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ON ORDERED DIVISION RINGS

BY

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Abstract. Prestel introduced a generalization of the notion of an ordering of a field, which is called a semiordering. Prestel's axioms for a semiordered field differ from the usual (Artin–Schreier) postulates in requiring only the closedness of the domain of positivity under $x \mapsto xa^2$ for non-zero a, in place of requiring that positive elements have a positive product. Our aim in this work is to study this type of ordering in the case of a division ring. We show that it actually behaves just as in the commutative case. Further, we show that the bounded subring associated with that ordering is a valuation ring which is preserved under conjugation, so one can associate with the semiordering a natural valuation.

1. Introduction. The investigation of ordered fields has a long tradition (see [1–4, 6]). They play an important part in many branches of mathematics. In [5], Prestel introduced a generalization of an ordering of a field, which is called a semiordering. Prestel's axioms for a semiordered field differ from the usual (Artin–Schreier) postulates in requiring only the closedness of the positive cone under $x \mapsto xa^2$ for non-zero a, in place of requiring that positive elements have a positive product. This generalization of positive cones and orderings is based on the following observation. Very often one only uses the property $x > 0 \Rightarrow xa^2 > 0$ of an ordering together with $1 > 0$. This is especially the case if one deals with quadratic forms. Our aim in this work is to study this type of ordering in the case of a division ring. We show that it actually behaves just as in the commutative case. For example, a division ring admits a semiordering if and only if -1 is not a sum of products of squares. In fact, we show that every semiordered division ring is ordered. Moreover, we show that every archimedean semiordered division ring is an ordered field.

Further, we are concerned with the existence of a natural valuation associated with the semiordering. This requires the study of the bounded subring associated with a given semiordering. We show that the bounded subring is a valuation ring which is preserved under conjugation, so one can associate with the semiordering a natural valuation.

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To study extensions of semiorderings to larger division rings, we investigate the notion of pre-semiordering in Section 4. Finally, we give an example of a semiordering which is not an ordering for a division ring. Throughout we denote by $\mathbb{Q}$ the field of rationals and by $\mathbb{Q}^+$ the positive rationals.

2. Preliminaries. Throughout this work, D denotes a (not necessarily commutative) division ring, and $D^{\cdot}$ denotes its multiplicative group of nonzero elements. We start by giving the definition of a semiordering.

Definition 2.1. By a *semiordering* (or *q-ordering*) of a division ring D, we mean an order relation $<$ such that D contains a subset P (the *positive cone*) satisfying

(1) $P + P \subset P$,
(2) $a \in P \Rightarrow ab^2 \in P$ for $0 \neq b \in D$,
(3) $0 \notin P$ and $1 \in P$,
(4) $P \cup \{0\} \cup -P = D$;

then $a > b \Leftrightarrow a - b \in P$, and $P = \{a \in D \mid a > 0\}$.

We first remark that this definition is the same as in the commutative case (see [5]). Clearly any ordering of D is also a semiordering.

If D is semiordered we denote P by P_D. Let C_D be the subset of all finite sums of elements of the form $a_1^2 a_2^2 \dots a_k^2$ in D with every $a_i \neq 0$. Clearly, C_D is closed under sums and products. Also, C_D contains inverses (for $c \in C_D, c^{-2} \in C_D \Rightarrow c^{-1} = cc^{-2} \in C_D$). If D is a semiordered division ring, then for $a \in P_D$ and $c \in C_D$ we have $ac \in P_D$ (by applying conditions (1) and (2) of the above definition several times). Since $1 \in P_D$, we clearly have $C_D \subset P_D$. Also, $-1 \notin P_D$ implies that $-1 \notin C_D$. So by [8, Theorem 1], D is an ordered division ring. Hence

Theorem 2.2. *Every semiordered division ring is ordered.*

We note that, although a semiordered division ring is ordered, there is no guarantee that the given semiordering is an ordering. At the end of this work, we will give an example of a semiordering which is not an ordering.

Corollary 2.3. *A division ring D admits a semiordering if and only if $-1 \notin C_D$.*

We now prove some properties of semiorderings.

