsymbol{\varepsilon}}(\chi) = (\overline{\varepsilon}_{ij}(\chi)) \in \mathcal{S}^2$:

- (i) is of class $C^{1,1}$ with respect to χ: $\overline{\varepsilon}_{ij}(\cdot) \in C^1(\mathbb{R})$ with $\overline{\varepsilon}'_{ij}(\cdot)$ Lipschitz continuous,
- (ii) satisfies growth conditions: there exists a constant $c > 0$ such that

$$|\overline{\boldsymbol{\varepsilon}}(\chi)| \le c(|\chi| + 1), \quad |\overline{\boldsymbol{\varepsilon}}'(\chi)| \le c \quad \forall \chi \in \mathbb{R}.$$

In view of (1.4) assumptions (A2), (A3) imply that the functions $W(\boldsymbol{\varepsilon},\chi)$, $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi)$, $W_{,\chi}(\boldsymbol{\varepsilon},\chi)$ are Lipschitz continuous with respect to $\boldsymbol{\varepsilon}$, χ, and satisfy the following growth conditions:

$$|W(\boldsymbol{\varepsilon},\chi)| \le c(|\boldsymbol{\varepsilon}|^2 + |\chi|^2 + 1), \tag{3.1}$$

$$|W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon},\chi)| \le c(|\boldsymbol{\varepsilon}| + |\chi| + 1), \tag{3.2}$$

$$|W_{,\chi}(\boldsymbol{\varepsilon},\chi)| \le c(|\boldsymbol{\varepsilon}|^2 + |\chi|^2 + 1), \quad \forall (\boldsymbol{\varepsilon},\chi) \in \mathcal{S}^2 \times \mathbb{R}. \tag{3.3}$$

(A4) The double-well potential $\Psi(\cdot) : \mathbb{R} \to \mathbb{R}$:

(i) is of class $C^{1,1}$: $\Psi(\cdot) \in C^1(\mathbb{R})$ with $\Psi'(\cdot)$ Lipschitz continuous,

(ii) satisfies a bound from below: there exist constants $c_1 > 0$, $c_2 \ge 0$ and a number $r > 2$ such that

$$\Psi(\chi) \ge c_1|\chi|^r - c_2 \quad \forall \chi \in \mathbb{R},$$

(iii) satisfies growth conditions: there exists a constant $c > 0$ such that

$$\Psi(\chi) \le c(|\chi|^{q/2+1} + 1), \quad \Psi'(\chi) \le c(|\chi|^{q/2} + 1), \quad \forall \chi \in \mathbb{R},$$

where $q \in [1,\infty)$ for $n = 2$, $q \in [1,6]$ for $n = 3$.

We note that $\Psi(\chi)$ defined by (1.6) satisfies

$$\Psi(\chi) \ge \tfrac{1}{8}\chi^4 - \tfrac{1}{2},$$

hence (A4)(ii) is satisfied, and clearly so is (A4)(iii) as well.

(A5) The gradient energy tensor $\mathbf{\Gamma}(\chi) = (\Gamma_{ij}(\chi)) : \mathbb{R}^n \to \mathbb{R}^n$:

(i) is a linear mapping, of class $C^{1,1}$ with respect to χ: $\Gamma_{ij}(\cdot) \in C^1(\mathbb{R})$ with $\Gamma'_{ij}(\cdot)$ Lipschitz continuous,

(ii) is symmetric: $\Gamma_{ij}(\cdot) = \Gamma_{ji}(\cdot)$,

(iii) is positive definite and bounded uniformly with respect to χ: there exist constants $0 < \underline{c}_\Gamma < \overline{c}_\Gamma$ such that

$$\underline{c}_\Gamma |\boldsymbol{\xi}|^2 \le \boldsymbol{\xi} \cdot \mathbf{\Gamma}(\chi)\boldsymbol{\xi} \le \overline{c}_\Gamma |\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n \text{ and } \chi \in \mathbb{R},$$

(iv) is such that $\mathbf{\Gamma}'(\chi) = (\Gamma'_{ij}(\chi)) : \mathbb{R}^n \to \mathbb{R}^n$ is uniformly bounded with respect to χ: there exists a constant $c_{\Gamma'} > 0$ such that

$$|\mathbf{\Gamma}'(\chi)\boldsymbol{\xi}| \le c_{\Gamma'}|\boldsymbol{\xi}| \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n \text{ and } \chi \in \mathbb{R}.$$

We recall that assumptions (A2)(iii), (A3)(ii), (A4)(ii), (A5)(iii) imply the following bound for the free energy:

$$f(\boldsymbol{\varepsilon},\chi,\nabla\chi) \ge c(|\boldsymbol{\varepsilon}|^2 + |\chi|^r + |\nabla\chi|^2) - c \qquad \forall (\boldsymbol{\varepsilon},\chi,\nabla\chi) \in \mathcal{S}^2 \times \mathbb{R} \times \mathbb{R}^n \tag{3.4}$$

with some constant $c > 0$. This is the main structural property used in the analysis of the problem.

The next assumption concerns the mobility matrix.

(A6) The mobility matrix $\mathbf{M}(\chi) = (M_{ij}(\chi)) : \mathbb{R}^n \to \mathbb{R}^n$:

(i) is a linear mapping, of class $C^{0,1}$ with respect to χ: $M_{ij}(\cdot) \in C^0(\mathbb{R})$ are Lipschitz continuous,

(ii) is symmetric: $M_{ij} = M_{ji}$,

(iii) is positive definite and bounded uniformly with respect to χ: there exist constants $0 < \underline{c}_M < \overline{c}_M$ such that

$$\underline{c}_M|\boldsymbol{\xi}|^2 \le \boldsymbol{\xi} \cdot \mathbf{M}(\chi)\boldsymbol{\xi} \le \overline{c}_M|\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n \text{ and } \chi \in \mathbb{R}.$$

The positive definiteness $\mathbf{M}(\chi)$ is the second main property used in the analysis.

