C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 92 2002 NO. 2

STABILITY AND FLATNESS IN ACTS OVER MONOIDS

BY

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Abstract. Our aim in this paper is to study the concept of *stability* for acts over monoids and in the process develop connections with flatness properties of acts and with some of the current techniques and construction used in the homological classification of monoids. We also present new proofs of some results relating to torsion free acts over monoids and to the embeddability of semigroup amalgams.

1. Introduction. The author [9, 11, 13] introduced the concept of *stability* for acts over monoids and showed some strong connections with flatness of monoid acts, with the embeddability of monoid amalgams and with the structure of amalgamated free products of monoids. Our main aim in this paper is to develop these connections and introduce some related properties of act monomorphisms. We also demonstrate that some of the current results and techniques from the homological classification of monoids are related to these properties.

Some connections between stability, flatness and amalgamation were presented by the author in [11] and [13]. We examine in more detail the connections between stability and flatness in Section 3. The results in this section then act as a "template" for Section 4 where we define a hierarchy of properties related to stability and also to the notion of *unitary* submonoid. Howie [4] has provided connections between the unitary properties and amalgamation whilst Bulman-Fleming [1] summarises various connections with one of the related properties that we introduce and the homological classification of monoids. Section 5 briefly looks at when monomorphisms into various types of *indecomposable* act are stable. In Section 6 we demonstrate some connections between stability and the various flatness concepts used in the homological classification theory before proving some interesting consequences in Section 7.

2. Preliminaries. Throughout this paper U and S will be monoids. For basic definitions and results on monoids and U-acts see for example [6]. Let X be a right U-act and Y a left U-act. Then it is well known that

2000 *Mathematics Subject Classification*: Primary 20M50.

$x \otimes y = x' \otimes y' \in X \otimes_U Y$ if and only if there exists a *U-scheme over X and Y joining (x, y) to (x', y')*, that is to say, a finite set of equations

$$\begin{aligned} x &= x_1 u_1, & u_1 y &= v_1 y_2, \\ x_1 v_1 &= x_2 u_2, & u_2 y_2 &= v_2 y_3, \\ &\dots & &\dots \\ x_{n-1} v_{n-1} &= x_n u_n, & u_n y_n &= v_n y', \\ x_n v_n &= x', \end{aligned}$$

where $u_1, \dots, u_n, v_1, \dots, v_n \in U,\ x_1, \dots, x_n \in X,\ y_2, \dots, y_n \in Y$. From now on, all tensor products, unless indicated, will be over U.

We say that a right U-act X is *flat* if for all left U-monomorphisms $f : A \to B$, the induced map $1_X \otimes f : X \otimes A \to X \otimes B$ is one-to-one. X is said to be *weakly flat* if for all left ideals I of U, $X \otimes I \to X \otimes U$ is one-to-one, while it is *principally weakly flat* if $X \otimes I \to X \otimes U$ is one-to-one for all principal left ideals I of U. It can be shown that X is flat if and only if whenever $x \otimes y = x' \otimes y'$ in $X \otimes Y$ then there is a U-scheme over X and $Uy \cup Uy'$ joining (x, y) to (x', y').

If $f : X \to Y$ is a right U-monomorphism then we say that f is (*right*) *pure* if for all left U-acts Z, the canonical map $f \otimes 1_Z : X \otimes Z \to Y \otimes Z$ is one-to-one. Recall also that we say that f *splits* if there exists a right U-morphism $g : Y \to X$ with $g \circ f = 1_X$. It is easy to check that if f splits then f is pure. Purity is a kind of "dual" condition to flatness and it is clear that if every right U-monomorphism is pure then every left U-act is flat.

Let $f : X \to Y$ be a right U-map and $g : A \to B$ be a left U-map. We say that the pair (f, g) is *stable* (or f *stabilises* g) if $\mathrm{im}(f \otimes 1) \cap \mathrm{im}(1 \otimes g) = \mathrm{im}(f \otimes g)$:

$$\begin{tikzcd} X \otimes A \arrow[r, "1\otimes g"] \arrow[d, "f\otimes 1"'] & X \otimes B \arrow[d, "f\otimes 1"] \\ Y \otimes A \arrow[r, "1\otimes g"'] & Y \otimes B \end{tikzcd}$$

Alternatively, whenever $y \otimes g(a) = f(x) \otimes b$ in $Y \otimes B$, there exist $x' \in X$, $a' \in A$ such that $y \otimes g(a) = f(x') \otimes g(a')$. We say that a right U-monomorphism f is *stable* if (f, g) is stable for all left U-monomorphisms g. It follows from [12, Theorem 3.1] that if f is pure then f is stable.

We say that a semigroup U is *right reversible* if any two principal left ideals intersect and that U is *left collapsible* if for all $u, v \in U$, there exists $s \in U$ with $su = sv$. Notice that if U is left collapsible then U is right reversible.

If $f : X \to Y$ is a right U-monomorphism, we can define a right U-congruence on Y by $\varrho_f = (\mathrm{im}(f) \times \mathrm{im}(f)) \cup 1_Y$. The quotient Y/ϱ_f

is denoted by Y/X and referred to as the *Rees quotient of* Y *by* X, and the element $y\varrho_f$ is denoted by $\overline{y}$.

A strong connection between flatness and stability is provided by the following result.

LEMMA 2.1 ([11, Corollary 4.9]). *Let* $f : X \to Y$ *be a right* U*-monomorphism.*

(1) *If* Y/X *is flat then* U *is right reversible and* f *is stable.*
(2) *If* Y *is flat*, U *is right reversible and* f *is stable then* Y/X *is flat.*

Note that flatness of Y is needed in (2). To see this, note that given any right U-act X we see that $X \to X \mathbin{\dot\cup} X = Y$ is pure and hence stable and if Y/X is flat then it follows that X is flat. So if flatness was not needed in (2) then we would be able to deduce the contradictory statement that "every right reversible monoid is right absolutely flat".

