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RIGID PATHS OF GENERIC 2-DISTRIBUTIONS WITH DEGENERATE POINTS ON 3-MANIFOLDS

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Abstract. We study rigid paths of generic 2-distributions with degenerate points on 3-manifolds. A complete description of such paths is obtained. For the proof, we construct separating surfaces of paths admissible for distributions.

0. Introduction. The purpose of this paper is to give a complete explicit description of rigid paths of generic 2-distributions, with degenerate points, on 3-manifolds. By a 2-*distribution with degenerate points* on a 3-manifold, we mean what is represented by either of the following objects:

(a) a Pfaffian equation $\{\omega = 0\}$, where ω is a smooth differential 1-form,

(b) a module $\langle X, Y\rangle$ of vector fields over the ring of smooth functions, which is generated by smooth vector fields X and Y.

In the following, we call such 2-distributions just "2-distributions" for convenience. For a usual plane field $\{\mathcal{D}_p \subset T_pM\}_{p\in M}$ on a 3-manifold M, a tangent plane $\mathcal{D}_p$ has the constant dimension $\dim \mathcal{D}_p = 2$ at any point $p \in M$. We note that a 2-distribution E, considered in this paper, may have a point $p \in M$ where $\dim E_p$ is $0, 1$ or 3. Such 2-distributions on 3-manifolds were studied by B. Jakubczyk and M. Ya. Zhitomirskiĭ in [JZh]. They gave a complete description of singularities of such distributions and a list of local normal forms in each case, (a) and (b).

Let M be a smooth connected manifold and E a 2-distribution on M. A path $\gamma : [\alpha, \beta] \to M$ is called *admissible* for E if it is tangent to E at any point $p \in \operatorname{Im}\gamma$. Let $a, b \in M$ be given two points. We denote the space of all admissible paths joining a to b by

$$\Omega_E(a, b) := \{\gamma : [0, 1] \to M \text{ admissible} \mid \gamma(0) = a,\ \gamma(1) = b\}.$$

A path $\gamma \in \Omega_E(a, b)$ is called *rigid* if any path in $\Omega_E(a, b)$ C^1-close enough to γ has the same image as γ. This notion was introduced in [BH]. Rigid paths of generic 2-distributions without degenerate points on 3-manifolds

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were studied by I. Zelenko and M. Ya. Zhitomirskiĭ in [ZZh]. They obtained a complete description of all rigid paths. We obtain a complete description of all rigid paths of generic 2-distributions with degenerate points on 3-manifolds. In particular, we study the local geometry near degenerate points.

In order to formulate the main theorem, we recall some fundamental notions. Throughout this paper, manifolds, distributions, functions etc. are assumed to be of class C^∞, and curves and paths are assumed to be of class C^1, unless mentioned otherwise.

Let E be a 2-distribution on a 3-manifold M. The dimension of E may be $2,1,0$ if E is a module of vector fields, and $2,3$ if E is a Pfaffian equation. A point $p \in M$ is called a *degenerate point* of a 2-distribution E if the dimension of E at p is not 2, $\dim E_p \neq 2$. The set of all degenerate points is denoted by $D = D(E)$.

A point $p \in M$ is called a *singular point* of E if it is a degenerate point of E or if $\dim E_p = 2$ and E is not a contact structure at p. Precisely, the latter statement means that $\dim \operatorname{span}\langle X_p, Y_p, [X,Y]_p\rangle < 3$ if E is a module of vector fields $\langle X, Y\rangle$, and $(\omega \wedge d\omega)_p = 0$ if E is a Pfaffian equation $\{\omega = 0\}$. We denote the set of all singular points by $S = S(E)$. For generic 2-distributions, $S(E)$ is a smooth surface if it is not empty (see Section 1 for precise genericity conditions). We sometimes call $S(E)$ the *Martinet surface.*

Now we define the characteristic vector field on the Martinet surface $S(E)$, which is determined by the generic 2-distribution E up to multiplications by non-vanishing functions (see [JZh]). First, we consider the case of a Pfaffian equation $E = \{\omega = 0\}$. Let Ω_S be a volume form on $S = S(E)$. (It is local if S is not orientable.) We define a vector field Z_E on S by the relation $Z_E \lrcorner \Omega_S = \omega|_{TS}$. This vector field Z_E is called the *characteristic vector field* of $E = \{\omega = 0\}$.

Next, we consider the case of a module of vector fields $E = \langle X, Y\rangle$. Let Ω be a volume form on M. (In the non-orientable case, we consider the local construction.) We define a differential 1-form ω corresponding to $E = \langle X, Y\rangle$ by $\omega(\cdot) := \Omega(X, Y, \cdot)$, and define a vector field $\widetilde{Z}$ on S by $\widetilde{Z} \lrcorner \Omega_S = \omega|_{TS}$. As the set $D(E)$ of degenerate points is a curve in $S(E)$ for a generic 2-distribution E, there is a function f on S which defines $D \subset S$. That is to say, $D = \{f = 0\}$ and $(df)_p \neq 0$ for all $p \in D$. We define the *characteristic vector field* Z_E of $E = \langle X, Y\rangle$ by $f \cdot Z_E = \widetilde{Z}$. Thus the characteristic vector field of a generic 2-distribution is defined. Singular points of Z_E are called *irregular points.* Non-degenerate non-singular points of Z_E are called *transversal points* of E, and the set of all those points is denoted by $\mathrm{Tr}(E)$.

According to the local behavior of these characteristic vector fields, singular points are classified into the following types, in [M], [Zh], [JP], and

[JZh] (see Theorem 1.2):

(a) E is a Pfaffian equation	(b) E is a module of vector fields
(0) transversal points	(0) transversal points
(1) hyperbolic points	(1) hyperbolic points
(2) elliptic points	(2) elliptic points
(3) saddle points	(3) saddle points
(4) node points	(4) node points
(5) focus points	(5) focus points
	(6) regular degenerate points

In the case of node points, there are two types of invariant manifolds of the characteristic vector fields. They correspond to the eigenvectors of linear differential equations for the characteristic vector fields. We call the invariant manifolds which correspond to the eigenvalues with largest absolute values the *rigid axes*. This notion is defined precisely for normal forms in Section 2.2.1.

In this paper, we consider immersed rigid paths. So by the term "rigid path", we mean a rigid path $\gamma : [0, 1] \to M$ whose tangent $\dot{\gamma}(t)$ is not zero at any point $t \in [0, 1]$. We note that even at the end points the tangents are well defined and not zero.

The following is the main theorem of this paper.