The last assumption concerns the data of the problem:

(A7) The initial data $\mathbf{u}_0$, $\mathbf{u}_1$, χ_0 and the force term $\mathbf{b}$ satisfy

$$\mathbf{u}_0 \in \mathbf{V}_0, \quad \mathbf{u}_1 \in \mathbf{L}_2(\Omega), \quad \chi_0 \in H^1(\Omega), \quad \mathbf{b} \in L_1(0,T;\mathbf{L}_2(\Omega)).$$

The main result of the paper is the following.

THEOREM 3.1. *Under assumptions* (A1)–(A7), *problem* (P_0) *has a measure-valued solution* ν_t. *Moreover*, *for a.a.* $t \in (0,T)$, *there exists a set* $M_t \subset \overline{\Omega}$ *with Lebesgue measure zero such that* ν_t *is continuous with respect to the Lebesgue measure in* $\overline{\Omega} \setminus M_t$.

4. Proof of Theorem 3.1.

The proof is split into several steps.

4.1. *The Faedo–Galerkin approximation of* (P_0). We follow the method applied in Bartkowiak and Pawłow (2005). Let $\{\mathbf{v}_j\}_{j\in\mathbb{N}}$ with $\mathbf{v}_j \in (C^\infty(\overline{\Omega}))^3$, $\mathbf{v}_j|_S = \mathbf{0}$ be an orthonormal basis of $\mathbf{V}_0$, and $\{z_j\}_{j\in\mathbb{N}}$ with $z_j \in C^\infty(\overline{\Omega})$ be an orthonormal basis of $H^1(\Omega)$. Without loss of generality we assume that $z_1 = 1$. For $m \in \mathbb{N}$ we set

$$\mathbf{V}_m = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}, \qquad V_m = \operatorname{span}\{z_1, \ldots, z_m\}.$$

The approximate problem $(P_0)^m$: For any $m \in \mathbb{N}$ find a triple $(\mathbf{u}^m, \chi^m, \mu^m)$ of the form

$$\begin{aligned} \mathbf{u}^m(\mathbf{x},t) &= \sum_{i=1}^m e_i^m(t)\mathbf{v}_i(\mathbf{x}), \\ \chi^m(\mathbf{x},t) &= \sum_{i=1}^m c_i^m(t) z_i(\mathbf{x}), \\ \mu^m(\mathbf{x},t) &= \sum_{i=1}^m d_i^m(t) z_i(\mathbf{x}) \end{aligned} \tag{4.1}$$

satisfying for a.e. $t \in [0,T]$ the system

$$(4.2)\qquad (\mathbf{u}^m_{tt}, \boldsymbol{\eta}^m)_{\mathbf{L}_2(\Omega)} + (W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}^m), \chi^m), \boldsymbol{\varepsilon}(\boldsymbol{\eta}^m))_{\mathbf{L}_2(\Omega)} = (\mathbf{b}, \boldsymbol{\eta}^m)_{\mathbf{L}_2(\Omega)} \quad \forall \boldsymbol{\eta}^m \in \mathbf{V}_m,$$

$$(4.3)\qquad (\chi^m_t, \xi^m)_{L_2(\Omega)} + (\mathbf{M}(\chi^m)\nabla\mu^m, \nabla\xi^m)_{\mathbf{L}_2(\Omega)} = 0 \quad \forall \xi^m \in V_m,$$

$$(4.4)\qquad (w^m, \zeta^m)_{L_2(\Omega)} - (\boldsymbol{\Gamma}(\chi^m)\nabla\chi^m, \nabla\zeta^m)_{\mathbf{L}_2(\Omega)} - (\tfrac{1}{2}\nabla\chi^m \cdot \boldsymbol{\Gamma}'(\chi^m)\nabla\chi^m + \Psi'(\chi^m) + W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}^m), \chi^m), \zeta^m)_{L_2(\Omega)} = 0 \quad \forall \zeta^m \in V_m,$$

$$(4.5)\qquad \mathbf{u}^m(0) = \mathbf{u}^m_0, \quad \mathbf{u}^m_t(0) = \mathbf{u}^m_1, \quad \chi^m(0) = \chi^m_0$$

where $\mathbf{u}^m_0$, $\mathbf{u}^m_1 \in \mathbf{V}_m$, $\chi^m_0 \in V_m$ satisfy, as $m \to \infty$,

$$(4.6)\qquad \begin{aligned} \mathbf{u}^m_0 &\to \mathbf{u}_0 && \text{strongly in } \mathbf{V}_0,\\ \mathbf{u}^m_1 &\to \mathbf{u}_1 && \text{strongly in } \mathbf{L}_2(\Omega),\\ \chi^m_0 &\to \chi_0 && \text{strongly in } H^1(\Omega). \end{aligned}$$

4.2. *A priori estimates for solutions of* $(P_0)^m$

LEMMA 4.1 (see Bartkowiak and Pawłow (2005), Lemma 5.2). *Assume that*:

(i) $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \chi)$, $W_{,\chi}(\boldsymbol{\varepsilon}, \chi)$, $\mathbf{M}(\chi)$, $\boldsymbol{\Gamma}'(\chi)$, $\Psi'(\chi)$ *are Lipschitz continuous functions of their arguments*,
(ii) $f(\boldsymbol{\varepsilon}, \chi, \nabla\chi)$ *satisfies the structure condition* (3.4),
(iii) $\mathbf{M}(\chi)$ *satisfies the uniform positive definiteness condition* (A6)(iii),
(iv) *the data satisfy* (A7).