Notice also that if U is right absolutely flat then every right U-monomorphism is stable.

Let X be a left U-act and define a U-congruence on X by $x \sim y$ if and only if either $x = y$ or there exist $x_1, \ldots, x_n \in X$, $u_1, \ldots, u_n, v_1, \ldots, v_n \in U$ with

$$x = u_1x_1, \quad v_1x_1 = u_2x_2, \quad \ldots, \quad v_nx_n = y.$$

We say that x is *connected* to y if $x \sim y$. Notice that U is right reversible if and only if for all $u, v \in U$, $u \sim v$ in $Uu \cup Uv$.

LEMMA 2.2 ([11, Lemma 2.3]). *Let* X *be a right* U*-act and* Y *a left* U*-act. If* $x \otimes y = x' \otimes y'$ *in* $X \otimes_U Y$, *then* $x \sim x'$ *in* X *and* $y \sim y'$ *in* Y.

We say that a right U-act F is *free* if it is isomorphic to a disjoint union of copies of U. A right U-act P is said to be *projective* if for all right U-epimorphisms $\beta : A \to B$ and all right U-maps $\alpha : P \to B$ there exists a right U-map $\gamma : P \to A$ such that the diagram

$$\begin{tikzcd}
& P \arrow[dl, "\gamma"'] \arrow[d, "\alpha"] \\
A \arrow[r, two heads, "\beta"'] & B
\end{tikzcd}$$

commutes. It can be shown that P is projective if and only if there exists a family $\{e_i : i \in I\}$ of idempotents of U such that $P \cong \dot{\bigcup}_{i\in I} e_iU$. A right U-act A is said to satisfy *condition* (P) if whenever $au = a'u'$ with $u, u' \in U$, $a, a' \in A$, there exist $a'' \in A$, $s, s' \in U$ with $a = a''s$, $a' = a''s'$ and $su = s'u'$, whilst A is said to satisfy *condition* (E) if whenever $au = au'$ with $a \in A$, $u, u' \in U$, there exist $a'' \in A$, $s \in U$ with $a = a''s$ and $su = su'$. A right U-act that satisfies conditions (P) and (E) is said to be *strongly flat.* Finally A is said to be *torsion free* if $au = a'u$ with u right cancellative implies

$a = a'$. Notice that we have the following strict implications:

$$\begin{aligned}&\text{free} \Rightarrow \text{projective} \Rightarrow \text{strongly flat} \Rightarrow \text{condition (P)} \Rightarrow \text{flat}\\ &\quad\Rightarrow \text{weakly flat} \Rightarrow \text{principally weakly flat} \Rightarrow \text{torsion free.}\end{aligned}$$

A great deal of work has been done in recent years to determine when one of these properties implies an other. See [1] for a useful survey of many of the main results in this area and [8] for a more comprehensive treatment.

The following elementary property of pushouts is easy to prove and will be used on a number of occasions without reference.

LEMMA 2.3 ([9, Lemma I.3.9]). *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "\alpha"] \\ C \arrow[r, "\beta"'] & D \end{tikzcd}$$

be a pushout diagram in the category of right U-acts. If $\alpha(b) = \beta(c)$ for some $b \in B$, $c \in C$ then there exist $a, a' \in A$ (not necessarily unique) such that $b = f(a)$ and $c = g(a')$. Also, if f and g are one-to-one then so are α and β and in that case, $a = a'$.

In fact we can show that $D = (B \mathbin{\dot{\cup}} C)/\sigma$ where $\sigma = \{(f(a), g(a)) : a \in A\}^{\#}$. Notice also that $D = \mathrm{im}(\alpha) \cup \mathrm{im}(\beta)$.

We refer the reader to [6, 10] for basic definitions and results concerning semigroup and monoid amalgams.

3. Flatness and stability. Let A be a sub-act of a right U-act B. In [9], the author defined A to be (*right*) *relatively unitary* in B if for all left ideals I of U, $A \cap BI = AI$. The notion of relatively unitary was first introduced by Howie in [4] and shown to be connected to embeddability of amalgams. If U is a subsemigroup of a semigroup S, then Howie called U *relatively unitary* in S if for all $u \in U$ and $s \in S$, $us \in U$ implies $us \in uU \cup \{u\}$, and $su \in U$ implies $su \in Uu \cup \{u\}$. He further showed that if (U, S) is a *weak amalgamation pair* then U is relatively unitary in S. The following is clear.

LEMMA 3.1. *Let $f : X \to Y$ be a right U-monomorphism. Then* $\mathrm{im}(f)$ *is right relatively unitary in Y if and only if for all left ideals I of U, the pair (f, ι_I) is stable, where $\iota_I : I \to U$ is the natural inclusion.*

In view of this last lemma, it seems appropriate to call a U-monomorphism $f : X \to Y$ *weakly stable* if $\mathrm{im}(f)$ is relatively unitary in Y. Notice that in fact f is weakly stable if and only if for all *principal* left ideals I of $U, (f, \iota_I)$ is stable. We shall use this fact later without reference.

We now deduce a "weak" version of Lemma 2.1 which first appeared in the author's PhD thesis but we include the proof here for completeness.