MAIN THEOREM. *Let E be a generic 2-distribution on a 3-manifold M. An immersed admissible path $\gamma : [\alpha, \beta] \to M$ is rigid if and only if*

(a) *in case $E = \{\omega = 0\}$ is a Pfaffian equation: the restriction of γ to the open interval (α, β) is a curve in $\mathrm{Tr}(E)$ and each of the end points $\gamma(\alpha)$, $\gamma(\beta)$ is of type $(0), (1), (3), (4)$ of the above table, and at least one of these ends is $(0), (1), (3)$, or (4) as an end of a rigid axis.*

(b) *in case $E = \langle X, Y \rangle$ is a module of vector fields: the restriction $\gamma|_{(\alpha,\beta)}$ lies in $\mathrm{Tr}(E)$ and the end points are of type $(0), (1), (6)$.*

I. Zelenko and M. Ya. Zhitomirskiĭ studied in [ZZh] rigid paths of generic 2-distributions on 3-manifolds. They considered generic 2-distributions without degenerate points. Such 2-distributions have singular points of type (a)-(0),(1),(2) only (or (b)-(0),(1),(2)). The result of this paper is an extension of their result to the case when generic 2-distributions have degenerate points.

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1. Classification of singularities and normal forms. First, we review the genericity condition for 2-distributions. Let $E = \langle X, Y \rangle$ or $\{\omega = 0\}$ be a 2-distribution on a 3-manifold M. We take a (local) volume form Ω

on M. A function H for E is defined as follows:

$$H = \Omega(X, Y, [X, Y]) \quad \text{if } E = \langle X, Y \rangle,$$
$$H \cdot \Omega = \omega \wedge d\omega \quad \text{if } E = \{\omega = 0\}.$$

We introduce the following notion to state the genericity condition. A linear operator in a 2-dimensional space is *non-resonant* if its eigenvalues λ_1, λ_2 satisfy $m_1\lambda_1 + m_2\lambda_2 \neq 0$ for any integers $m_1 \geq -1,\ m_2 \geq -1$.

We assume the following genericity conditions.

(G1) $j^1_p H \neq 0$ for any $p \in M$.

(G2) The linearization of Z_E at $p \in S(E)$ is non-degenerate if p is a singular point of Z_E. In addition, if p is a degenerate point of E, then the linearization is non-resonant.

Condition (G1) guarantees the structural smoothness of $S(E)$ as $S(E)$ is a level set of H. It is known that 2-distributions which satisfy (G1) and (G2) are generic (see [JZh]).

Now, we define some types of singular points. Let E be a generic 2-distribution on a 3-manifold M, and Z_E its characteristic vector field. The following is well known for non-degenerate points.

PROPOSITION 1.1 (Martinet, [M]). *Let $p \in S(E)$ be a non-degenerate irregular point. Then the sum of the eigenvalues of the linearization of Z_E at p is* 0.

As a consequence, we can distinguish the following three types of non-degenerate irregular points. A non-degenerate irregular point $p \in S(E)$ is called *hyperbolic* if the eigenvalues of Z_E at p are real and non-zero, *elliptic* if they are purely imaginary, and *parabolic* if both are 0. Parabolic points cannot appear because of condition (G2). According to Proposition 1.1 and this fact, any non-degenerate irregular point is hyperbolic or elliptic.

For degenerate singular points, it is known that there is no restriction on the eigenvalues of Z_E (see [JZh]). A degenerate irregular singular point p is called a *node* if the eigenvalues of Z_E at p are real non-zero and of the same sign, a *saddle* if they are real non-zero and of different signs, and a *focus* if they are of the form $a + b\sqrt{-1}$, $a \neq 0$, $b \neq 0$, $a, b \in \mathbb{R}$. By the genericity condition (G2), any degenerate irregular point is of one of the above three types.

All singular points are classified and normal forms are given as follows (see [M], [Zh], [JP], [JZh]):

THEOREM 1.2. *Any singular point of a generic* 2-*distribution E on a* 3-*manifold M is of one of the following types, and is equivalent to one of the normal forms below, where b, θ, λ are moduli.*

(a) $E = \{\omega = 0\}$ *is a Pfaffian equation*:

(0) *A regular non-degenerate point*: $\{dz + x^2 \cdot dy = 0\}$.
(1) *A hyperbolic point*: $\{dz + (xz + x^2 y + bx^3 y^2) \cdot dy = 0\}$.
(2) *An elliptic point*: $\{dz + (xz + x^3/3 + xy^2 + bx^3 y^2) \cdot dy = 0\}$.
(3) *A saddle point*: $\{z \cdot dz - \theta x \cdot dx + (\theta x + y) \cdot dy = 0\}$, $\theta > 0$.
(4) *A node point*: $\{z \cdot dz - \theta x \cdot dx + (\theta x + y) \cdot dy = 0\}$, $\theta < -4$.
(5) *A focus point*: $\{z \cdot dz - \theta x \cdot dx + (\theta x + y) \cdot dy = 0\}$, $-4 < \theta < 0$.

(b) $E = \langle X, Y \rangle$ *is a module of vector fields*:

(0) *A regular non-degenerate point*: $\langle (\partial/\partial x), (\partial/\partial y) - x^2 \cdot (\partial/\partial z) \rangle$.
(1) *A hyperbolic point*:
$\langle (\partial/\partial x), (\partial/\partial y) - (xz + x^2 y + bx^3 y^2) \cdot (\partial/\partial z) \rangle$.
(2) *An elliptic point*:
$\langle (\partial/\partial x), (\partial/\partial y) - (xz + x^3/3 + xy^2 + bx^3 y^2) \cdot (\partial/\partial z) \rangle$.
(3) *A saddle point*:
$\langle (\partial/\partial x), x \cdot (\partial/\partial y) + (z + \lambda x^2 + y^2) \cdot (\partial/\partial z) \rangle$, $\lambda > 0$.
(4) *A node point*:
$\langle (\partial/\partial x), x \cdot (\partial/\partial y) + (z + \lambda x^2 + y^2) \cdot (\partial/\partial z) \rangle$, $\lambda < -4$.
(5) *A focus point*:
$\langle (\partial/\partial x), x \cdot (\partial/\partial y) + (z + \lambda x^2 + y^2) \cdot (\partial/\partial z) \rangle$, $-4 < \lambda < 0$.
(6) *A regular degenerate point*: $\langle (\partial/\partial x), x \cdot (\partial/\partial y) + y \cdot (\partial/\partial z) \rangle$.

2. Local behavior of admissible paths. It is well known that rigid paths exist in $S(E)$ if E is a subbundle of TM (see [LS]). We note that the set $D(E)$ consists of isolated points if $E = \{\omega = 0\}$ is a Pfaffian equation. On the other hand, $D(E)$ is a curve in $S(E)$ and $D(E)$ itself is not admissible if $E = \langle X, Y \rangle$ is a module of vector fields (see [JZh]). Therefore, rigid paths exist in $S(E)$ also for generic 2-distributions. Thus we have only to investigate admissible paths in $S(E)$.

In this section we study the local existence of rigid paths. As mentioned above, rigid paths live on the Martinet surface $S(E)$. We consider neighborhoods of singular points of each type by using the normal forms of Theorem 1.2.