Lemma 2.4. *Let D be any semiordered division ring, $a \in D$ and C_D be the set defined above. Then*

(1) $a > 0$ *if and only if* $a^{-1} > 0$,
(2) *if* $a > 0$, *then* $d^2 a > 0$ *for* $0 \neq d \in D$ *and hence* $ca > 0$ *for* $c \in C_D$,
(3) *if* $a > 0$, *then* $ra > 0$ *for every* $r \in \mathbb{Q}^+$,

(4) *if* $a > 1$, *then* $a^{-1} < 1$,
(5) *if* $0 < a < b$ *and* $a, b \in C_D$, *then* $a^{-1} > b^{-1}$,
(6) *if* $0 < a < b$ *and* $a \in C_D$, *then* $a^2 < b^2$,
(7) *if* $0 < a < b$ *and* $b \in C_D$, *then* $a^2 < b^2$,
(8) *if* $a > 0$, *then* $xax^{-1} > 0$ *for all* $x \in D^{\cdot}$.

Proof. (1) Clearly $a > 0$ implies $a^{-1} = aa^{-2} > 0$.

(2) If $a > 0$, then $ab^2 > 0$, and $b^{-2}a^{-1} > 0$ by (1). So $b^{-2}a = b^{-2}a^{-1}a^2 > 0$. This shows that $d^2a > 0$ for every $d \in D^{\cdot}$. For the rest of (2), use arguments similar to the proof of Theorem 1.

(3) This is evident.

(4) Since $a - 1 > 0$, we have $1 - a^{-1} = (a-1)a^{-2} + (1-a^{-1})^2 > 0$, and $a^{-1} < 1$.

(5) Since $b - a > 0$ and $a^{-1} \in C_D$, we have $ba^{-1} - 1 = (b-a)a^{-1} > 0$, i.e., $ba^{-1} > 1$. So $a^{-1} > b^{-1}$, because $b^{-1} \in C_D$.

(6) Since $b - a > 0$ and $a \in C_D$, we have $(b-a)a^{-1} > 0$ and hence $ba^{-1} > 1$. Using (4) gives $ab^{-1} = (ba^{-1})^{-1} < 1$. So $b^{-1} < a^{-1}$ (because $a^{-1} \in C_D$). From $b^2b^{-1}a^2 < b^2a^{-1}a^2$, it follows that $ba^2 < b^2a$ so that $ba < b^2$. Also, $a^2 < ba$ (because $a < b$ and $a \in C_D$). Thus $a^2 < ba < b^2$.

(7) Use a similar argument.

(8) This follows from $xax^{-1} = a(a^{-1})^2(ax)^2(x^{-1})^2$. ∎

THEOREM 2.5. *A semiordering* $<$ *is an ordering of* D *if and only if for all* $a, b \in D$, $0 < a < b$ *implies* $a^2 < b^2$.

Proof. Assume $<$ is a semiordering which satisfies the condition in the statement of the theorem. We prove that $<$ is an ordering. We first claim that for all $a, b \in D$ with $a, b > 0$ we have $ba + ab > 0$. Since $a - b > 0$ or $b - a > 0$, we may assume that $0 < a < b$. Clearly, $a + b > b > b - a > 0$ and so $(a+b)^2 > b^2 > (b-a)^2$. Hence $2(ba + ab) > 0$. By Lemma 2.4(3), $ba + ab > 0$.

Next we claim that $bab > 0$ and $bab^{-1} > 0$ for all $a, b > 0$ in D. If $bab < 0$, then $-bab > 0$ and $a > 0$ implies $(-bab)a + a(-bab) > 0$. Thus $-(ba)^2 - (ab)^2 > 0$, which is a contradiction. Hence $bab > 0$ and $bab^{-1} = babb^{-2} > 0$.