Theorem 3.2 ([9, Theorem IV.1.6]). *Let $f : X \to Y$ be a right U-monomorphism.*

(1) *If Y/X is weakly flat then U is right reversible and f is weakly stable.*

(2) *If Y is weakly flat, U is right reversible and f is weakly stable then Y/X is weakly flat.*

Proof. For simplicity, we assume that f is an inclusion map. Let I be a left ideal of U and consider the maps

$$\alpha : Y/X \otimes I \to (Y/X)I, \qquad \beta : Y/X \otimes I \to (YI)/(XI),$$
$$\gamma : (Y/X)I \to (YI)/(X \cap YI), \qquad \delta : (YI)/(XI) \to (YI)/(X \cap YI)$$

defined by

$$\alpha(\overline{y} \otimes i) = \overline{y}i, \quad \beta(\overline{y} \otimes i) = \overline{yi}, \quad \gamma(\overline{y}i) = \overline{yi}, \quad \delta(\overline{yi}) = \overline{yi}.$$

It is not too hard to check that these maps are all well-defined, that α, β and δ are surjections and that γ is a bijection. We therefore have a commutative diagram

$$\begin{tikzcd} Y/X \otimes I \arrow[r, "\beta"] \arrow[d, "\alpha"'] & YI/XI \arrow[d, "\delta"] \\ (Y/X)I \arrow[r, "\gamma"'] & (YI)/(X \cap YI) \end{tikzcd}$$

It is also not too hard to check that

1. $Y/X \otimes I \to Y/X \otimes U$ is one-to-one if and only if α is a bijection,
2. α is a bijection if and only if β and δ are bijections,
3. δ is a bijection if and only if $X \cap YI = XI$ i.e. $(f, I \to U)$ is stable.

The result will therefore follow if we can show

Lemma 3.3. *If β is a bijection then for all $i, j \in I$, $i \sim j$ in I. Conversely, if $Y \otimes I \to Y \otimes U$ is one-to-one and if for all $i, j \in I$, $i \sim j$ in I, then β is a bijection.*

Proof. Suppose that β is a bijection and let $i, j \in I$. Then it is clear that if $x \in X$ then $\overline{xi} = \overline{xj}$ in $(YI)/(XI)$ and so $\overline{x} \otimes i = \overline{x} \otimes j$ in $Y/X \otimes I$. Hence $i \sim j$ in I by Lemma 2.2.

Conversely, suppose that $\overline{yi} = \overline{y'j}$ in $(YI)/(XI)$ with $i, j \in I$, $y, y' \in Y$. Then we have two possibilities: either (i) $yi = y'j$ in YI, or (ii) $yi = xk$, $y'j = x'k'$ for some $x, x' \in X$, $k, k' \in I$. In the former case, we see that $y \otimes i = y' \otimes j$ in $Y \otimes I$ since $Y \otimes I \to Y \otimes U$ is one-to-one. Hence $\overline{y} \otimes i = \overline{y'} \otimes j$ in $Y/X \otimes I$ as required. In the latter case, again since $Y \otimes I \to Y \otimes U$ is one-to-one, we have $y \otimes i = x \otimes k$ and $y' \otimes j = x' \otimes k'$ in $Y \otimes I$. But since $k \sim k'$ in I we deduce from [11, Lemma 4.1] that $\overline{y} \otimes i = \overline{y'} \otimes j$ in $Y/X \otimes I$ as required. ∎

The proof of Theorem 3.2 is thus complete. ∎

Notice that the proof of the above result also allows us to deduce

THEOREM 3.4. *Let $f : X \to Y$ be a right U-monomorphism.*

(1) *If Y/X is principally weakly flat then f is weakly stable.*

(2) *If Y is principally weakly flat and f is weakly stable then Y/X is principally weakly flat.*

Later we shall consider similar results for the other flatness conditions mentioned earlier but for now we wish to study the connections between (weak) stability and (weak) flatness in more detail.

LEMMA 3.5. *Let U be a monoid and $u \in U$. Then the inclusion $uU \to U$ is right weakly stable if and only if u is regular.*

Proof. Since $(uU \to U, Uu \to U)$ is stable we have $u \in uUu$ and u is regular. Conversely, if u is regular and if $ux = yv$ for any $x, y, v \in U$ then $ux = uu^{-1}ux = uu^{-1}yv$ and so $(uU \to U, Uv \to U)$ is stable. ∎

Notice that if $f : X \to Y$ is any right U-monomorphism and if $e \in E(U)$, then $(f, Ue \to U)$ is stable. Consequently we can deduce the following result.

THEOREM 3.6. *Every* (*right*) *U-monomorphism is weakly stable if and only if U is regular.*

THEOREM 3.7. *Let U be a monoid and J a right ideal of U. Then the following are equivalent*:

(1) *$J \to U$ is stable*,
(2) *$J \to U$ is weakly stable*,
(3) *for all left ideals I of U, $J \cap I = JI$*,
(4) *for all finitely generated left ideals I of U, $J \cap I = JI$*,
(5) *for all $k \in U$, $J \cap Uk = Jk$*,
(6) *for all $j \in J$, $j \in Jj$.*

Proof. (2)⇒(3). Since $(J \to U, I \to U)$ is stable, $J \cap UI = JI$.

(6)⇒(1). If $\lambda : A \to B$ is a left U-monomorphism and if $\lambda(a) = jb$ for some $a \in A$, $j \in J$, $b \in B$ then since $j = kj$ for some $k \in J$ we see that $\lambda(a) = k\lambda(a)$ and so $(J \to U, \lambda)$ is stable as required.

The other implications are clear. ∎

The following corollary is probably well known.

COROLLARY 3.8. *Let U be a monoid and J a proper right ideal of U. Then U/J is flat if and only if it is weakly flat.*

Note that Bulman-Fleming [1] calls a right ideal J *left stable* if $j \in Jj$ for all $j \in J$. This condition is important in the homological classification theory of monoids.