2.1. *Admissible paths containing non-degenerate singular points.* First we observe that there are no elliptic points on rigid paths. This is shown in [ZZh]. Near an elliptic point, the characteristic vector field Z_E has a weakly non-degenerate focus (spiral) type phase portrait. Thus admissible paths containing elliptic points have infinite lengths. As a rigid path is compact, rigid paths never have elliptic points.

Admissible paths containing hyperbolic points are studied in [ZZh]. There is no rigid path containing hyperbolic points in its interior.

PROPOSITION 2.1 (Zelenko–Zhitomirskiĭ, [ZZh]). *Let E be a generic 2-distribution. Admissible paths in $S(E)$ containing hyperbolic points in their interior are not rigid.*

2.2. *Admissible paths containing degenerate points of Pfaffian equations*

2.2.1. *Saddle and node points.* Let $E = \{\omega = 0\}$ be a Pfaffian equation, and p a saddle or node point of E. First, we choose a proper normal form. According to Theorem 1.2, there is a coordinate neighborhood (U, φ) of p with the normal form $\varphi^{-1*}\omega = z{\cdot}dz - \theta x{\cdot}dx + (\theta x + y){\cdot}dy$, with $\theta < -4$ or $\theta > 0$. Applying the coordinate transformation $f_\theta : (x, y, z) \mapsto (X, Y, z)$ defined by

$$(2.1) \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \theta + \sqrt{\theta^2 + 4\theta} & \theta - \sqrt{\theta^2 + 4\theta} \\ 2\theta & 2\theta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

we obtain the normal form

$$(2.2) \qquad \overline{\omega} := (f_\theta \circ \varphi)^{-1*}\omega = z{\cdot}dz + \Theta_1 Y{\cdot}dX + \Theta_2 X{\cdot}dY,$$

where $\Theta_1 := 2\theta^2(\theta - \sqrt{\theta^2 + 4\theta} + 4)$, $\Theta_2 := 2\theta^2(\theta + \sqrt{\theta^2 + 4\theta} + 4)$ are constants. We note that $\Theta_1 > 0$ if $\theta > 0$, $\Theta_1 < 0$ if $\theta < -4$, and $\Theta_2 > 0$. In these local coordinates, $E = \{\omega = z{\cdot}dz + \Theta_1 Y{\cdot}dX + \Theta_2 X{\cdot}dY = 0\}$, $S(E) = \{z = 0\}$, and

$$(2.3) \qquad Z_E = \Theta_2 X{\cdot}(\partial/\partial X) - \Theta_1 Y{\cdot}(\partial/\partial Y).$$

In the case when p is a saddle point ($\theta > 0$), the phase portrait of Z_E (see (2.3)) is of saddle type. The X- and Y-axes are separatrices. Then we obtain the following.

PROPOSITION 2.2. *Let $p \in S(E)$ be a saddle point of E, and $L \subset S(E)$ an invariant manifold of Z_E containing p. Then one of the following holds.*

- *There exist local coordinates (X, Y, z) and a constant $\theta > 0$ for which E is represented as a Pfaffian equation for a 1-form (2.2). In addition, L lies on the X-axis.*
- *There exist local coordinates (X', Y', z') and a constant $\theta' > 0$ for which E is represented as a Pfaffian equation for a 1-form (2.2). In addition, L lies on the Y-axis.*

In the case of node points ($\theta < -4$), the phase portrait of Z_E (see (2.3)) is of node type. We note that $0 < \Theta_2 < -\Theta_1$. Therefore there are many orbits of Z_E C^1-close to the X-axis. We call the X-axis the *flexible* axis, and the Y-axis the *rigid* axis.

PROPOSITION 2.3. *Let $p \in S(E)$ be a node point of E, and $L \subset S(E)$ an admissible curve of E containing p, whose image consists of orbits of Z_E. Then one of the following holds.*

• *There exist local coordinates* (X, Y, z) *and a constant* $\theta < -4$ *for which* E *is represented as a Pfaffian equation for a* 1*-form* (2.2)*. In addition*, L *lies on the* X*-axis.*

• *There exist local coordinates* (X', Y', z') *and a constant* $\theta' < -4$ *for which* E *is represented as a Pfaffian equation for a* 1*-form* (2.2)*. In addition*, L *lies on the* Y*-axis.*

Proof. Now we consider the case when L is neither the X- nor Y-axis with respect to the normal form $\overline{\omega}$ in (2.2). As L consists of orbits of $Z_E = \Theta_2 X\cdot(\partial/\partial X) - \Theta_1 Y\cdot(\partial/\partial Y)$, L lies on the curve $\{Y = C\cdot X^{-\Theta_1/\Theta_2},\ z = 0\}$ for some constant C. We note that $-\Theta_1/\Theta_2 > 1$. Consider the coordinate transformation

$$h : (X, Y, z) \mapsto (\widetilde{X}, \widetilde{Y}, \widetilde{z}) = (X, Y - CX^{-\Theta_1/\Theta_2}, z).$$

It preserves the 2-distribution $E = \{\overline{\omega} = 0\}$, in particular the 1-form $\overline{\omega}$ of (2.2):

$$\begin{aligned} h^*\overline{\omega} &= z\cdot dz + \Theta_1(Y - CX^{-\Theta_1/\Theta_2})\cdot dX + \Theta_2 X\cdot d(Y - CX^{-\Theta_1/\Theta_2}) \\ &= z\cdot dz + \Theta_1(Y - CX^{-\Theta_1/\Theta_2})\cdot dX \\ &\quad + \Theta_2 X\cdot\left(dY + C\,\frac{\Theta_1}{\Theta_2}\,X^{-\Theta_1/\Theta_2 - 1}\cdot dX\right) \\ &= z\cdot dz + \Theta_1 Y\cdot dX + \Theta_2 X\cdot dY = \overline{\omega}. \end{aligned}$$

The admissible path L, considered now, is mapped onto the $\widetilde{X}$-axis $\{\widetilde{Y} = 0, \widetilde{z} = 0\}$ by this transformation. ∎

The purpose of this section is to prove the following.

Proposition 2.4. *Let* E *be a generic Pfaffian equation, and* p *a saddle or node point of* E*. Then an immersed admissible path in* $S(E)$ *is not rigid if it contains* p *in its interior.*

This proposition is reduced to the following lemma.

Lemma 2.5. *Let* E *be a distribution on* $\mathbb{R}^3$ *defined by the normal form* (a)-(3), (4) *of Theorem* 1.2*. Consider admissible paths* $\mu^{\pm} : [-a, a] \to \mathbb{R}^3$ *defined by* $\mu^{\pm}(t) := ((\theta \pm \sqrt{\theta^2 + 4\theta})t, 2\theta t, 0)$*. Then there exist families* $\{\mu^{\pm}_{\varepsilon}\}$ *of admissible paths each of which has the following properties*:

(i) *the path* $\mu^{\pm}_{\varepsilon}$ *is not a reparametrization of* $\mu^{\pm}$ *for any* ε,
(ii) *the family* $\{\mu^{\pm}_{\varepsilon}\}$ *is* C^1*-convergent to* $\mu^{\pm}$ *as* ε *tends to* 0,
(iii) $\mu^{\pm}_{\varepsilon}(\pm a) = \mu^{\pm}(\pm a) = (\pm(\theta \pm \sqrt{\theta^2 + 4\theta})a, \pm 2\theta a, 0)$ *for any* ε,
(iv) $\mu^{\pm}_{\varepsilon}(t) - \mu^{\pm}(t)$ *is a flat function at* $t = \pm a$*.*

A function is called *flat* at a point if it vanishes at this point together with all its derivatives.