Now assume that there exist $a, b > 0$ such that $ab < 0$. Then $-ab > 0$ and $-ba = b(-ab)b^{-1} > 0$. So $-ab - ba > 0$, which is a contradiction. Hence $ab > 0$ for all $a, b > 0$ and the semiordering is an ordering. ∎

We call a semiordering $<$ on a division ring D *archimedean* if for every $a \in D$, there is some natural number n such that $a < n$.

PROPOSITION 2.6. *A semiordering* $<$ *on a division ring* D *is archimedean if and only if* $\mathbb{Q}$ *is dense in* D *with respect to* $<$ (*i.e., if* $a < b$ *then there is* $r \in \mathbb{Q}$ *such that* $a < r < b$).

The proof of Proposition 2.6 is similar to the one in the case of a field (see [5]).

THEOREM 2.7. *Every archimedean semiordered division ring D is an ordered field.*

Proof. It is known that every archimedean ordered division ring D is an ordered field. Hence, to prove the theorem, it remains to show that D is ordered.

Let $a, b \in D$ with $a > b > 0$. Then $a + b > a - b > 0$, and there is some $r \in \mathbb{Q}$ such that $a+b > r > a-b$. By Lemma 2.4, $(a+b)^2 > r^2 > (a-b)^2$. As in the proof of Theorem 2.5, one can show that $ab > 0$, i.e., D is ordered. ∎

3. Order valuation. Our next goal is to study the notion of the order valuation of a semiordered division ring D. We first define the notion of a bounded element in D. Call $a \in D$ *bounded* if $a^2 \le r$ for some $r \in \mathbb{Q}^+$. If $a^2 < r$ for every $r \in \mathbb{Q}^+$, we call a an *infinitesimal.* Let V_D denote the set of all bounded elements of D, and let J_D denote the set of all infinitesimals in D. We first establish that V_D is a valuation ring in D and the multiplicative group U_D of invertible elements in V_D is formed by precisely those a such that $r_1 \le a^2 \le r_2$ for some positive rationals r_1 and r_2 (we call these elements *units*). Three more remarks will be needed.

REMARK 3.1. *For non-zero elements $a, b \in D$ and $r \in \mathbb{Q}^+$:*

(1) $(a \pm b)^2 \le 2(a^2 + b^2)$,
(2) $a^2 < r^2$ *if and only if* $-r < a < r$.

Proof. (1) This follows from the identity

$$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2),$$

and the fact that every non-zero square is positive.

(2) Assume $-r < a < r$; then $r - a > 0$ and $(r + a)^2(r - a) > 0$. Also $a + r > 0$ implies $(r - a)^2(r + a) > 0$. Hence

$$2r(r^2 - a^2) = (r + a)^2(r - a) + (r - a)^2(r + a) > 0,$$

i.e., $a^2 < r^2$. Conversely, if $a^2 < r^2$, then

$$r \pm a = \frac{1}{2r}[(r \pm a)^2 + (r^2 - a^2)] > 0.$$

Therefore $-r < a < r$. ∎

The following two remarks follow immediately from the definition and Remark 3.1.

REMARK 3.2. (1) *The set V_D of bounded elements is an additive subgroup of D.*

(2) *The set J_D of infinitesimals is an additive subgroup.*

Remark 3.3. *If a is a positive element in a semiordered division ring D and $0 \neq x \in D$, then*:

(1) *a is bounded if and only if $a < r$ for some $r \in \mathbb{Q}^+$*,

(2) *a is a unit if and only if $r_2 < a < r_1$ for some $r_1, r_2 \in \mathbb{Q}^+$*,

(3) *a is infinitesimal if and only if $a < r$ for every $r \in \mathbb{Q}^+$, and*

(4) *x is bounded (resp. unit, infinitesimal) if and only if x^2 is bounded (resp. unit, infinitesimal).*

Theorem 3.4. *Let D be a semiordered division ring. Under the notations introduced above we have*

(1) *V_D is a total subring of D, i.e., V_D is a subring which contains x or x^{-1} for every $x \in D^{\cdot}$.*

(2) *The set of non-units of the ring V_D is precisely the ideal of infinitesimals and consequently, J_D is the unique maximal ideal of V_D.*