We thus deduce

Corollary 3.9 ([7, Lemma 4]). *Let U be a monoid and J a right ideal of U. Then U/J is (weakly) flat if and only if U is right reversible and $j \in Jj$ for all $j \in J$. Also, U/J is principally weakly flat if and only if $j \in Jj$ for all $j \in J$.*

Lemma 3.10. *Let J be a right ideal of a monoid U and suppose that $u \in U$ is such that $J \cup uU \to U$ is (weakly) stable. If u is not regular then $u \in J$.*

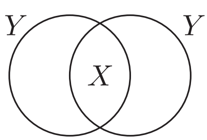
Proof. Since $u \in J \cup uU$ and since u is not regular, there exists $k' \in J$ with $u = k'u$ and so $u \in J$ as required. ■

Corollary 3.11. *If J is a finitely generated right ideal of a monoid U then $J \to U$ is (weakly) stable if and only if J is generated by regular elements.*

Let $f : X \to Y$ be a right U-monomorphism and consider the following pushout diagram:

$$(*) \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\alpha"] \\ Y \arrow[r, "\beta"'] & P \end{tikzcd}$$

We shall also make use of this diagram in later sections. Notice that if $\alpha(y) = \beta(y')$ in P then $y = y' = f(x)$ for some $x \in X$. It may be useful to view P pictorially as the union of two copies of Y amalgamating X:



Lemma 3.12 ([13, Lemma 1.5]). *Suppose that $f : X \to Y$ is a right U-monomorphism and consider the pushout diagram $(*)$. Then α and β split (and hence are pure).*

The flat case of the next result was proved in [13], the proof of the weakly flat and principally weakly flat cases are similar and are left to the reader.

Theorem 3.13 (cf. [13, Lemma 1.6]). *Suppose that $f : X \to Y$ is a right U-monomorphism and consider the pushout diagram $(*)$. Then P is (resp. weakly, principally weakly) flat if and only if Y is (resp. weakly, principally weakly) flat and f is (resp. weakly, weakly) stable.*

Recall [10] that a right U-monomorphism $f : X \to Y$ is said to be *perfect* if f is right pure and Y is right flat. It follows that if f is perfect then so are α and β. Perfect monomorphisms were essentially introduced by Hall [3] and later recast in terms of acts and tensor products by Howie [5]. They are closely connected with embeddability of semigroup amalgams.

In the case $Y = U$ and $X = J$, a proper right ideal of U, the pushout is sometimes denoted by $A(J)$ (see [1]). So we see that $A(J)$ is flat (equivalently weakly or principally weakly flat) if and only if for all $j \in J$, $j \in Jj$. In this case, Theorem 3.13 has a number of interesting applications in the homological classification of monoids. We provide in Section 6 a few similar results for some of the other flatness conditions that appear in that area.

To summarise this section, we have looked at the following questions: when are monomorphisms of the form $J \to U$ (weakly) stable? what is the connection between (weak) stability of $f : X \to Y$ and (weak) flatness of Y/X? what is the connection between (weak) stability of f and (weak) flatness of the pushout P?

4. Stable and unitary monomorphisms. We introduce five more properties of U-monomorphisms related to stability and consider similar questions to those above. These properties have been used in the past in connection with semigroup amalgams or with homological classification techniques.

Let $f : X \to Y$ be a right U-monomorphism. We shall say that f is *unitary* if $y \in \operatorname{im}(f)$ whenever $yu \in \operatorname{im}(f)$ and $u \in U$. It is not too hard to deduce that if f is unitary then f is (right) pure and hence stable.

In some respects the unitary property can be viewed as a kind of "strong stability", an observation that is strengthened by Theorem 6.1 below. Consider now the following related properties:

P-unitary: whenever $y, y' \in Y$, $u, u' \in U$ are such that $yu \neq y'u'$ but $yu, y'u' \in \operatorname{im}(f)$ then either $y \in \operatorname{im}(f)$ or $y' \in \operatorname{im}(f)$; equivalently

$$(\forall y, y' \in Y \setminus \operatorname{im}(f))(\forall u, u' \in U)(yu, y'u' \in \operatorname{im}(f) \Rightarrow yu = y'u').$$

E-unitary: whenever $y \in Y$, $u, u' \in U$ are such that $yu \neq yu'$ but $yu, yu' \in \operatorname{im}(f)$ then $y \in \operatorname{im}(f)$; equivalently

$$(\forall y \in Y \setminus \operatorname{im}(f))(\forall u, u' \in U)(yu, yu' \in \operatorname{im}(f) \Rightarrow yu = yu').$$

In [10], the author defined the concept of quasi-unitary subsemigroups (a generalisation of Howie's almost unitary condition) and provided a connection with this property and amalgamation. We generalise this concept here by defining a right U-monomorphism $f : X \to Y$ to be *quasi-unitary* if there exists a right U-morphism $\phi : Y \to Y$ such that

1. $\phi^2 = \phi$,
2. $\phi \circ f = f$,
3. for all $y \in Y$ and $u \in U$, $yu \in \operatorname{im}(f) \Rightarrow \phi(y) \in \operatorname{im}(f)$.

For example if f splits then f is quasi-unitary. Notice that $\mathrm{im}(\phi) \cap \mathrm{im}(f)$ is unitary in $\mathrm{im}(f)$.

Finally, we shall say that a right U-monomorphism $f : X \to Y$ is *c-unitary* if $y \in \mathrm{im}(f)$ whenever $yu \in \mathrm{im}(f)$ with u right cancellative.

Clearly, unitary implies P-unitary implies E-unitary. Note that the implications are in general strict: let G be a group and let $U = G^0$. Let $X = \{x\}$, $Y = \{x, y\}$ and define a U-act structure on Y by $zg = z$, $z0 = x$ for all $z \in Y$ and all $g \in G$. Then X is a subact of Y such that $X \to Y$ is P-unitary but not unitary. Moreover the natural inclusion $X \dot{\cup} X \to Y \dot{\cup} Y$ is E-unitary but not P-unitary.

Notice that if $X \to Y$ is P-unitary but not unitary then X contains a fixed point. Later we shall see that P-unitary is related to property (P) while E-unitary to property (E).