Then there exists a solution $(\mathbf{u}^m, \chi^m, \mu^m)$ *to problem* $(P_0)^m$ *on the interval* $[0,T]$, *satisfying the energy estimates*

$$(4.7)\qquad \|\mathbf{u}^m_t\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\boldsymbol{\varepsilon}(\mathbf{u}^m)\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\chi^m\|_{L_\infty(0,T;L_r(\Omega))} + \|\nabla\chi^m\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} + \|\nabla\mu^m\|_{\mathbf{L}_2(\Omega^T)} \le c,$$

with a constant c *depending only on the data and independent of* m.

We note that by Korn's inequality and Sobolev's imbedding, (4.7) implies

$$(4.8)\qquad \|\mathbf{u}^m\|_{L_\infty(0,T;\mathbf{V}_0)} \le c,$$

$$(4.9)\qquad \|\chi^m\|_{L_\infty(0,T;H^1(\Omega))} + \|\chi^m\|_{L_\infty(0,T;L_{q_n}(\Omega))} \le c.$$

We recall also

LEMMA 4.2 (see Bartkowiak and Pawłow (2005), Lemmas 5.3, 5.4). *Under the assumptions of Lemma* 4.1, *suppose that*:

(i) $W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}, \chi)$ *and* $W_{,\chi}(\boldsymbol{\varepsilon}, \chi)$ *satisfy the growth conditions* (3.2), (3.3),
(ii) $\Psi'(\chi)$ *satisfies the growth condition* (A4)(iii),
(iii) $\boldsymbol{\Gamma}'(\chi)$ *satisfies the uniform bound* (A5)(iv),
(iv) $\mathbf{M}(\chi)$ *satisfies the uniform bound* (A6)(iii).

Then

$$\|\mu^m\|_{L_2(0,T;H^1(\Omega))} \le c, \tag{4.10}$$

$$\|\mathbf{u}^m_{tt}\|_{L_2(0,T;\mathbf{V}_0')} \le c, \tag{4.11}$$

$$\|\chi^m_t\|_{L_2(0,T;V')} \le c, \tag{4.12}$$

with a constant c depending only on the data and independent of m.

We note that in view of the growth conditions (A2)(iii), (iv) on $\mathbf{A}(\chi)$, (A3)(ii) on $\overline{\boldsymbol{\varepsilon}}(\chi)$, (A4)(iii) on $\Psi(\chi)$, (A5)(iii), (iv) on $\boldsymbol{\Gamma}(\chi)$, and (A6)(iii) on $\mathbf{M}(\chi)$, the estimates (4.7)–(4.9) imply the following uniform bounds:

$$\begin{aligned}
&\|\overline{\boldsymbol{\varepsilon}}(\chi^m)\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} \le c,\\
&\|\mathbf{A}(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m))\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} \le c,\\
&\|\overline{\boldsymbol{\varepsilon}}'(\chi^m) \cdot \mathbf{A}(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m))\|_{L_\infty(0,T;L_2(\Omega))} \le c,\\
&\|\Psi'(\chi^m)\|_{L_\infty(0,T;L_2(\Omega))} \le c,\\
&\|\boldsymbol{\Gamma}(\chi^m)\nabla\chi^m\|_{L_\infty(0,T;\mathbf{L}_2(\Omega))} \le c,\\
&\|\mathbf{M}(\chi^m)\nabla\mu^m\|_{\mathbf{L}_2(\Omega^T)} \le c,
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
&\|\nabla\chi^m \cdot \boldsymbol{\Gamma}'(\chi^m)\nabla\chi^m\|_{L_\infty(0,T;L_1(\Omega))} \le c,\\
&\|(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m)) \cdot \mathbf{A}'(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m))\|_{L_\infty(0,T;L_1(\Omega))} \le c,
\end{aligned} \tag{4.14}$$

with a constant c depending only on the data.

4.3. *Weak formulation of $(P_0)^m$.* Using the identity

$$\int_0^T \langle\phi_t, \eta\rangle_{V',V}\, dt = -\int_0^T (\phi, \eta_t)_{L_2(\Omega)}\, dt - (\phi_0, \eta(0))_{L_2(\Omega)},$$

which holds true for all $\phi \in L_2(0,T;V) \cap H^1(0,T;V')$ with $\phi(0) = \phi_0$, and $\eta \in L_2(0,T;V) \cap H^1(0,T;L_2(\Omega))$ with $\eta(T) = 0$, we introduce the following weak formulation of $(P_0)^m$ analogous to (2.1)–(2.3):

$$\begin{aligned}
&\int_0^T (\mathbf{u}^m, \boldsymbol{\eta}^m_{tt})_{\mathbf{L}_2(\Omega)} dt + (\mathbf{u}^m_0, \boldsymbol{\eta}^m_t(0))_{\mathbf{L}_2(\Omega)} - (\mathbf{u}^m_1, \boldsymbol{\eta}^m(0))_{\mathbf{L}_2(\Omega)}\\
&\qquad + \int_0^T (W_{,\boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}^m)\chi^m), \boldsymbol{\varepsilon}(\boldsymbol{\eta}^m))_{\mathbf{L}_2(\Omega)}\, dt = \int_0^T (\mathbf{b}, \boldsymbol{\eta}^m)_{\mathbf{L}_2(\Omega)}\, dt
\end{aligned} \tag{4.15}$$

for all $\boldsymbol{\eta}^m(\mathbf{x},t) \equiv \vartheta_1(t)\boldsymbol{\eta}^m_0(\mathbf{x})$ where $\vartheta_1 \in C^\infty_c([0,T))$ with $\vartheta_1(T) = 0$, $\vartheta_1'(T) = 0$, and the $\boldsymbol{\eta}^m_0 \in \mathbf{V}_m$ satisfy $\boldsymbol{\eta}^m_0 \to \boldsymbol{\eta}_0$ in $(C^1(\overline{\Omega}))^3$ as $m \to \infty$;