We can deduce Proposition 2.4 from Lemma 2.5 as follows. Let $p \in S(E)$ be a saddle or node point, and $\gamma : [\alpha, \beta] \to S(E)$ an immersed admissible path with $\gamma(t_0) = p$ for $t_0 \in (\alpha, \beta)$. On account of Theorem 1.2 and Propositions 2.2, 2.3, there exist a neighborhood U and local coordinates (X, Y, z) on U for which E is defined by the normal form (a)-(3) or (4), $p = (0, 0, 0)$, and the restriction of γ to a sufficiently small neighborhood of t_0 lies on the X- or Y-axis. We consider the case of the X-axis. We may assume, by a reparametrization, that γ is defined on $[-1, 1]$ and $p = \gamma(0)$, and that $\gamma|_{[-\delta,\delta]}$ is a path in U of the form $\gamma(t) = (X(t), Y(t), z(t)) = (t, 0, 0)$ for a small δ. Applying the coordinate transformation (2.1), we obtain $\gamma(t) = (x(t), y(t), z(t)) = ((\theta + \sqrt{\theta^2 + 4\theta})t, 2\theta t, 0)$. It is sufficient to show that this $\gamma(t)$ is not rigid. We set $a := \delta/2$. We apply Lemma 2.5 to $\gamma|_{[-a,a]}$, and denote the resulting family of admissible paths by μ_ε^+. We define a family of admissible paths γ_ε on $[-1, 1]$ by

$$\gamma_\varepsilon(t) := \begin{cases} \gamma(t), & -1 \le t \le -a,\ a \le t \le 1, \\ \mu_\varepsilon^+(t), & -a \le t \le a. \end{cases}$$

It is clear that γ_ε is not a reparametrization of γ for any $\varepsilon \neq 0$, that the end points of γ_ε coincide with those of γ, and that γ_ε is C^1-convergent to γ. Therefore, γ is not rigid.

In the case of the Y-axis, we can apply the same argument with μ_ε^- in place of μ_ε^+. ■

Proof of Lemma 2.5. We show the existence of $\{\mu_\varepsilon^+\}$. The existence of $\{\mu_\varepsilon^-\}$ can be proved in the same manner.

Let $f : [-a, a] \to \mathbb{R}$ be an odd function flat at $\pm a$. We define a function $g : [-a, a] \to \mathbb{R}$ by

$$g(t)g'(t) = -2\theta^2\varepsilon\{(\theta + \sqrt{\theta^2 + 4\theta} + 4)tf'(t) + (\theta - \sqrt{\theta^2 + 4\theta} + 4)f(t)\}$$

with the initial condition $g(-a) = 0$. Note that the function $\varphi(t) := g(t)g'(t)$ is odd as $f(t)$ is odd. Then $\partial g(t)^2/\partial t = 2\varphi(t)$ is odd, and $g(-a)^2 = 0$. This implies $g(t)^2$ is even and $g(a) = 0$. Now we define a family $\{\mu_\varepsilon^+\}$ of admissible paths by

$$\mu_\varepsilon^+(t) := ((\theta + \sqrt{\theta^2 + 4\theta})t + (\theta - \sqrt{\theta^2 + 4\theta})\varepsilon f(t), 2\theta(t + \varepsilon f(t)), g(t)).$$

According to the above argument, it satisfies conditions (i)–(iv). Direct calculations show that μ_ε^+ is admissible for each ε. ■

2.2.2. *Focus points.* Let $E = \{\omega = 0\}$ be a Pfaffian equation, and p a focus point of E. First, we choose a proper normal form. According to Theorem 1.2, there is a coordinate neighborhood (U, φ) of p with the normal form $\varphi^{-1*}\omega = z\cdot dz - \theta x\cdot dx + (\theta x + y)\cdot dy$, with a constant θ, $-4 < \theta < 0$.

The coordinate transformation $g_\theta : (x, y, z) \mapsto (X, Y, z)$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -\theta & \sqrt{-\theta^2 - 4\theta} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \tag{2.4}$$

leads to the normal form

$$(g_\theta \circ \varphi)^{-1*}\omega = z{\cdot}dz + (\Theta_1 X + \Theta_2 Y){\cdot}dX + (-\Theta_2 X + \Theta_1 Y){\cdot}dY,$$

where $\Theta_1 := -\theta^2 - 4\theta > 0$, $\Theta_2 := -\theta\sqrt{-\theta^2 - 4\theta} > 0$ are constants. Furthermore, in the cylindrical coordinates $X = r\cos\phi$, $Y = r\sin\phi$, we obtain the following normal form:

$$\widetilde{\omega} = z{\cdot}dz + \Theta_1 r{\cdot}dr - \Theta_2 r^2{\cdot}d\phi. \tag{2.5}$$

Then we have $E = \{z{\cdot}dz + \Theta_1 r{\cdot}dr - \Theta_2 r^2{\cdot}d\phi = 0\}$, $S(E) = \{z = 0\}$, $Z_E = \Theta_2 r^2{\cdot}(\partial/\partial r) + \Theta_1 r{\cdot}(\partial/\partial\phi)$ in these local coordinates. We note that the orbits of Z_E are logarithmic spirals. This normal form is invariant under rotations around the z-axis. Thus, the following holds.

PROPOSITION 2.6. *Let $p \in S(E)$ be a focus point of E, and L an admissible curve of E containing p. Then there exist local cylindrical coordinates (r, ϕ, z) and a constant θ, $-4 < \theta < 0$, for which E is represented as a Pfaffian equation for a 1-form* (2.5), *and $L = \{\phi = (\Theta_1/\Theta_2)\log r\}$.*

We note that an immersed path is defined as a mapping $\gamma : [0, 1] \to M$ on a compact interval with nonzero tangents $\dot\gamma(t) = 0$ at any point $t \in [0, 1]$. At a focus point, a nonzero tangent cannot be defined. Thus any immersed admissible path in $S(E)$ contains no focus points.

2.3. *Admissible paths containing degenerate points of modules of vector fields*

2.3.1. *Irregular degenerate points*

PROPOSITION 2.7. *Let E be a generic module of vector fields and $p \in S(E)$ an irregular degenerate point. Then there is no admissible path in $S(E)$ containing p.*

Proof. On account of Theorem 1.2, there are local coordinates (x, y, z) near p for which E is represented by one of the normal forms (b)-(3), (4), (5). For this normal form, the Martinet surface is $S(E) = \{z = \lambda x^2 - y^2\}$, and the characteristic vector field is $Z_E = -2((\lambda x + y){\cdot}(\partial/\partial x) + \lambda x{\cdot}(\partial/\partial y))$.