THEOREM 4.1. *If $f : X \to Y$ is a right P-unitary U-monomorphism then f is quasi-unitary.*

Proof. If f is in fact unitary then we can take $\phi = 1_Y$ and the result is trivial. Otherwise, there exist $y_0 \in Y \backslash X$ and $u_0 \in U$ with $y_0 u_0 = f(x_0) \in \mathrm{im}(f)$ and x_0 is unique with respect to this property. Notice that x_0 is then a fixed point in X. In this case define

$$\phi(y) = \begin{cases} y & \text{if } y \in \mathrm{im}(f), \\ f(x_0) & \text{otherwise.} \end{cases}$$

It is straightforward to check that ϕ has the required properties. ■

The proof in fact shows that f splits.

THEOREM 4.2. *Let U be a left reversible monoid and suppose that $f : X \to Y$ is E-unitary. Then f is quasi-unitary.*

Proof. The proof is similar to the previous one. We assume that f is not unitary and so there exists a fixed point in $\mathrm{im}(f)$. Choose one of these fixed points and call it $f(x_0)$. Now define $\phi : Y \to Y$ by

$$\phi(y) = \begin{cases} y & \text{if } y \in \mathrm{im}(f), \\ f(x_1) & \text{if } y \notin \mathrm{im}(f),\ yu = f(x_1) \text{ for some } u \in U, \\ f(x_0) & \text{otherwise.} \end{cases}$$

It is left to the reader to see that ϕ has the required properties. The only point worth noting is that if $yu \notin \mathrm{im}(f)$ and $yuU \not\subseteq \mathrm{im}(f)$ then $yU \not\subseteq \mathrm{im}(f)$ since U is left reversible. ■

THEOREM 4.3. *Let $f : X \to Y$ be a right quasi-unitary monomorphism. Then f is pure.*

Proof. Let B be a left U-act and suppose that $f(x) \otimes b = f(x') \otimes b'$ in $Y \otimes B$ so that we have a U-scheme over Y and B:

$$\begin{aligned} f(x) &= y_1u_1, & u_1b &= v_1b_2, \\ y_1v_1 &= y_2u_2, & u_2b_2 &= v_2b_3, \\ &\dots & &\dots \\ y_{n-1}v_{n-1} &= y_nu_n, & u_nb_n &= v_nb'. \\ y_nv_n &= f(x'), \end{aligned}$$

We apply ϕ to the left hand equations to get

$$\begin{aligned} f(x) &= \phi(y_1)u_1, & u_1b &= v_1b_2, \\ \phi(y_1)v_1 &= \phi(y_2)u_2, & u_2b_2 &= v_2b_3, \\ &\dots & &\dots \\ \phi(y_{n-1})v_{n-1} &= \phi(y_n)u_n, & u_nb_n &= v_nb'. \\ \phi(y_n)v_n &= f(x'), \end{aligned}$$

But $\phi(y_1) \in \mathrm{im}(f)$ and so $\phi(y_2) \in \mathrm{im}(f)$. Continuing in this way we see that this new scheme is actually over $\mathrm{im}(f)$ and B and so $x \otimes b = x' \otimes b'$ in $X \otimes B$ as required. ∎

THEOREM 4.4. *Let $f : X \to Y$ be an E-unitary right U-monomorphism. Then f is stable.*

Proof. Suppose that $\lambda : A \to B$ is a left U-monomorphism and that $y \otimes \lambda(a) = f(x) \otimes b$ in $Y \otimes B$. If $y \in \mathrm{im}(f)$ then there is nothing to show, so we can suppose that the following is a minimal length U-scheme over Y and B connecting $(y, \lambda(a))$ to $(f(x'), b')$ for some $x' \in X$, $b' \in B$:

$$\begin{aligned} y &= y_1u_1, & u_1\lambda(a) &= v_1b_2, \\ y_1v_1 &= y_2u_2, & u_2b_2 &= v_2b_3, \\ &\dots & &\dots \\ y_{n-1}v_{n-1} &= y_nu_n, & u_nb_n &= v_nb'. \\ y_nv_n &= f(x'), \end{aligned}$$

Now because of the minimality of the scheme, $y_n \notin \mathrm{im}(f)$ and so since f is E-unitary we deduce that $f(x')$ is a fixed point in Y and hence the following is a U-scheme over Y and B joining $(y, \lambda(a))$ and $(f(x'), \lambda(a))$ as required:

$$\begin{aligned} y &= y_1u_1, & u_1\lambda(a) &= v_1b_2, \\ y_1v_1 &= y_2u_2, & u_2b_2 &= v_2b_3, \\ &\dots & &\dots \end{aligned}$$

$$
\begin{aligned}
y_{n-1}v_{n-1}, &= y_n u_n, & u_n b_n &= v_n b', \\
y_n v_n &= f(x')v_n, & v_n b' &= u_n b_n, \\
f(x')u_n &= f(x')v_{n-1}, & v_{n-1}b_n &= u_{n-1}b_{n-1}, \\
&\dots & &\dots \\
f(x')u_2 &= f(x')v_1, & v_1 b_2 &= u_1\lambda(a). \ \blacksquare \\
f(x')u_1 &= f(x'),
\end{aligned}
$$

We therefore have a sequence of implications

$$
\begin{tikzcd}
 & & \text{quasi-unitary} \arrow[r, Rightarrow] & \text{pure} \arrow[dr, Rightarrow] & & \\
\text{unitary} \arrow[r, Rightarrow] & \text{P-unitary} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & & \text{stable} \arrow[r, Rightarrow] & \text{weakly stable} \\
 & & \text{E-unitary} \arrow[urr, Rightarrow] & & &
\end{tikzcd}
$$

It is also easy to verify

LEMMA 4.5. *Let U be a monoid and $f : X \to Y$ a right U-monomorphism with Y torsion free. If f is weakly stable then f is c-unitary.*

With the possible exceptions of weakly stable $\Rightarrow$ stable, E-unitary $\Rightarrow$ pure and E-unitary $\Rightarrow$ quasi-unitary, there are no other possible implications between these properties as will be shown later in this section.