Proof. (1) Let $a, b \in V_D$, i.e., $a^2 \leq r_1$ and $b^2 \leq r_2$ for some $r_1, r_2 \in \mathbb{Q}^+$. Then

$$(a-b)^2 \leq 2(a^2+b^2) \leq 2(r_1+r_2) = r \qquad \text{for some } r \in \mathbb{Q}^+.$$

Hence, from $a^2+b^2-(ab+ba) = (a-b)^2 \in V_D$, it follows that $ab+ba \in V_D$. Thus,

$$bab = \tfrac{1}{2}[(ab+ba)b + b(ab+ba) - (b^2a+ab^2)] \in V_D.$$

Similarly, $ba^2b, ab^2a \in V_D$, and $(ab)^2+(ba)^2 = a(bab)+(bab)a \in V_D$; so that

$$(ab-ba)^2 = (ab)^2+(ba)^2-[ab^2a+ba^2b] \in V_D.$$

By Remark 3.3, $ab-ba \in V_D$. Finally, $2ab = (ab+ba)+(ab-ba) \in V_D$ and so $ab \in V_D$.

If $x \in D$ and $x \notin V_D$, then $x^2 > r$ for all $r \in \mathbb{Q}^+$, and hence $1 - rx^{-2} = (x^2-r)x^{-2} > 0$. Then $x^{-2} < 1/r$ and so $x^{-1} \in V_D$. Thus V_D is a total subring.

(2) We show here that the units are precisely the invertible elements in V_D. If x is a unit, then $x \in V_D$ and $x^2 \geq r$ for some $r \in \mathbb{Q}^+$. Then $(x^2-r)x^{-2} \geq 0$ and so $x^{-2} \leq 1/r$. Hence $x^{-1} \in V_D$ and x is invertible in V_D. Conversely, if x is invertible in V_D, then $x^2 \leq r_1$ for some $r_1 \in \mathbb{Q}^+$. Also, $x^{-1} \in V_D$ implies that $x^{-2} \leq r_2$ for some $r_2 \in \mathbb{Q}^+$. So, as above, $x^2 \geq 1/r_2 = r_2'$. Hence $r_2' \leq x^2 \leq r_1$ and x is a unit. ∎

Theorem 3.5. *If D is a semiordered division ring, then the bounded subring V_D is preserved under conjugation. Therefore, V_D is a valuation subring of D.*

Proof. Let $a \in V_D$, $a > 0$. Then by Remark 3.3, $a < r$ for some $r \in \mathbb{Q}^+$. By Lemma 2.4(8), $x(r-a)x^{-1} > 0$ for every $x \in D^{\cdot}$, so that $xax^{-1} < r$. Since $xax^{-1} > 0$, it follows that $xax^{-1} \in V_D$ for every $x \in D^{\cdot}$. If $a < 0$

in V_D, then $-a > 0$. Hence $-xax^{-1} \in V_D$ and also $xax^{-1} \in V_D$ for every $x \in D^{\cdot}$. ∎

The bounded subring V_D of a semiordered division ring D is now a valuation ring. Let G denote the multiplicative group $D^{\cdot}/U_D$, written additively. We set $\overline{a} = aU_D > 0$ in G if and only if $a \in J_D$. We write $\overline{a} > \overline{b} \Leftrightarrow \overline{a} - \overline{b} > 0$. Then we have a group ordering on G. Define $\omega : D^{\cdot} \to G$ by $\omega(a) = \overline{a}$. For all $a, b \in D^{\cdot}$, we have

$$\omega(ab) = \omega(a) + \omega(b), \qquad \omega(a+b) \geq \min(\omega(a), \omega(b)),$$

i.e., ω is a valuation with valuation ring precisely the bounded subring V_D (see [7]). This valuation is naturally called the *order valuation.*

THEOREM 3.6. *In any semiordered division ring D, the residue division ring $\overline{D} = V_D/J_D$ has a semiordering which is archimedean, so $\overline{D}$ is an archimedean ordered field.*

Proof. Let $\overline{P} = \{a + J_D \mid a \text{ is a positive unit in } D\}$. Clearly, $\overline{1} = 1 + J_D \in \overline{P}$ and $\overline{0} \notin \overline{P}$. It is straightforward to check that $\overline{P}$ is a positive cone of some semiordering in $\overline{D}$.