In the case of focus points ($-4 < \lambda < 0$), the integral curves of the characteristic vector field Z_E are logarithmic spirals, by a similar argument to that in Section 2.2.2. Thus any immersed admissible path contains no focus points.

Next, we consider the cases of saddle and node points ($\lambda < -4$, $\lambda > 0$, respectively). The invariant manifolds of Z_E are

$$E_1 = \left\langle (\lambda + \sqrt{\lambda^2 + 4\lambda}) \cdot \frac{\partial}{\partial x} + 2\lambda \cdot \frac{\partial}{\partial y} \right\rangle,$$
$$E_2 = \left\langle (\lambda - \sqrt{\lambda^2 + 4\lambda}) \cdot \frac{\partial}{\partial x} + 2\lambda \cdot \frac{\partial}{\partial y} \right\rangle.$$

At a degenerate point $p = (0, 0, 0)$, $E_p = \langle (\partial/\partial x) \rangle$ is transversal to E_1 and E_2. Thus there is no admissible curve in $S(E)$ containing a saddle or node point p. ■

2.3.2. *Regular degenerate points.* The aim of this section is to show the following proposition.

PROPOSITION 2.8. *Let E be a generic module of vector fields, and $p \in S(E)$ a regular degenerate point of E. Then an admissible path in $S(E)$ which contains p in its interior is not rigid.*

There is a local coordinate neighborhood $(U; x, y, z)$ for which E is defined by the normal form (b)-(6): $\langle (\partial/\partial x), x(\partial/\partial y) + y(\partial/\partial z) \rangle$, according to Theorem 1.2. Then, at the origin, $S = \{y = 0\}$, $D = \{x = 0, y = 0\}$, $Z_E = (\partial/\partial x)$.

By a similar argument to that in Section 2.2, using the normal form (b)-(6), the proposition above is reduced to the following lemma.

LEMMA 2.9. *Let E be a module of vector fields on $\mathbb{R}^3$ defined by the normal form* (b)-(6). *Consider an admissible path $\mu : [-a, a] \to \mathbb{R}^3$ defined by $\mu(t) := (t, 0, 0)$. Then there exists a family $\{\mu_\varepsilon\}$ of admissible paths with the following properties*:

(i) *the path μ_ε is not a reparametrization of μ for any $\varepsilon > 0$,*
(ii) *the family $\{\mu_\varepsilon\}$ is C^1-convergent to μ as ε tends to* 0,
(iii) $\mu_\varepsilon(\pm a) = \mu(\pm a) = (\pm a, 0, 0)$ *for any ε,*
(iv) $\mu_\varepsilon(t) - \mu(t)$ *is a flat function at $t = \pm a$.*

Proof. Let $f : [-a, a] \to \mathbb{R}$ be an odd function flat at $\pm a$. We define a function $g_\varepsilon : [-a, a] \to \mathbb{R}$ by the differential equation

$$\frac{d}{dt}\{g_\varepsilon(t)^2\} = 2\varepsilon t f'$$

with the initial condition $g_\varepsilon(-a) = 0$. We note that we can take f so that g_ε can be defined. Then we have $g_\varepsilon(a) = 0$. In fact, $g_\varepsilon(t)^2$ is an even function as f is odd.

Now, we define a family $\{\mu_\varepsilon(t)\}$ of paths by

$$\mu_\varepsilon(t) = (t, g_\varepsilon(t), \varepsilon f(t)).$$

According to the construction of g_ε above, this family satisfies the conditions (i)–(iv). These paths are admissible because

$$(\mu'_\varepsilon(t))_{\mu_\varepsilon(t)} = \left(\frac{\partial}{\partial x}\right)_{\mu_\varepsilon(t)} + \frac{\varepsilon f'(t)}{g_\varepsilon(t)}\left(x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)_{\mu_\varepsilon(t)}. \blacksquare$$

3. Separating surfaces. We introduce the notion of the separating surface defined in [ZZh]. Its existence implies the rigidity of paths.

3.1. *Definition of the separating surface.* Let Γ be a path defined on an interval $[\alpha,\beta]$, and γ a path defined on a subinterval $[a,b]\subset[\alpha,\beta]$. We set $\tau := \inf\{t\in[a,b] \mid \gamma(t)\notin \operatorname{Im}\Gamma\}$ if $\operatorname{Im}\gamma\not\subset\operatorname{Im}\Gamma$, and $\tau=b$ if $\operatorname{Im}\gamma\subset\operatorname{Im}\Gamma$. The image $\gamma((\tau,b])$ is called the *splitting part* of γ with respect to Γ and denoted by $\mathrm{SP}(\gamma,\Gamma)$.

Let E be a generic 2-distribution on a 3-manifold M, $\Gamma:[\alpha,\beta]\to M$ an admissible path, and $U\subset M$ a neighborhood of $\operatorname{Im}\Gamma$. Consider a 2-surface G in U containing $\operatorname{Im}\Gamma$, which divides U into two open connected parts U^+ and U^-.

Definition 3.1. The surface G is called a *separating surface* of Γ in U if $\mathrm{SP}(\gamma,\Gamma)$ is contained in U^+ for any admissible path γ defined on a subinterval $[a,b]\subset[\alpha,\beta]$, which is C^1-close to $\Gamma|_{[a,b]}$ and starts at a point of $\operatorname{Im}\Gamma$. We call the set U^+ the *positive side* of G.

It is clear from this definition that Γ is rigid if it has a separating surface. Further, we introduce stronger and weaker notions.

Definition 3.2. The surface G is called a *strongly separating surface* of Γ in U if G is a separating surface and $\operatorname{Im}\gamma\subset U^+$ for any admissible path γ defined on a subinterval $[a,b]\subset[\alpha,\beta]$, which is C^1-close enough to $\Gamma|_{[a,b]}$ and starts at a point of U^+.

Definition 3.3. The surface G is called a *weakly separating surface* of Γ in U if $\mathrm{SP}(\gamma,\Gamma)$ is contained in U^+ or $S(E)$ for any admissible path γ defined on a subinterval $[a,b]\subset[\alpha,\beta]$, which is C^1-close to $\Gamma|_{[a,b]}$ and starts at a point of $\operatorname{Im}\Gamma$.

3.2. *Local models of separating surfaces.* In this section we find local models of separating surfaces. Normal forms near transversal and hyperbolic points are considered in [ZZh]. Here we consider normal forms near degenerate points.