$$(4.16)\qquad -\int_0^T(\chi^m,\xi^m_t)_{L_2(\Omega)}dt-(\chi^m_0,\xi^m(0))_{L_2(\Omega)} + \int_0^T(\mathbf{M}(\chi^m)\nabla\mu^m,\nabla\xi^m)_{\mathbf{L}_2(\Omega)}\,dt=0$$

for all $\xi^m(\mathbf{x}^m,t)\equiv\vartheta_2(t)\xi^m_0(\mathbf{x})$ where $\vartheta_2\in C^\infty_c([0,T))$ with $\vartheta_2(T)=0$ and the $\xi^m_0\in V_m$ satisfy $\xi^m_0\to\xi_0$ in $C^1(\overline{\Omega})$ as $m\to\infty$; and

$$(4.17)\qquad \int_0^T(\mu^m,\zeta^m)_{L_2(\Omega)}\,dt-\int_0^T(\mathbf{\Gamma}(\chi^m)\nabla\chi^m,\nabla\zeta^m)_{\mathbf{L}_2(\Omega)}\,dt - \frac{1}{2}\int_0^T(\nabla\chi^m\cdot\mathbf{\Gamma}'(\chi^m)\nabla\chi^m,\zeta^m)_{L_2(\Omega)}\,dt - \int_0^T(\Psi'(\chi^m),\zeta^m)_{L_2(\Omega)}\,dt-\int_0^T(W_{,\chi}(\boldsymbol{\varepsilon}(\mathbf{u}^m),\chi^m),\zeta^m)_{L_2(\Omega)}\,dt=0$$

for all $\zeta^m(\mathbf{x},t)\equiv\vartheta_3(t)\zeta^m_0(\mathbf{x})$ where $\vartheta_3\in C^\infty([0,T])$ and the $\zeta^m_0\in V_m$ satisfy $\zeta^m_0\to\zeta_0$ in $C^1(\overline{\Omega})$ as $m\to\infty$.

In view of the forms of the test functions $\boldsymbol{\eta}^m$, ξ^m, ζ^m, the identities (4.15)–(4.17) can be rewritten as follows:

$$(4.18)\qquad \int_0^T\vartheta_1''(t)\int_\Omega\mathbf{u}^m\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx\,dt + \int_0^T\vartheta_1(t)\int_\Omega\mathbf{A}(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m)-\overline{\boldsymbol{\varepsilon}}(\chi^m))\cdot\boldsymbol{\varepsilon}(\boldsymbol{\eta}_0(\mathbf{x}))\,dx\,dt = \int_0^T\vartheta_1(t)\int_\Omega\mathbf{b}\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx\,dt+\int_\Omega\vartheta_1(0)\mathbf{u}_1\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx - \int_\Omega\vartheta_1'(0)\mathbf{u}_0\cdot\boldsymbol{\eta}_0(\mathbf{x})\,dx+R^I_m(\mathbf{u}^m,\chi^m)(\vartheta_1,\boldsymbol{\eta}_0)$$

for all $\boldsymbol{\eta}_0\in(C^\infty(\overline{\Omega}))^3$, $\vartheta_1\in C^\infty_c([0,T))$ with $\boldsymbol{\eta}_0|_S=\mathbf{0}$, $\vartheta_1(T)=0$, $\vartheta_1'(T)=0$, where

$$R^I_m(\mathbf{u}^m,\chi^m)(\vartheta_1,\boldsymbol{\eta}_0)\equiv\int_0^T\vartheta_1''(t)\int_\Omega\mathbf{u}^m\cdot(\boldsymbol{\eta}_0(\mathbf{x})-\boldsymbol{\eta}^m_0(\mathbf{x}))\,dx\,dt + \int_0^T\vartheta_1(t)\int_\Omega\mathbf{A}(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m)-\overline{\boldsymbol{\varepsilon}}(\chi^m))\cdot\boldsymbol{\varepsilon}(\boldsymbol{\eta}_0(\mathbf{x})-\boldsymbol{\eta}^m_0(\mathbf{x}))\,dx\,dt$$

$$
\begin{aligned}
&- \int_0^T \vartheta_1(t) \int_\Omega \mathbf{b} \cdot (\boldsymbol{\eta}_0(\mathbf{x}) - \boldsymbol{\eta}_0^m(\mathbf{x}))\, dx\, dt \\
&- \int_\Omega \vartheta_1(0)(\mathbf{u}_1 \cdot \boldsymbol{\eta}_0(\mathbf{x}) - \mathbf{u}_1^m \cdot \boldsymbol{\eta}_0^m(\mathbf{x}))\, dx \\
&+ \int_\Omega \vartheta_1'(0)(\mathbf{u}_0 \cdot \boldsymbol{\eta}_0(\mathbf{x}) - \mathbf{u}_0^m \cdot \boldsymbol{\eta}_0^m(\mathbf{x}))\, dx;
\end{aligned}
$$

$$
\begin{aligned}
(4.19)\qquad -\int_0^T \vartheta_2'(t) \int_\Omega \chi^m \xi_0(\mathbf{x})\, dx\, dt + \int_0^T \vartheta_2(t) \int_\Omega \mathbf{M}(\chi^m)\nabla\mu^m \cdot \nabla\xi_0(\mathbf{x})\, dx\, dt \\
= \int_\Omega \vartheta_2(0)\chi_0\xi_0(\mathbf{x})\, dx + R_m^{II}(\chi^m, \mu^m)(\vartheta_2, \xi_0)
\end{aligned}
$$

for all $\xi_0 \in C^\infty(\Omega)$, $\vartheta_2 \in C_c^\infty([0,T))$ with $\vartheta_2(T) = 0$, where