Consider the element $\overline{a} = a + J_D \in \overline{P}$, where $r_2 \leq a \leq r_1$ for some $r_1, r_2 \in \mathbb{Q}^+$. Then $\overline{r}_2 = r_2 + J_D \leq \overline{a} \leq \overline{r}_1 = r_1 + J_D$ and we can find a natural number n such that $\overline{a} < n$. Hence, $\overline{D}$ is an archimedean semiordered division ring. By Theorem 2.7, $\overline{D}$ is an archimedean ordered field. ∎

4. Semiorderings under division ring extensions. Let us now discuss extensions of a semiordering of a division ring D to larger division rings. As in the commutative case (see [5]), we have to use the notion of a pre-positive cone. A subset $P \subset D$ is called a *pre-positive cone* if it satisfies:

(1) $P + P \subset P$,
(2) $a \in P \Rightarrow ab^2 \in P$ for $0 \neq b \in D$,
(3) $0 \notin P$ and $1 \in P$.

A pre-positive cone P induces an order relation on D, which we call a *pre-semiordering.* Any positive cone of a semiordering is clearly a pre-positive cone. Also, any intersection of positive cones of D is a pre-positive cone of D. In this section we assume that D is semiordered, that is, $-1 \notin C_D$, or equivalently $0 \notin C_D$; then C_D is a pre-positive cone with the following features: $C_D \subset P$ and $C_D P = P C_D = P$, for each pre-positive cone P.

THEOREM 4.1. *If P is a pre-positive cone of D and $a \notin P$, then there is a pre-positive cone P' of D containing P with $-a \in P'$.*

Proof. Let $P' = P \cup -aC_D \cup (P + (-a)C_D)$. Clearly $-a \in P'$. We check axioms (1) to (3) for P'. Since P and C_D are additive, it follows that $P' + P' \subset P'$. Clearly $1 \in P'$, and to show that $0 \notin P'$, it suffices to show that $0 \notin P + (-a)C_D$. If $0 \in P + (-a)C_D$, then $p - ac = 0$ for some $p \in P$ and $c \in C_D$. Hence $a = pc^{-1} \in PC_D = P$, which is a contradiction. Axiom (2) is evident. ∎

THEOREM 4.2. *Any pre-positive cone P_0 of D can be extended to some positive cone P.*

Proof. By Zorn's lemma, the set of all pre-positive cones extending P_0 contains some maximal pre-positive cone P. If $a \notin P$ for some $a \in D$, it follows by Theorem 4.1 that there is a pre-positive cone P' containing P and such that $-a \in P'$. The maximality of P yields $P' = P$, so that $-a \in P$. Thus P is a positive cone. ∎

COROLLARY 4.3. *A pre-positive cone P of D is maximal (with respect to set theoretical inclusion) if and only if P is positive cone.*

THEOREM 4.4. *Let E be any division ring extension of D. Let P_D be a positive cone of D. Let P_1 be the set of elements in E which are expressible as sums of elements of the form $\prod_i a_{j_i} c_{j_i}$ ($a_{j_i} \in P_D$ and $c_{j_i} \in C_E =$ the set of all finite sums of products of squares in E). If $0 \notin P_1$, then the set P_D can be enlarged to some positive cone of E.*

Proof. Since $0 \notin P_1$, it follows that $0 \notin C_E$, and E is ordered. One can show that P_1 is a pre-positive cone of E. Thus, by Theorem 4.2, P_1 can be extended to some positive cone of E which contains P. ∎

Exactly as for a semiordering, one can define the bounded elements, the infinitesimals and the units for a given pre-semiordering of the division ring D. For P_0 a pre-positive cone of some pre-semiordering, let V_0, J_0 denote the sets of all bounded elements and infinitesimals respectively.