Lemma 3.4 ([ZZh]). *Let E be a distribution on $\mathbb{R}^3$ defined by* (a)-(0) *or* (b)-(0), *and $\Gamma(t)=(\overline{x}(t),\overline{y}(t),\overline{z}(t))=(0,t,0)$ an admissible path defined on an interval $[\alpha,\beta]$. Then the surface $G=\{z=0\}$ is a strongly separating surface of Γ in $U=\mathbb{R}^3$. The positive side is $U^+=\{z<0\}$.*

LEMMA 3.5 ([ZZh]). *Let E be a distribution on $\mathbb{R}^3$ defined by* (a)-(1) *or* (b)-(1), *and $\Gamma(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t)) = (0, t, 0)$ an admissible path defined on an interval $[0, \beta]$, $\beta > 0$. (Note that the origin is a hyperbolic point.) Then the surface $G = \{z = 0\}$ is a separating surface of Γ in $U = \mathbb{R}^3$. The positive side is $U^+ = \{z < 0\}$.*

In a similar way, we construct a local model near saddle points.

LEMMA 3.6. *Let E be a distribution on $\mathbb{R}^3$ defined by* (a)-(3), *and let*

$$\Gamma_1(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t)) = ((\theta + \sqrt{\theta^2 + 4\theta})t, 2\theta t, 0),$$
$$\Gamma_2(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t)) = ((\theta - \sqrt{\theta^2 + 4\theta})t, 2\theta t, 0)$$

be admissible paths defined on an interval $[0, \beta]$, $\beta > 0$, for $\theta > 0$. (Note that the origin is a saddle point.) Then the surface

$$G_1 = \left\{ z^2 + \frac{\theta + 4 + \sqrt{\theta^2 + 4\theta}}{\theta + 4}\,(y^2 - \theta x(x - y)) = 0 \right\}$$

is a separating surface of Γ_1 in $U = \mathbb{R}^3$. The positive side is

$$U_1^+ = \left\{ z^2 + \frac{\theta + 4 + \sqrt{\theta^2 + 4\theta}}{\theta + 4}\,(y^2 - \theta x(x - y)) < 0 \right\}.$$

The surface

$$G_2 = \left\{ z^2 + \frac{\theta + 4 - \sqrt{\theta^2 + 4\theta}}{\theta + 4}\,(y^2 - \theta x(x - y)) = 0 \right\}$$

is a separating surface of Γ_2, and the positive side is

$$U_2^+ = \left\{ z^2 + \frac{\theta + 4 - \sqrt{\theta^2 + 4\theta}}{\theta + 4}\,(y^2 - \theta x(x - y)) > 0 \right\}.$$

Proof. We apply the change of coordinates defined by (2.1). Then we have

$$\begin{aligned}
&E = \{z{\cdot}dz + \Theta_1 Y{\cdot}dX + \Theta_2 X{\cdot}dY = 0\},\\
&\Gamma_1(t) = (\overline{X}(t), \overline{Y}(t), \overline{z}(t)) = (t, 0, 0),\\
&\Gamma_2(t) = (\overline{X}(t), \overline{Y}(t), \overline{z}(t)) = (0, t, 0),\\
&G_1 = \{z^2 + 2\Theta_2 XY = 0\}, \qquad U_1^+ = \{z^2 + 2\Theta_2 XY < 0\},\\
&G_2 = \{z^2 + 2\Theta_1 XY = 0\}, \qquad U_2^+ = \{z^2 + 2\Theta_1 XY > 0\}.
\end{aligned}$$

We note that $\Theta_2 > \Theta_1 > 0$. We prove the lemma by using these coordinates.

(1) First, we consider the X-axis Γ_1. Let $\gamma(t) = (X(t), Y(t), z(t))$ be an admissible path defined on a subinterval $[a, b] \subset [0, \beta]$, which is C^1-close to $\Gamma_1|_{[a,b]}$ and starts at $\gamma(a) \in \operatorname{Im}\Gamma_1$. We note that $X(t) > 0$ for any $t \in (a, b]$

because of the C^1-closeness. On account of the admissibility of $\gamma(t)$, we have

$$Y'(t) = -\frac{1}{\Theta_2 X(t)}(z(t)z'(t) + \Theta_1 Y(t)X'(t)). \tag{3.1}$$

We define a function

$$Q(t) := \exp\left(\frac{\Theta_1}{\Theta_2}\int_0^t \frac{X'(s)}{X(s)}\,ds\right).$$

Then, by (3.1), we have

$$(Q(t)Y(t))' = -\frac{1}{\Theta_2}\cdot\frac{Q(t)}{X(t)}\,z(t)z'(t).$$

Integrating gives

$$\begin{aligned}
Q(t)Y(t) &= \int_a^t (Q(s)Y(s))'\,ds \\
&= -\frac{1}{2\Theta_2}\int_a^t \frac{Q(s)}{X(s)}\,(z(s)^2)'\,ds \\
&= -\frac{1}{2\Theta_2}\left\{\frac{Q(t)}{X(t)}\,z(t)^2 - \int_a^t \left(\frac{Q(s)}{X(s)}\right)' z(s)^2\,ds\right\} \\
&= -\frac{1}{2\Theta_2}\left\{\frac{Q(t)}{X(t)}\,z(t)^2 + \frac{\Theta_2-\Theta_1}{\Theta_2}\int_a^t \frac{Q(s)X'(s)z(s)^2}{X(s)^2}\,ds\right\}.
\end{aligned}$$

By the definitions, C^1-closeness of γ, and the assumption $\theta > 0$, we have

$$Q(t) > 0, \qquad X(t) > 0, \qquad X'(t) > 0, \qquad \Theta_2 > 0, \qquad \Theta_2 - \Theta_1 > 0$$

for $t > 0$. Then we obtain

$$Q(t)Y(t) \le -\frac{1}{2\Theta_2}\cdot\frac{Q(t)}{X(t)}\,z(t)^2$$

for any $t \in [a,b]$, and the fact that $Q(t_0)Y(t_0) = -(Q(t_0)/(2\Theta_2 X(t_0)))z(t_0)^2$ for some $t_0 \in (a,b]$ is equivalent to $z(s) = 0$ for any $s \le t_0$.

We note that admissible curves in the Martinet surface $S(E) = \{z = 0\}$ are orbits of the characteristic vector field $Z_E = \Theta_2 X\cdot(\partial/\partial X) - \Theta_1 Y\cdot(\partial/\partial Y)$. The origin is a saddle type singular point of Z_E, and the invariant manifolds are the X- and Y-axes. From this fact and the inequality $Q(t) > 0$, the statement above implies $z(t)^2 + 2\Theta_2 X(t)Y(t) \le 0$ for any $t \in [a,b]$, and the condition that $z(t_0)^2 + 2\Theta_2 X(t_0)Y(t_0) = 0$ for some $t_0 \in (a,b]$ is equivalent to $Y(s) = 0$ for any $s \le t_0$. Therefore, $\{z^2 + 2\Theta_2 XY = 0\} = G$ is a separating surface of γ in $U = \mathbb{R}^3$, and the positive side is $\{z^2 + 2\Theta_2 XY < 0\}$.