$$
\begin{aligned}
R_m^{II}(\chi^m, \mu^m)(\vartheta_2, \xi_0) \equiv\ & - \int_0^T \vartheta_2'(t) \int_\Omega \chi^m(\xi_0(\mathbf{x}) - \xi_0^m(\mathbf{x}))\, dx\, dt \\
&+ \int_0^T \vartheta_2(t) \int_\Omega \mathbf{M}(\chi^m)\nabla\mu^m \cdot \nabla(\xi_0(\mathbf{x}) - \xi_0^m(\mathbf{x}))\, dx\, dt \\
&- \int_\Omega \vartheta_2(0)(\chi_0\xi_0(\mathbf{x}) - \chi_0^m\xi_0^m(\mathbf{x}))\, dx;
\end{aligned}
$$

and

$$
\begin{aligned}
(4.20)\qquad & \int_0^T \vartheta_3(t) \int_\Omega \mu^m \zeta_0(\mathbf{x})\, dx\, dt - \int_0^T \vartheta_3(t) \int_\Omega \boldsymbol{\Gamma}(\chi^m)\nabla\chi^m \cdot \nabla\zeta_0(\mathbf{x})\, dx\, dt \\
&- \frac{1}{2}\int_0^T \vartheta_3(t) \int_\Omega \nabla\chi^m \cdot \boldsymbol{\Gamma}'(\chi^m)\nabla\chi^m \zeta_0(\mathbf{x})\, dx\, dt - \int_0^T \vartheta_3(t) \int_\Omega \Psi'(\chi^m)\zeta_0(\mathbf{x})\, dx\, dt \\
&+ \int_0^T \vartheta_3(t) \int_\Omega \overline{\boldsymbol{\varepsilon}}'(\chi^m) \cdot \mathbf{A}(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m))\zeta_0(\mathbf{x})\, dx\, dt \\
&- \frac{1}{2}\int_0^T \vartheta_3(t) \int_\Omega (\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m)) \cdot \mathbf{A}'(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m))\zeta_0(\mathbf{x})\, dx\, dt \\
&= R_m^{III}(\mathbf{u}^m, \chi^m, \mu^m)(\vartheta_3, \zeta_0)
\end{aligned}
$$

for all $\zeta_0 \in C^\infty(\overline{\Omega})$, $\vartheta_3 \in C^\infty([0,T])$, where

$$
\begin{aligned}
R_m^{III}(\mathbf{u}^m, \chi^m, \mu^m)(\vartheta_3, \zeta_0) \equiv\ & \int_0^T \vartheta_3(t) \int_\Omega \mu^m(\zeta_0(\mathbf{x}) - \zeta_0^m(\mathbf{x}))\, dx\, dt \\
&- \int_0^T \vartheta_3(t) \int_\Omega \boldsymbol{\Gamma}(\chi^m)\nabla\chi^m \cdot \nabla(\zeta_0(\mathbf{x}) - \zeta_0^m(\mathbf{x}))\, dx\, dt
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T \vartheta_3(t)\int_\Omega (\nabla\chi^m\cdot\boldsymbol{\Gamma}'(\chi^m)\nabla\chi^m(\zeta_0(\mathbf{x})-\zeta_0^m(\mathbf{x}))\,dx\,dt\\
&-\int_0^T \vartheta_3(t)\int_\Omega \Psi'(\chi^m)(\zeta_0(\mathbf{x})-\zeta_0^m(\mathbf{x}))\,dx\,dt\\
&+\int_0^T \vartheta_3(t)\int_\Omega \overline{\boldsymbol{\varepsilon}}'(\chi^m)\cdot\mathbf{A}(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m)-\overline{\boldsymbol{\varepsilon}}(\chi^m))(\zeta_0(\mathbf{x})-\zeta_0^m(\mathbf{x}))\,dx\,dt\\
&-\frac{1}{2}\int_0^T \vartheta_3(t)\int_\Omega (\boldsymbol{\varepsilon}(\mathbf{u}^m)-\overline{\boldsymbol{\varepsilon}}(\chi^m))\cdot\mathbf{A}'(\chi^m)\\
&\cdot(\boldsymbol{\varepsilon}(\mathbf{u}^m)-\overline{\boldsymbol{\varepsilon}}(\chi^m))(\zeta_0(\mathbf{x})-\zeta_0^m(\mathbf{x}))\,dx\,dt.
\end{aligned}
$$

In view of the uniform estimates (4.7), (4.13), (4.14), the convergences (4.6), and the C^1-convergences of $\boldsymbol{\eta}_0^m$, ξ_0^m and ζ_0^m to $\boldsymbol{\eta}_0$, ξ_0 and ζ_0, it follows that, as $m\to\infty$,

$$
\begin{aligned}
&R_m^I(\mathbf{u}^m,\chi^m)(\vartheta_1,\eta_0)\to 0,\\
&R_m^{II}(\chi^m,\mu^m)(\vartheta_2,\xi_0)\to 0,\\
&R_m^{III}(\mathbf{u}^m,\chi^m,\mu^m)(\vartheta_3,\zeta_0)\to 0
\end{aligned}
\tag{4.21}
$$

for all test functions mentioned above.

4.4. *Letting $m\to\infty$ in $(P_0)^m$.* From estimates (4.7)–(4.12) it follows that there exists a triple $(\mathbf{u},\chi,\mu)$ with

$$
\begin{aligned}
&\mathbf{u}\in L_\infty(0,T;\mathbf{V}_0), && \mathbf{u}_t\in L_\infty(0,T;\mathbf{L}_2(\Omega)), && \mathbf{u}_{tt}\in L_2(0,T;\mathbf{V}_0'),\\
&\chi\in L_\infty(0,T;H^1(\Omega)), && \chi_t\in L_2(0,T;V'), && \mu\in L_2(0,T;H^1(\Omega)),
\end{aligned}
\tag{4.22}
$$

and a subsequence $(\mathbf{u}^m,\chi^m,\mu^m)$ of solutions to $(P_0)^m$ (denoted by the same indices) such that, as $m\to\infty$:

$$
\begin{aligned}
&\mathbf{u}^m\to\mathbf{u} && \text{weak}^* \text{ in } L_\infty(0,T;\mathbf{V}_0),\\
&\mathbf{u}_t^m\to\mathbf{u}_t && \text{weak}^* \text{ in } L_\infty(0,T;\mathbf{L}_2(\Omega)),\\
&\mathbf{u}_{tt}^m\to\mathbf{u}_{tt} && \text{weakly in } L_2(0,T;\mathbf{V}_0'(\Omega)),
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
&\chi^m\to\chi && \text{weak}^* \text{ in } L_\infty(0,T;H^1(\Omega)),\\
&\chi_t^m\to\chi_t && \text{weakly in } L_2(0,T;V'),
\end{aligned}
\tag{4.24}
$$

$$
\mu^m\to\mu \quad \text{weakly in } L_2(0,T;H^1(\Omega)).
\tag{4.25}
$$

Then by the standard compactness results (see Simon (1987), Corollary 4) it follows in particular that

$$
\begin{aligned}
&\mathbf{u}^m \to \mathbf{u} && \text{strongly in } L_2(0,T;\mathbf{L}_q(\varOmega)) \cap C([0,T],\mathbf{L}_q(\varOmega)) \\
&&& \text{and a.e. in } \varOmega^T, \\
&\mathbf{u}_t^m \to \mathbf{u}_t && \text{strongly in } C([0,T];\mathbf{V}_0'),
\end{aligned} \tag{4.26}
$$

$$
\begin{aligned}
&\chi^m \to \chi && \text{strongly in } L_2(0,T;L_q(\varOmega)) \cap C([0,T];L_q(\varOmega)) \\
&&& \text{and a.e. in } \varOmega^T,
\end{aligned} \tag{4.27}
$$

where $q \in [1,\infty)$ for $n = 2$ and $q \in [1,6]$ for $n = 3$. Hence,

$$
\begin{aligned}
&\mathbf{u}^m(0) = \mathbf{u}_0^m \to \mathbf{u}(0) && \text{strongly in } \mathbf{L}_q(\varOmega), \\
&\mathbf{u}_t^m(0) = \mathbf{u}_1^m \to \mathbf{u}_t(0) && \text{strongly in } \mathbf{V}_0', \\
&\chi^m(0) = \chi_0^m \to \chi(0) && \text{strongly in } L_q(\varOmega),
\end{aligned} \tag{4.28}
$$

which together with the convergences (4.6) implies that

$$
\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{u}_t(0) = \mathbf{u}_1, \quad \chi(0) = \chi_0. \tag{4.29}
$$

Our goal is to pass to the limit $m \to \infty$ in the identities (4.18)–(4.20). Clearly, by the weak convergences (4.23)–(4.25) the linear terms in (4.18)–(4.20) converge to the corresponding limits. Also,

$$
\begin{aligned}
\boldsymbol{\varepsilon}(\mathbf{u}^m) &\to \boldsymbol{\varepsilon}(\mathbf{u}) && \text{weak}^* \text{ in } L_\infty(0,T;L_2(\varOmega)), \\
\nabla\chi^m &\to \nabla\chi && \text{weak}^* \text{ in } L_\infty(0,T;\mathbf{L}_2(\varOmega)), \\
\nabla\mu^m &\to \nabla\mu && \text{weakly in } L_2(\varOmega^T).
\end{aligned} \tag{4.30}
$$

Consider now the nonlinear terms in (4.18)–(4.20). By the uniform bounds (4.13) and the convergence $\chi^m \to \chi$ a.e. in $\varOmega^T$ we can apply the classical weak convergence result for the nonlinear terms (see Lions (1969), Chap. 1, Lemma 1.3) to conclude that

$$
\begin{aligned}
\overline{\boldsymbol{\varepsilon}}(\chi^m) &\to \overline{\boldsymbol{\varepsilon}}(\chi) && \text{weak}^* \text{ in } L_\infty(0,T;\mathbf{L}_2(\varOmega)), \\
\varPsi'(\chi^m) &\to \varPsi'(\chi) && \text{weak}^* \text{ in } L_\infty(0,T;L_2(\varOmega)).
\end{aligned} \tag{4.31}
$$

Due to the regularity and the boundedness assumptions on $\mathbf{A}(\chi)$, $\overline{\boldsymbol{\varepsilon}}'(\chi)$, $\mathbf{M}(\chi)$ and $\mathbf{\Gamma}(\chi)$ we have

$$
\begin{aligned}
&\mathbf{A}(\chi^m) \to \mathbf{A}(\chi), \quad \overline{\boldsymbol{\varepsilon}}'(\chi^m) \to \overline{\boldsymbol{\varepsilon}}'(\chi), \\
&\mathbf{M}(\chi^m) \to \mathbf{M}(\chi), \quad \mathbf{\Gamma}(\chi^m) \to \mathbf{\Gamma}(\chi) \quad \text{a.e. in } \varOmega^T.
\end{aligned} \tag{4.32}
$$

Therefore, in view of (4.30)–(4.32),

$$
\begin{aligned}
&\mathbf{A}(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m)) \to \mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{weak}^* \text{ in } L_\infty(0,T;\mathbf{L}_2(\varOmega)), \\
&\overline{\boldsymbol{\varepsilon}}'(\chi^m) \cdot \mathbf{A}(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m)) \to \overline{\boldsymbol{\varepsilon}}'(\chi) \cdot \mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{weak}^* \text{ in } L_\infty(0,T;\mathbf{L}_2(\varOmega)), \\
&\mathbf{M}(\chi^m)\nabla\mu^m \to \mathbf{M}(\chi)\nabla\mu \quad \text{weakly in } L_2(\varOmega^T), \\
&\mathbf{\Gamma}(\chi^m)\nabla\chi^m \to \mathbf{\Gamma}(\chi)\nabla\chi \quad \text{weak}^* \text{ in } L_\infty(0,T;\mathbf{L}_2(\varOmega)).
\end{aligned} \tag{4.33}
$$