Let U be a submonoid of a monoid S. Hall [3] (using the language of representations rather than acts) defined U to have the *orbit preserving extension property* in S if for all U-acts X, there exists an S-act Z with a unitary U-monomorphism $f : X \to Z$ (see also [10]). He proved [3, Theorem 27] that if $[U; S, T]$ is an amalgam of monoids such that U has the orbit preserving extension property in S and T then the amalgam is strongly embeddable and U has the orbit preserving extension property in $S *_U T$.

Recall [10] that a monoid U is said to have the *extension property* in a containing monoid S if for all right U-acts X and all left U-acts Y the canonical morphism $X \otimes_U Y \to X \otimes_U S \otimes_U Y$ is one-to-one. The following is not too hard to prove:

THEOREM 4.6. *Let U be a submonoid of a monoid S. Then U has the extension property in S if and only if for every right U-act X there exists a right S-act Z and a right pure U-monomorphism $f : X \to Z$.*

Proof. If U has the extension property in S then $X = X \otimes_U U \to X \otimes_U S \otimes_U U = X \otimes_U S$ is one-to-one. It is then clear that it is a right pure monomorphism.

Conversely, if X is any right U-act and Y any left U-act then there exists a right S-act Z and a right pure U-monomorphism $f : X \to Z$. Define $\phi : X \otimes_U S \otimes_U Y \to Z \otimes_U Y$ by $\phi(x \otimes s \otimes y) = f(x)s \otimes y$ and check that the diagram

$$\begin{tikzcd} X \otimes_U Y \arrow[r] \arrow[d] & X \otimes_U S \otimes_U Y \arrow[dl] \\ Z \otimes_U Y & \end{tikzcd}$$

commutes. The result then follows on noting that the vertical map is one-to-one. ∎

The author [10, Theorem 6.1] proved that if $[U; S, T]$ is a monoid amalgam with U having the extension property in S and T then the amalgam is embeddable and U has the extension property in $S *_U T$. Moreover [10, Theorem 6.11], U is an amalgamation base if and only if U has the extension property in every containing monoid.

Consequently, there are very strong connections between these types of monomorphisms and amalgamation.

The following is clear:

THEOREM 4.7. *Let J be a proper right ideal of a monoid U. The following are equivalent*:

(1) *$J \to U$ is P-unitary*,
(2) *$J \to U$ is E-unitary*,
(3) *$|J| = 1$.*

LEMMA 4.8. *Let J be a right ideal of a monoid U. The following are equivalent*:

(1) *$J \to U$ splits*,
(2) *$J \to U$ is quasi-unitary*,
(3) *there exists an idempotent $e \in J$ with $j = ej$ for all j in J.*

Proof. (1)⇒(2) is clear.

(2)⇒(3). Suppose that $J \to U$ is quasi-unitary with related morphism $\phi : U \to U$. Then let $e = \phi(1)$ and note that $e^2 = \phi(1)\phi(1) = \phi(1\phi(1)) = \phi^2(1) = \phi(1) = e$, that $j = \phi(j) = \phi(1)j = ej$ and that $e \in J$ since $j.1 \in J$ for all j in J, as required.

(3)⇒(1). Suppose now that there exists an idempotent $e \in J$ with $j = ej$ for all j in J. Then define a right U-map $\phi : U \to J$ by $\phi(u) = eu$ and note that $J \to U$ splits. ∎

COROLLARY 4.9. *For any monoid U and any $u \in U$, the inclusion $uU \to U$ is quasi-unitary if and only if u is regular.*

We then easily deduce the following

THEOREM 4.10. *Let U be a monoid. Then every inclusion $J \to U$ of right ideals J is quasi-unitary if and only if U is regular and every right ideal is principal.*

LEMMA 4.11. *Let J be a right ideal of a monoid U and suppose that $u \in U$ is such that $J \cup uU \to U$ is quasi-unitary. If u is not regular then $u \in J$.*

Proof. There exists $e \in J \cup uU$ with $u = eu$ and since u is not regular we have $e \in J$ from which it follows that $u \in U$. ∎

We then deduce the following result (which also follows from Corollary 3.11):

COROLLARY 4.12. *If J is a finitely generated right ideal of a monoid U such that $J \to U$ is quasi-unitary then J is generated by regular elements.*

If $x, y \in U$ then we shall denote by $\lambda(x, y)$ the smallest left U-congruence on U which contains (x, y). Notice that $(u, v) \in \lambda(x, y)$ if and only if either $u = v$ or there exist $s_1, \ldots, s_n, x_1, \ldots, x_n, y_1, \ldots, y_n \in U$ with $\{x_i, y_i\} = \{x, y\}$ for $i = 1, \ldots, n$ such that

$$\begin{aligned} u &= s_1 x_1, \\ s_1 y_1 &= s_2 x_2, \\ &\cdots \\ s_n y_n &= v. \end{aligned}$$

Notice that if B is any left U-act in which there exists $b \in B$ with $xb = yb$ then $ub = vb$ for any $u, v \in U$ with $(u, v) \in \lambda(x, y)$.

Let J be a right ideal of a monoid U. We shall say that $J \to U$ is *strongly connected* if for all $x, y \in J$, there exists a sequence of equalities over J (as a U-act)

$$x = x_1 u_1, \quad x_1 v_1 = x_2 u_2, \quad \ldots, \quad x_n v_n = y$$

such that for $i = 1, \ldots, n$, $(x_i v_i, x) \in \lambda(x, y)$. In this case we shall also say that x and y are *strongly connected.* Notice that this holds if $x \otimes \overline{1} = y \otimes \overline{1}$ in $J \otimes U/\lambda(x, y)$.