(2) Next, we consider the Y-axis Γ_2. Let $\widetilde{\gamma}(t) = (X(t), Y(t), z(t))$ be an admissible path defined on a subinterval $[a,b] \subset [0,\beta]$, which is C^1-close to

$\Gamma_2|_{[a,b]}$ and starts at $\gamma(a) \in \operatorname{Im} \Gamma_2$. We note that $Y(t) > 0$ for any $t \in (a, b]$ because of the C^1-closeness. On account of the admissibility of $\widetilde{\gamma}(t)$, we have

$$X'(t) = -\frac{1}{\Theta_1 Y(t)}(z(t)z'(t) + \Theta_2 X(t)Y'(t)).$$

A similar argument to one above with the function

$$\widetilde{Q}(t) := \exp\left(\frac{\Theta_2}{\Theta_1}\int_0^t \frac{Y'(s)}{Y(s)}\,ds\right)$$

shows that

$$\widetilde{Q}(t)X(t) = -\frac{1}{2\Theta_1}\left\{\frac{\widetilde{Q}(t)}{Y(t)}\,z(t)^2 + \frac{\Theta_1 - \Theta_2}{\Theta_1}\int_a^t \frac{\widetilde{Q}(s)Y'(s)z(s)^2}{Y(s)^2}\,ds\right\}.$$

We note $\Theta_1 - \Theta_2 < 0$. Then we have

$$z(t)^2 + 2\Theta_1 X(t)Y(t) \ge 0,$$

and if equality holds for some t_0, then $z(t) = 0$ for any $t \le t_0$. This implies that $G_2 = \{z^2 + 2\Theta_1 XY = 0\}$ is a separating surface, and $U_2^+ = \{z^2 + 2\Theta_1 XY \ge 0\}$ is the positive side. ■

Next, we consider node points. We note that $\Theta_1 < 0$. A similar argument to one above gives the following. As Γ_1 lies on the flexible axis of a node point, there exist C^1-close admissible paths on $S(E)$. Therefore the separating surface is weak.

LEMMA 3.7. *Let E be a distribution on $\mathbb{R}^3$ defined by* (a)-(4), *and let*

$$\Gamma_1(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t)) = ((\theta + \sqrt{\theta^2 + 4\theta})t, 2\theta t, 0),$$
$$\Gamma_2(t) = (\overline{x}(t), \overline{y}(t), \overline{z}(t)) = ((\theta - \sqrt{\theta^2 + 4\theta})t, 2\theta t, 0)$$

be admissible paths defined on an interval $[0, \beta]$, $\beta > 0$, *for* $\theta < -4$. (*Note that the origin is a node point.*) *Then the surface*

$$G_1 = \left\{z^2 + \frac{\theta + 4 + \sqrt{\theta^2 + 4\theta}}{\theta + 4}\,(y^2 - \theta x(x - y)) = 0\right\}$$

is a weakly separating surface of Γ_1 in $U = \mathbb{R}^3$. The positive side is

$$U_1^+ = \left\{z^2 + \frac{\theta + 4 + \sqrt{\theta^2 + 4\theta}}{\theta + 4}\,(y^2 - \theta x(x - y)) < 0\right\}.$$

The surface

$$G_2 = \left\{z^2 + \frac{\theta + 4 - \sqrt{\theta^2 + 4\theta}}{\theta + 4}\,(y^2 - \theta x(x - y)) = 0\right\}$$

is a (*non-weakly*) *separating surface of* Γ_2 *with positive side*

$$U_2^+ = \left\{ z^2 + \frac{\theta + 4 - \sqrt{\theta^2 + 4\theta}}{\theta + 4} \, (y^2 - \theta x(x - y)) > 0 \right\}.$$

Next, we find a normal model of separating surfaces near a regular degenerate point of modules of vector fields.

LEMMA 3.8. *Let* E *be a distribution on* $\mathbb{R}^3$ *defined by* (b)-(6). *Take an admissible path* $\Gamma(t) := (\overline{x}(t), \overline{y}(t), \overline{z}(t)) = (t, 0, 0)$ *defined on an interval* $[0, \beta]$, $\beta > 0$. (*Note that the origin is a regular degenerate point.*) *Then the surface* $G := \{z = 0\}$ *is a separating surface of* Γ *in* $U = \mathbb{R}^3$. *The positive side is* $U^+ = \{z > 0\}$.

Proof. Let $\gamma(t) = (X(t), Y(t), z(t))$ be an admissible path defined on a subinterval $[a, b] \subset [0, \beta]$, which is C^1-close to $\Gamma|_{[a,b]}$ and satisfies $\gamma(a) \in \operatorname{Im}\Gamma$. We note that $x(t) > 0$ for any $t \in (a, b]$ because of C^1-closeness. Then we obtain, from the admissibility of $\gamma(t)$, $z'(t) = y(t)y'(t)/x(t)$. Integration gives

$$z(t) = \frac{1}{2} \left\{ \frac{y(t)^2}{x(t)} + \int_0^t \frac{x'(s)}{x(s)^2} \, y(s)^2 \, ds \right\}.$$

On account of the C^1-closeness of $\gamma(t)$ to $\Gamma(t)$, we have $x(t) > 0$ and $x'(t) > 0$ for any $t \in (a, b]$. Further, the condition that there exists $t_0 \in (a, b]$ for which $z(t)$ vanishes is equivalent to $y(s) = 0$ for any $s \leq t_0$. Admissible curves on the Martinet surface $S(E) = \{y = 0\}$ are orbits of the characteristic vector field $Z_E = (\partial/\partial x)$. Therefore the statement above implies $z(t) \geq 0$ for any $t \in [a, b]$, and the condition that there exists $t_0 \in (a, b]$ for which $z(t)$ vanishes is equivalent to $z(s) = 0$ for any $s \leq t_0$. ∎

3.3. *Globalization of local separating surfaces.* In this section, we present a method of globalizing local separating surfaces. As a consequence, we obtain the proof of the Main Theorem.

Global separating surfaces of admissible paths in $\operatorname{Tr}(E)$ were constructed in [ZZh]. Let E be a generic 2-distribution on a 3-manifold M, and $\gamma : [\alpha, \beta] \to S(E)$ an immersed simple non-closed admissible path. We suppose that $\operatorname{Im}\gamma \subset \operatorname{Tr}(E)$. Let Γ be an admissible curve in $S(E)$ defined on an open interval whose image contains that of γ. Take any line subdistribution L of E which is transversal to $S(E)$ at any point of $\operatorname{Im}\gamma$. We define $G(\Gamma, L)$ as the set which consists of the points of all leaves of L crossing $\operatorname{Im}\gamma$.

PROPOSITION 3.9 ([ZZh]). (1) *There exists a neighborhood* $U \subset M$ *of* $\operatorname{Im}\gamma$ *for which* $G := U \cap G(\Gamma, L)$ *is a strongly separating surface of* γ *in* U.