THEOREM 4.5. *Let $\{P_i\}_{i \in I}$ be the family of positive cones containing a given pre-positive cone P_0 of the division ring D. Let V_i, J_i be the subring of bounded elements and the ideal of infinitesimals respectively, attached to the semiordering induced by P_i. Let U_i be the group of units of the ring V_i. Then*

(i) $\bigcap_i V_i = V_0$,
(ii) $\bigcap_i J_i = J_0$,
(iii) $\bigcap_i U_i = U_0$.

Proof. We prove (i); for (ii) and (iii) use similar arguments. Clearly $V_0 \subset \bigcap_i V_i$. Conversely, if $a \notin V_0$, we show that $a \notin V_i$ for some i. From $a \notin V_0$, it follows that $a^2 > r$ for every positive rational r, that is, $a^2 - r \in P_0$ for every rational r. Let $P_r = P_0 \cup (a^2 - r)C \cup (P_0 + (a^2 - r)C)$ and $P_0' = \bigcup_r P_r$. One can show that P_0' is a pre-positive cone of D containing P_0 and $a^2 - r$ for every rational r. By Theorem 4.2, P_0' can be extended to some positive cone P. Clearly $P_0 \subset P$ and $a^2 - r \in P$ for every rational r. Thus a is not bounded in P, i.e., $a \notin V_i$ for some i. ∎

To a certain extent Theorem 4.5 reduces the treatment of pre-semiorderings to that of semiorderings. For instance one has the following fact.

COROLLARY 4.6. *For any pre-positive cone P_0 of D, the bounded subring V_0 is preserved under conjugation.*

5. An example. Finally, we give an example of a semiordering which is not an ordering for a division ring. Start with a semiordered commutative field F (e.g., $\mathbb{R}$). Construct the field $F((x))$ of formal Laurent series in one indeterminate x over F. Next form the division ring D of formal Laurent series in an indeterminate y with coefficients in $F((x))$ written on the left, according to the relation $yx = 2xy$. We note that the characteristic of F is not 2 (actually the characteristic of any ordered field is zero). Clearly, the centre of D is F. We will show that D has a semiordering which extends that of F, and this semiordering is not an ordering.

Let $G = \mathbb{Z} \times \mathbb{Z}$ be an abelian group under componentwise addition, ordered lexicographically by

$$(m,n) > 0 \text{ or } < 0 \quad \text{according as} \quad m > 0 \text{ or } m < 0$$

and

$$(0,n) > 0 \text{ or } < 0 \quad \text{according as} \quad n > 0 \text{ or } n < 0.$$

For $\alpha \in D$ define $\omega(\alpha) = (p,q) \in G$, where $x^q y^p$ is the monomial of smallest $p+q$ in the element α. This is a valuation on D whose residue field $\overline{D}$ can be identified with the centre F. We now lift the semiordering of the residue field F to D.

From the proof of Theorem 3.6, we expect the positive cone P_D of D to contain all α such that $\alpha + J_D$ is positive in $\overline{D} = F$ and $\alpha = u$ a unit in D. In fact, every $\alpha \in D$, where $\omega(\alpha) = (p,q)$, can be written in the form $ux^q y^p$ for a unit u in D. Since $yx = 2xy$, it follows that

$$x^2 y^2 = \left(\frac{1}{\sqrt{2}} xy\right)^2.$$

So every $\alpha \in D$ can be written as a product of an element of the form u, ux, uy or uxy (where u is a unit in D) and a non-zero square in D.

Let $M = \{u, ux, uy : u \text{ is a unit in } D \text{ and } u + J_D > 0 \text{ in } \overline{D}\} \cup \{uxy : u \text{ is a unit in } D \text{ and } u + J_D < 0 \text{ in } \overline{D}\}$. Take $P_D = M \cup MD^{\cdot 2}$. It is a routine matter to check that P_D is a positive cone of a semiordering of D which extends the semiordering of F. Clearly, $x > 0$ and $y > 0$ but $xy < 0$ and hence P_D is not closed under products, i.e., P_D is not an ordering.

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