THEOREM 4.13. *Let J be a right ideal of a monoid U. Then $J \to U$ is pure if and only if $J \to U$ is stable and strongly connected.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$) Suppose that the given condition holds and that B is any left U-act such that $ib = jb'$ in B for some $i, j \in J$. By assumption, there exist $k, k' \in J$ with $i = ki$, $j = k'j$ and a collection of equations over J

$$k = x_1 u_1, \quad x_1 v_1 = x_2 u_2, \quad \ldots, \quad x_n v_n = k'$$

such that for each $l = 1\dots,n$, $(x_l v_l, k) \in \lambda(k,k')$. Let $b'' = ib$ and notice that since $kb'' = k'b''$ we have $x_l u_l b'' = x_l v_l b'' = kb''$. Also, for each x_l there exists $k_l \in J$ with $x_l = k_l x_l$ and so we can construct a U-scheme over J and B:

$$\begin{aligned} i &= k_1 i, & ib &= x_1 v_1 b'', \\ k_1 x_1 v_1 &= k_2 x_2 u_2, & x_2 u_2 b'' &= x_2 v_2 b'', \\ k_2 x_2 v_2 &= k_3 x_3 u_3, & x_3 u_3 b'' &= x_3 v_3 b'', \\ &\dots & &\dots \\ k_{n-1} x_{n-1} v_{n-1} &= k_n x_n u_n, & x_n u_n b'' &= jb', \\ k_n j &= j, \end{aligned}$$

as required. ■

LEMMA 4.14. *Let U be a monoid and let $u \in U$. Then $uU \to U$ is strongly connected.*

Proof. Since $uv = u.v$, $u.w = uw$ and $(uv, uw) \in \lambda(uv, uw)$ for all $v, w \in U$, the result follows. ■

It is easy to check

THEOREM 4.15. *Let U be a monoid and let $x \neq y \in U$. Then $xU \cup yU \to U$ is strongly connected if and only if there exists $z \in xU \cap yU$ such that $z\lambda(x,y)x$.*

Proof. Let $J = xU \cup yU$ and suppose that $J \to U$ is strongly connected. Then $x = x_1 u_1,\ x_1 v_1 = x_2 u_2,\ \dots,\ x_n v_n = y$, with $(x_i v_i, x) \in \lambda(x,y)$. If all $x_i \in xU$ then put $z = y$ else let i be the smallest index with $x_i u_i \in yU$ and put $z = x_i u_i$. The result then follows.

Conversely, we see from the proof of Lemma 4.14 that we need only show that xu is strongly connected to yv for any $u, v \in U$. But if $z = xu_0 = yv_0$ is such that $z\lambda(x,y)x$ then $xu = x.u, x.u_0 = y.v_0,\ y.v = yv$ and so $J \to U$ is strongly connected. ■

COROLLARY 4.16. *Let J be a right ideal of a monoid U. Then $J \to U$ is strongly connected if for all $x, y \in J$, $xU \cup yU \to U$ is strongly connected.*

Proof. Let $x, y \in J$ and note that x is strongly connected to y in $xU \cup yU$ and so in J. Hence $J \to U$ is strongly connected. ■

COROLLARY 4.17. *Let U be a monoid. Then $J \to U$ is strongly connected for all right ideals J of U if and only if for all $x, y \in U$ there exists $z \in xU \cap yU$ such that $z\lambda(x,y)x$.*

From Corollary 3.11 we see that

COROLLARY 4.18. *If J is a finitely generated right ideal of a monoid U such that $J \to U$ is pure then J is generated by regular elements.*

Notice that in general stable $\not\Rightarrow$ pure since if U is regular then $J \to U$ is stable for all right ideals J of U but not every regular monoid is left absolutely weakly flat. Also if U is inverse then U is left absolutely flat and so $J \to U$ is pure for all right ideals J of U but since not every right ideal of U is principal it follows that in general pure $\not\Rightarrow$ quasi-unitary.

It would be of interest to have a useful description of $J \to U$ being c-unitary. Certainly, if it is and if J is a proper right ideal of U then J cannot contain any right cancellative elements. Moreover, if J contains *all* the non-right cancellative elements of U then $J \to U$ is c-unitary as is the case if J consists entirely of regular elements or even if for every $j \in J$ there exist $k \in J$, $u \in U$ with k regular and $j = ku$. However, at the moment we do not even have a satisfactory characterisation of $uU \to U$ to be c-unitary.

As a summary, notice that we have

COROLLARY 4.19. *Let U be a monoid and $u \in U$ and consider the inclusion $\iota : uU \to U$. Then the following are equivalent*:

(1) *u is regular*,
(2) *ι splits*,
(3) *ι is quasi-unitary*,
(4) *ι is pure*,
(5) *ι is stable*,
(6) *ι is weakly stable.*

LEMMA 4.20. *Suppose that $f : X \to Y$ is a right U-monomorphism and consider the pushout diagram* $(*)$. *Then α is E-unitary* (*resp. P-unitary*, *c-unitary*, *unitary*) *if and only if f is.*

Proof. We only prove the E-unitary case, the others being similar.

($\Rightarrow$) Suppose that $yu = f(x)$, $yv = f(x')$ and $yu \neq yv$ with $y \in Y$, $x, x' \in X$, $u, v \in U$. Then $\beta(y)u \neq \beta(y)v$ but $\beta(y)u = \beta(yu) = \beta(f(x)) = \alpha(f(x))$ and similarly $\beta(y)v = \alpha(f(x'))$. Now $\beta(y) \in \mathrm{im}(\alpha)$ and hence $y \in \mathrm{im}(f)$.