(2) *Let U^+ be a positive side of G with respect to γ, and $\widetilde{\gamma}$ a reparametrization of γ with the inverse orientation. Then G is also a strongly separating surface of $\widetilde{\gamma}$, but U^+ is a negative side.*

I. Zelenko and M. Ya. Zhitomirskiĭ also constructed separating surfaces for admissible paths which may have hyperbolic points as their end points. They introduced a method of pasting separating surfaces to globalize local separating surfaces.

PROPOSITION 3.10 ([ZZh]). *Let $\gamma : [\alpha, \beta] \to S(E)$ be an admissible path. Take $t_1,\ t_2$ with $\alpha < t_2 < t_1 < \beta$. Set $\gamma_1 := \gamma|_{[\alpha,t_1]}$ and $\gamma_2 := \gamma|_{[t_2,\beta]}$. Suppose that γ_1 has a (resp. strongly) separating surface G_1 in a neighborhood U_1 of $\mathrm{Im}\,\gamma_1$ with positive side U_1^+, and that γ_2 has a strongly separating surface G_2 in a neighborhood U_2 of $\mathrm{Im}\,\gamma_2$ with positive side U_2^+, which satisfy $G_2 \cap (U_1 \cap U_2) = G_1 \cap (U_1 \cap U_2)$. Then $G := G_1 \cup G_2$ is a (resp. strongly) separating surface of γ in $U := U_1 \cup U_2$ with positive side $U^+ := U_1^+ \cup U_2^+$.*

REMARK 3.11. We note that this argument also holds for weakly separating surfaces. If local separating surfaces of both ends are weak, then the resulting separating surface is also weak.

According to the argument in Section 2, to complete the proof of the Main Theorem, it remains to consider the cases when admissible paths have as their ends saddle or node points of Pfaffian equations, and regular degenerate points of modules of vector fields.

(a-3) *Saddle points.* Let $\gamma : [0, 1] \to S(E)$ be an admissible path, one of whose ends $p := \gamma(0) \in S(E)$ is a saddle point of a Pfaffian equation. There is a coordinate neighborhood $(U_1; X, Y, z)$ of p in which $p = (0, 0, 0)$, $E|_{U_1} = \{\omega := z{\cdot}dz + \Theta_1 Y{\cdot}dX + \Theta_2 X{\cdot}dY = 0\}$, where $\Theta_1 := 2\theta^2(\theta - \sqrt{\theta^2 + 4\theta} + 4)$, $\Theta_2 := 2\theta^2(\theta + \sqrt{\theta^2 + 4\theta} + 4)$ for $\theta > 0$. According to Proposition 2.2, we may assume that $\mathrm{Im}\,\gamma \cap U_1$ lies on the X-axis $\{Y{=}z{=}0\}$ or the Y-axis $\{X{=}z{=}0\}$. In the following, we assume that $\mathrm{Im}\,\gamma \cap U_1$ lies on the X-axis and $\mathrm{Im}\,\widetilde{\gamma} \cap U_1$ on the Y-axis. We take $\delta \in (0, 1/2)$ for which $\gamma_1 := \gamma|_{[0,2\delta]}$ and $\widetilde{\gamma}_1 := \widetilde{\gamma}|_{[0,2\delta]}$ are paths in U_1. Then, by Lemma 3.6, $G_1 := \{z^2 + 2\Theta_2 XY = 0\}$ is a separating surface of γ_1 in U_1 with positive side $\{z^2 + 2\Theta_2 XY < 0\}$, and $\widetilde{G}_1 = \{z^2 + 2\Theta_1 XY = 0\}$ is a separating surface of $\widetilde{\gamma}_1$ with positive side $\{z^2 + 2\Theta_1 XY > 0\}$. We set $L_1 := \langle \Theta_2 X{\cdot}(\partial/\partial z) - z{\cdot}(\partial/\partial Y)\rangle \subset E|_{U_1}$ and $\widetilde{L}_1 := \langle \Theta_1 Y{\cdot}(\partial/\partial z) - z{\cdot}(\partial/\partial X)\rangle \subset E|_{U_1}$. It is a line subdistribution of E transverse to the Martinet surface $S(E) = \{z = 0\}$ at any point $q \in (\mathrm{Im}\,\gamma_1 \setminus \{p\}) \cap U_1$ or $\widetilde{q} \in (\mathrm{Im}\,\widetilde{\gamma}_1 \setminus \{p\}) \cap U_1$. We note that $G(\gamma_1, L_1) \subset G_1$ and $G(\widetilde{\gamma}_1, \widetilde{L}_1) \subset \widetilde{G}_1$ are local separating surfaces, by Proposition 3.9. Then, using Proposition 3.10, we can apply to this neighborhood the same argument as in [ZZh].

(a-4) *Node points.* On account of Proposition 2.3 and Lemma 3.7, we can apply a similar argument to that above.

Consequently, an admissible path in the Martinet surface of a generic Pfaffian equation has a separating surface if its interior points are transversal points and ends are transversal or hyperbolic or saddle or node points.

REMARK 3.12. We note that the separating surface obtained is not strong but weak if both ends are node points as ends of flexible axes, according to Remark 3.11. Then there exist C^1-close admissible paths in $S(E)$, that is, it is not rigid.

This implies statement (a) of the Main Theorem.

(b-6) *Regular degenerate points.* Similarly, let $\gamma:[0,1]\to S(E)$ be an admissible path, one of whose ends $p:=\gamma(0)\in S(E)$ is a regular degenerate point of a module of vector fields. There is a coordinate neighborhood $(U_3;x,y,z)$ of p in which $p=(0,0,0)$, $E|_{U_3}=\langle(\partial/\partial x), x\cdot(\partial/\partial y)+y\cdot(\partial/\partial z)\rangle$. We take $\delta\in(0,1/2)$ for which $\gamma_3:=\gamma|_{[0,2\delta]}$ is a path in U_3. In this case γ_3 lies on the x-axis. Then, by Lemma 3.8, $G_3:=\{z=0\}$ is a separating surface of γ_3 in U_3 with positive side $\{z>0\}$. We set $L_3:=\langle x\cdot(\partial/\partial y)+y\cdot(\partial/\partial z)\rangle\subset E|_{U_3}$. It is a line subdistribution which is transversal to the Martinet surface $S(E)=\{y=0\}$ at any point $q\in(\operatorname{Im}\gamma_3\setminus\{p\})\cap U_3$. We note that $G(\gamma_3,L_3)$ is a local separating surface by Proposition 3.9. Moreover, the positive side of $G(\gamma_3,L_3)$ is included in the positive side of G_3. Therefore G_3 is also a local separating surface of $\gamma_3\setminus\{p\}$. Thus, using Proposition 3.10, we can apply to this neighborhood the same argument as in [ZZh]. Consequently, an admissible path in the Martinet surface of a generic module of vector fields has a separating surface if its interior points are transversal points and ends are transversal or hyperbolic or regular degenerate points.

This completes the proof of the Main Theorem.

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