We now turn to the remaining two crucial terms in the identity (4.20),

$$\tfrac{1}{2}\nabla\chi^m \cdot \mathbf{\Gamma}'(\chi^m)\nabla\chi^m \quad \text{and} \quad \tfrac{1}{2}(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m)) \cdot \mathbf{A}'(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m)).$$

The convergence of these terms follows by repeating the arguments used in Neustupa (1993). Namely, from the uniform estimates (4.14) it follows that for a given $\phi \in C(\overline{\Omega})$ there exist functions $f_{\mathbf{\Gamma}'}(\phi) \in L_\infty(0,T)$, $f_{\mathbf{A}'}(\phi) \in L_\infty(0,T)$ and a subsequence $(\mathbf{u}^m, \chi^m, \mu^m)$ (denoted by the same indices) such that

$$\begin{aligned} &\int_\Omega \nabla\chi^m \cdot \mathbf{\Gamma}'(\chi^m)\nabla\chi^m \phi(\mathbf{x})\,dx \to f_{\mathbf{\Gamma}'}(\phi) \quad \text{weak}^* \text{ in } L_\infty(0,T), \\ &\int_\Omega (\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m)) \cdot \mathbf{A}'(\chi^m)(\boldsymbol{\varepsilon}(\mathbf{u}^m) - \overline{\boldsymbol{\varepsilon}}(\chi^m))\phi(\mathbf{x})\,dx \to f_{\mathbf{A}}(\phi) \\ &\hspace{20em} \text{weak}^* \text{ in } L_\infty(0,T). \end{aligned} \tag{4.34}$$

Moreover, there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned} &|f_{\mathbf{\Gamma}'}(\phi)(t)| \le c_1\|\phi\|_{C(\overline{\Omega})}, \\ &|f_{\mathbf{A}'}(\phi)(t)| \le c_2\|\phi\|_{C(\overline{\Omega})} \quad \text{for a.a. } t \in (0,T). \end{aligned} \tag{4.35}$$

For $t \in (0,T)$, $f \in Y$ (the space of functions on $\mathbb{R}^M$ of the form (2.10)) and $\phi \in C(\overline{\Omega})$ we set

$$\begin{aligned} \mathcal{A}_{t,f}(\phi) = {}& \mathbf{d}_1 \cdot \int_\Omega \mathbf{u}\,\phi(\mathbf{x})\,dx + \mathbf{B} \cdot \int_\Omega \mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u} - \overline{\boldsymbol{\varepsilon}}(\chi))\phi(\mathbf{x})\,dx \\ &+ a_1 \int_\Omega \chi\phi(\mathbf{x})\,dx + \mathbf{d}_2 \cdot \int_\Omega \mathbf{M}(\chi)\nabla\mu\;\phi(\mathbf{x})\,dx \\ &+ a_2 \int_\Omega \mu\phi(\mathbf{x})\,dx + \mathbf{d}_3 \cdot \int_\Omega \mathbf{\Gamma}(\chi)\nabla\chi\;\phi(\mathbf{x})\,dx \\ &+ a_3 \int_\Omega \Psi'(\chi)\phi(\mathbf{x})\,dx + a_4 \int_\Omega \overline{\boldsymbol{\varepsilon}}'(\chi) \cdot \mathbf{A}(\chi)(\boldsymbol{\varepsilon}(\mathbf{u}) - \overline{\boldsymbol{\varepsilon}}(\chi))\phi(\mathbf{x})\,dx \\ &+ a_5 f_{\mathbf{\Gamma}'}(\phi) + a_6 f_{\mathbf{A}'}(\phi), \end{aligned} \tag{4.36}$$

where $\mathbf{d} \in \mathbb{R}^n$, $\mathbf{B} \in \mathbb{R}^{n^2}$ and $a_1, \ldots, a_6 \in \mathbb{R}$. From the properties (4.22), (4.31), (4.33) and (4.35) it follows that for a.a. $t \in (0,T)$ and a given $f \in Y$, $\mathcal{A}_{t,f}$ is a bounded linear functional on $C(\overline{\Omega})$.

At this point we can repeat the arguments used by Neustupa (1993) based on the representation theorems for bounded linear functionals. First, by the Riesz theorem (see e.g. Rudin (1974)), there exists a regular Borel measure $\theta_{t,f}$ on $\overline{\Omega}$ so that

$$\mathcal{A}_{t,f}(\phi) = \int_\Omega \phi(\mathbf{x})\,d\theta_{t,f}(\mathbf{x}).$$

Secondly, if E is a Borel set in $\overline{\Omega}$ then $\theta_{t,f}(E)$ (in its dependence on f) is a linear functional on Y. Thirdly, there exists a regular Borel measure $\nu_{t,E}$ on $\mathbb{R}^M$ such that

$$\theta_{t,f}(E) = \int_{\mathbb{R}^M} f(\boldsymbol{\lambda})\, d\nu_{t,E}(\boldsymbol{\lambda}).$$

The measure $\nu_{t,E}$ can be extended to a regular Borel measure ν_t on $\overline{\Omega}\times\mathbb{R}^M$ so that $\nu_t(E\times I) = \nu_{t,E}(I)$ for all Borel sets E in $\overline{\Omega}$ and I in $\mathbb{R}^M$. Consequently, the functional $\mathcal{A}_{t,f}$ can be represented in the form

$$\mathcal{A}_{t,f}(\phi) = \int_{\Omega} \phi(\mathbf{x}) \int_{\mathbb{R}^M} f(\boldsymbol{\lambda})\, d\nu_t(\mathbf{x},\boldsymbol{\lambda}). \tag{4.37}$$

