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SOME REMARKS ON DUALITY OF STATIONARY SEQUENCES

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Abstract. The paper clarifies the connection between Urbanik's and Miamee and Pourahmadi's concepts of duality for univariate weakly stationary random sequences. Some of Urbanik's results are proved in an alternative way and at the same time generalized to the multivariate case.

1. Introduction. In [8] K. Urbanik introduced a concept of duality for univariate weakly stationary random sequences over the group of integers and studied relations between linear prediction problems of the sequence in question and its dual. His main tool was the one-sided moving average representation of a completely non-deterministic sequence and his proofs were essentially "spectral free".

In [5] A. G. Miamee and M. Pourahmadi introduced a notion of dual sequences of univariate weakly stationary random sequences (in fact, even of certain harmonizable stable sequences) which have spectral densities whose inverses are integrable. Their method uses the spectral domain of a sequence and is also applicable if the group of integers is replaced by an arbitrary discrete abelian group. Miamee and Pourahmadi's method was generalized to the multivariate case in [2].

It turns out that the duality concept of Urbanik reduces to that of Miamee and Pourahmadi if the spectral measure of the sequence has the above mentioned properties. The aim of the present short note is to point out this connection and this way obtain alternative proofs of some of Urbanik's results. At the same time we will study multivariate processes where some results of Makagon and Weron (see [3], [4]) will be helpful.

2. Preliminaries. Let G be an (additively written) discrete abelian group, Γ its dual group, and σ the normalized Haar measure of Γ. For a subset Q of G denote by Q^{c} the complement $Q^{\mathrm{c}} := G \setminus Q$. Set $G_g := \{g\}^{\mathrm{c}}$, $g \in G$. If $Q \subseteq G_0$, let Q^* be the complement $Q^* := G_0 \setminus Q$.

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For $q \in \mathbb{N}$, let $\mathcal{M}_q$ be the algebra of $q \times q$-matrices with entries from the field of complex numbers $\mathbb{C}$. If $A \in \mathcal{M}_q$, the symbol A^+ stands for the Moore–Penrose inverse of A (see [6, p. 406]).

The class of all q-variate weakly stationary random processes over G will be denoted by $\mathcal{S}_q(G)$. For some basic facts concerning such processes we refer to [7].

Let $X := \{X_g\}_{g \in G} \in \mathcal{S}_q(G)$. If $Q \subseteq G$, denote by $[X, Q]$ the left Hilbert-$\mathcal{M}_q$-module spanned by X_g, $g \in Q$. Set $[X, G] =: [X]$. If L is a submodule of $[X]$, let $L^\perp$ be the (Gramian) orthogonal complement of L with respect to $[X]$. For a subset Q of G_0, it is an important problem of linear prediction theory to calculate the orthogonal projection $\widehat{X}_0(Q)$ of X_0 onto $[X, Q]$ as well as the prediction error matrix

$$\Delta(X, Q) := \langle X_0 - \widehat{X}_0(Q), X_0 - \widehat{X}_0(Q) \rangle.$$

Here $\langle \cdot, \cdot \rangle$ stands for the Gramian matrix.

We recall that a process X of $\mathcal{S}_q(G)$ is said to be $\mathcal{J}_0$-*regular* if

$$\bigcap_{g \in G} [X, G_g] = \{0\}$$

and $\mathcal{J}_0$-*singular* if

$$[X, G_g] = [X] \quad \text{for all } g \in G$$

(cf. [7, Definition 2.10] and [4, p. 133]).

Lemma 2.1 (cf. [7, Theorem 2.13]). *For $X \in \mathcal{S}_q(G)$, there exists a unique Wold decomposition $X = U + V$ into a $\mathcal{J}_0$-regular process U and a $\mathcal{J}_0$-singular process V.*

Let F be the (non-stochastic) spectral measure of X and F' the derivative of its absolutely continuous part with respect to σ. We will say that X *has a spectral density* if F is absolutely continuous and in that case F' is said to be the spectral density of X.

Lemma 2.2 (cf. [4, Theorem 5.3]). *A process X of $\mathcal{S}_q(G)$ is $\mathcal{J}_0$-regular if and only if it has a spectral density F' with the following two properties*:

(i) *The range of F' is constant σ-a.e.*
(ii) *The integral $\int_\Gamma F'^+ \, d\sigma$ exists.*

3. Dual processes. Following Urbanik ([8, p. 155]) we introduce a concept of a dual process. Note that our definitions of the dual process as well as of the sets K and K_0 below slightly differ from Urbanik's. Let P denote the orthogonal projector in $\mathbb{C}^q$ onto the range of $\Delta(X, G_0)$.