Using the convergences (4.23)–(4.25), (4.33) and (4.34) we can let $m\to\infty$ in (4.18)–(4.20) to deduce the identities (2.7)–(2.9). Consider, for example, the term

$$\int_0^T \vartheta_3(t) \int_{\Omega} \nabla\chi^m \cdot \mathbf{\Gamma}'(\chi^m)\nabla\chi^m \zeta_0(\mathbf{x})\, dx\, dt.$$

In this case we choose

$$f(\boldsymbol{\lambda}) = \boldsymbol{\lambda}_{\nabla\chi} \cdot \mathbf{\Gamma}'(\lambda_\chi)\boldsymbol{\lambda}_{\nabla\chi}.$$

By the convergence $(4.34)_1$, definition (4.36) of $\mathcal{A}_{t,f}$ and its representation (4.37), it follows that as $m\to\infty$,

$$\begin{aligned}
&\int_0^T \vartheta_3(t) \int_{\Omega} \nabla\chi^m \cdot \mathbf{\Gamma}'(\chi^m)\nabla\chi^m \zeta_0(\mathbf{x})\, dx\, dt\\
&\qquad\to \int_0^T \vartheta_3(t) f_{\mathbf{\Gamma}'}(\zeta_0)(t)\, dt = \int_0^T \vartheta_3(t)\mathcal{A}_{t,f}(\zeta_0)\, dt\\
&\qquad= \int_0^T \vartheta_3(t) \int_{\Omega}\int_{\mathbb{R}^M} \boldsymbol{\lambda}_{\nabla\chi} \cdot \mathbf{\Gamma}'(\lambda_\chi)\boldsymbol{\lambda}_{\nabla\chi}\zeta_0(\mathbf{x})\, d\nu_t(\mathbf{x},\boldsymbol{\lambda})\, dt.
\end{aligned}$$

In the same way we can show that as $m\to\infty$, all other terms in (4.18)–(4.20), except R_m^I, R_m^{II}, R_m^{III} which tend to zero, converge to the corresponding terms in the identities (2.7)–(2.9). Clearly, by construction, condition (2.11) is also satisfied. We conclude that ν_t is a solution of (P_0) in the sense of Definition 2.1.

The statement concerning the continuity of ν_t follows by the same arguments as in Neustupa (1993). This completes the proof. ∎

Remark. We should underline that from the abstract theorems in Alibert and Bouchitté (1997) and Müller (1999) a result more precise than Theorem 3.1 can be deduced.

First, Theorem 2.5 of Alibert and Bouchitté (1997) guarantees the convergence of nonlinearities not only in finite-dimensional subspace of continuous functions (as in (2.10)) but also in the infinite-dimensional space $\mathcal{F}$. Due to that convergence some physical relations can be satisfied in the measure sense.

Secondly, the measure ν_t can be split into a probability measure $\nu_{(t,x)}$ on $\mathbb{R}^d$ with $d = n + 1 + M$, a probability measure $\nu_{(t,x)}^{\infty}$ on the unit sphere S^{d-1} in $\mathbb{R}^d$, and a Radon measure m defined only on $\Omega \times [0, T)$. Here $\nu_{(t,x)}$ is the standard Young measure; the above three measures are usually called DiPerna–Majda measures.

Finally, Corollary 3.4 of Müller (1999) allows one to select variables with defect of strong convergence in tensorial distribution of the measure $\nu_{(t,x)}$.

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REFERENCES

J. J. Alibert and G. Bouchitté (1997), *Non-uniform integrability and generalized Young measures*, J. Convex Anal. 4, 129–147.

Ł. Bartkowiak and I. Pawłow (2005), *The Cahn–Hilliard–Gurtin system coupled with elasticity*, Control Cybernet. 34, 1005–1043.

E. Bonetti, P. Colli, W. Dreyer, G. Giliardi, G. Schimperna and J. Sprekels (2002), *On a model for phase separation in binary alloys driven by mechanical effects*, Phys. D 165, 48–65.

H. Garcke (2000), *On mathematical models of phase separation in elastically stressed solids*, Habilitation Thesis, Univ. Bonn.

H. Garcke (2003a), *On Cahn–Hilliard systems with elasticity*, Proc. Roy. Soc. Edinburgh 133, 307–331.

H. Garcke (2003b), *On a Cahn–Hilliard model for phase separation with elastic misfit*, Ann. Inst. H. Poincaré Anal. Non Linéaire 22, 165–185.

H. Garcke, M. Rumpf and U. Weikard (2001), *The Cahn–Hilliard equation with elasticity—finite element approximation and qualitative studies*, Interfaces Free Bound. 3, 101–118.

M. E. Gurtin (1996), *Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance*, Phys. D 92, 178–192.

D. Kröner and W. M. Zajączkowski (1996), *Measure-valued solutions of the Euler equations for ideal compressible polytropic fluids*, Math. Methods Appl. Sci. 19, 235–252.

J.-L. Lions (1969), *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris.

J. Málek, J. Nečas, M. Rokyta and M. Růžička (1996), *Weak and Measure-Valued Solutions to Evolutionary PDEs*, Appl. Math. and Math. Comput. 13, Chapman and Hall, London.

A. Miranville (2003), *Generalized Cahn–Hilliard equations based on a microforce balance*, J. Appl. Math. 4, 165–185.

S. Müller (1999), *Variational models for microstructure and phase transitions*, in: Calculus of Variations and Geometric Evolution Problems (Cetraro, 1996), Lecture Notes in Math. 1713, Springer, 1999, 85–210.

J. Neustupa (1993), *Measure-valued solutions of the Euler and Navier–Stokes equations for compressible barotropic fluids*, Math. Nachr. 163, 217–227.

W. Rudin (1974), *Real and Complex Analysis*, 2nd ed., McGraw-Hill, New York.

J. Simon (1987), *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl. 146, 65–97.

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