Definition 3.1. Let $X \in \mathcal{S}_q(G)$. A process X^* of $\mathcal{S}_q(G)$ is called the *dual process* of X if $[X^*] \subseteq [X]$, $\langle X_g^*, X_h \rangle = 0$ for $g \neq h$ and $\langle X_g^*, X_g \rangle = P$, $g, h \in G$.

The dual process exists and is unique. It can be described more explicitly. In fact, if Y_g denotes the orthogonal projection of X_g onto $[X, G_g]^{\perp}$ and $Z_g := \langle Y_g, Y_g\rangle^{+} Y_g$, $g \in G$, then the process $Z := \{Z_g\}_{g\in G}$ coincides with X^*. From results of Makagon and Weron ([3, p. 49 and formula (4.12)]) we now obtain the following assertion.

LEMMA 3.2. *Let F be the spectral measure of the process X of $\mathcal{S}_q(G)$. Then X^* has the spectral density $PF'^{+}P$.*

DEFINITION 3.3. Set

$$K := \{X \in \mathcal{S}_q(G) : [X] = \{0\} \text{ or } [X^*] \neq \{0\}\},$$
$$K_0 := \{X \in K : [X^*] = [X]\}.$$

LEMMA 3.4. *A process X of $\mathcal{S}_q(G)$ belongs to K_0 if and only if it is $\mathcal{J}_0$-regular.*

Proof. First note that Z_g spans $[X, G_g]^{\perp}$, $g \in G$. Hence, $[X^*]^{\perp} = [Z]^{\perp} = \{0\}$ if and only if $\bigcap_{g\in G}[X, G_g] = \{0\}$.

PROPOSITION 3.5. *Let $X \in \mathcal{S}_q(G)$. If X^* is its dual, then $X^* \in K_0$.*

Proof. Let U be the $\mathcal{J}_0$-regular part of the Wold decomposition of X. By Lemma 3.4, $U \in K_0$. Hence, $U^* \in K_0$ (cf. [8, Proposition 2.1]). On the other hand, it is obvious that $X^* = U^*$.

Let $X \in K_0$, $[X] \neq \{0\}$. Using Lemmas 3.4 and 2.2 and considering instead of X the process TXT^* for a suitable unitary T of $\mathcal{M}_q$, without loss of generality, we can assume that X has a spectral density of the form

$$\begin{pmatrix} F'_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

where F'_{11} is invertible σ-a.e. and F'^{-1}_{11} is integrable with respect to σ. Then by Lemma 3.2 we conclude that the spectral density of X^* is equal to

$$\begin{pmatrix} F'^{-1}_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

σ-a.e. Of course, we can confine ourselves to the upper left corner and thus apply results of [2].

Let $X \in \mathcal{S}_q(G)$ and $Q \subseteq G$. It is obvious that $[X, Q] \subseteq [X^*, Q^{\mathrm{c}}]^{\perp}$. We denote by $\Lambda(X)$ the set of all subsets Q of G such that $[X, Q] = [X^*, Q^{\mathrm{c}}]^{\perp}$.

THEOREM 3.6. *Let $X \in \mathcal{S}_q(G)$ and $Q \subseteq G_0$. Then*

$$\Delta(X, Q) \geq \Delta(X^*, Q^*)^{+}. \tag{3.1}$$

If, moreover, $X \in K_0$ and $Q \in \Lambda(X)$, then

$$\Delta(X, Q) = \Delta(X^*, Q^*)^{+}. \tag{3.2}$$

Proof. Let U be the $\mathcal{J}_0$-regular part of the Wold decomposition of X. Then $\Delta(X,Q) \geq \Delta(U,Q)$. Since $U \in K_0$, we have $\Delta(U,Q) \geq \Delta(U^*,Q^*)^+$ by Theorem 2.2 of [2]. Using these two inequalities and the fact that $X^* = U^*$, we immediately get (3.1). Equality (3.2) is a consequence of Theorem 2.2 of [2].

Theorem 3.6 shows that conditions for Q belonging to $\Lambda(X)$ are of interest. Some results can be found in [5] and [8] if $q = 1$ and in [2] if q is an arbitrary positive integer. Moreover, in [1], [2] and [5] duality is used to conclude some explicit linear prediction results.

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