($\Leftarrow$) Suppose that $pu \neq pv$ but $pu = \alpha(y_1)$, $pv = \alpha(y_2)$ and $p = \beta(y)$ for $y, y_1, y_2 \in Y$, $u, v \in U$. Then $yu = f(x_1)$, $yv = f(x_2)$ for some $x_1, x_2 \in X$ and $yu \neq yv$. Hence $y \in \mathrm{im}(f)$ and so $p \in \mathrm{im}(\alpha)$. ∎

THEOREM 4.21. *For any monoid U, the following are equivalent*:

(1) *every right U-monomorphism is E-unitary*,
(2) *every right weakly stable U-monomorphism is E-unitary*,
(3) *every right stable U-monomorphism is E-unitary*,
(4) *every right pure U-monomorphism is E-unitary*,
(5) *every right quasi-unitary U-monomorphism is E-unitary*,
(6) *every inclusion $J \to U$ with J a right ideal of U is E-unitary*,
(7) *U is a group or a 0-group.*

Proof. (1)⇒(2)⇒(3)⇒(4)⇒(5) are clear.

(5)⇒(6). This follows from Lemmas 3.12 and 4.20.

(6)⇒(7). U can have at most one proper right ideal $\{0\}$ and so U is a group or a 0-group.

(7)⇒(1). If U is a group then every right U-monomorphism is unitary and so is E-unitary. Suppose then that U is a 0-group and let $f : X \to Y$ be a right U-monomorphism and suppose also that $yu \neq yv$ but $yu = f(x)$, $yv = f(x')$. If $y \notin \mathrm{im}(f)$ then it follows that $u = v = 0$, giving a contradiction. ∎

In a similar way we have

THEOREM 4.22. *For any monoid U, the following are equivalent*:

(1) *every right U-monomorphism is P-unitary*,
(2) *every right weakly stable U-monomorphism is P-unitary*,
(3) *every right stable U-monomorphism is P-unitary*,
(4) *every right pure U-monomorphism is P-unitary*,
(5) *every right quasi-unitary U-monomorphism is P-unitary*,
(6) *U is a group.*

Proof. Again (1)⇒(2)⇒(3)⇒(4)⇒(5) are clear.

(5)⇒(6). As before, U is either a group or a 0-group. But from a previous example we see that for every 0-group there exists a U-monomorphism that is not P-unitary and so from Lemmas 3.12 and 4.20 there exists a quasi-unitary U-monomorphism that is not P-unitary.

(6)⇒(1). If U is a group then every right U-monomorphism is unitary and so is P-unitary. ∎

Also, it is easy to verify

THEOREM 4.23. *For any monoid U, the following are equivalent*:

(1) *every right U-monomorphism is unitary*,
(2) *every right weakly stable U-monomorphism is unitary*,
(3) *every right stable U-monomorphism is unitary*,
(4) *every right pure U-monomorphism is unitary*,
(5) *every right quasi-unitary U-monomorphism is unitary*,
(6) *every inclusion $J \to U$ with J a right ideal of U is unitary*,
(7) *U is a group.*

THEOREM 4.24. *For any monoid U, the following are equivalent*:

(1) *every right U-monomorphism is c-unitary*,
(2) *every right weakly stable U-monomorphism is c-unitary*,
(3) *every right stable U-monomorphism is c-unitary*,
(4) *every right pure U-monomorphism is c-unitary*,
(5) *every right quasi-unitary U-monomorphism is c-unitary*,

(6) *every inclusion* $J \to U$ *with* J *a right ideal of* U *is c-unitary*,

(7) *every right cancellative element in* U *is right invertible.*

Proof. (1)⇒(2)⇒(3)⇒(4)⇒(5)⇒(6) are clear.

(6)⇒(7). Let c be a right cancellative element of U and consider the inclusion $cU \to U$. By the c-unitary property, it follows that $cU = U$ and the result follows.

(7)⇒(1). Easy. ∎

THEOREM 4.25. *Let* U *be a monoid. The following are equivalent*:

(1) *every right P-unitary* U*-monomorphism is unitary*,

(2) *every right E-unitary* U*-monomorphism is P-unitary*,

(3) U *is a group.*

Proof. (1)⇒(3). Let $u_0 \in U$ and let ϱ be the right U-congruence on U generated by $\{(u_0, u_0u) : u \in U\}$. Then it is easy to see that $\{[u_0]_\varrho\} \to U/\varrho$ is a P-unitary monomorphism and hence is unitary. Consequently, $\varrho = \nabla$, the universal congruence on U. But this means that U is right simple and hence a group.

(2)⇒(3). As is the previous case, let $u_0 \in U$ and consider the E-unitary monomorphism $\{[u_0]_\varrho\}\dot{\cup}\{[u_0]_\varrho\} \to U/\varrho\dot{\cup}U/\varrho$. But this cannot be P-unitary unless $\varrho = \nabla$ and so again U is a group.

The other implications follow from Theorems 4.22 and 4.23. ∎

5. Monomorphisms into indecomposable acts. Recall that a U-act A is called *locally cyclic* if for every $a, a' \in A$ there exists $a'' \in A$ with $a, a' \in a''U$, and A is *indecomposable* if A is not the coproduct (disjoint union) of any two subacts. It is well known that A is indecomposable if and only if $a \sim a'$ for all $a, a' \in A$. It is easy to establish that cyclic ⇒ locally cyclic ⇒ indecomposable.

In this section, ϱ will denote a right U-congruence on U so that U/ϱ is a cyclic right U-act. Note that in general, if X is a subact of U/ϱ then X need not be cyclic but X will consist of ϱ-classes, $[u]_\varrho$, for some $u \in U$. Let $J = \{u \in U : [u]_\varrho \in X\}$ and note that J is a right ideal of U. Clearly $X \cong J/(\varrho|_{J\times J})$, which we shall write simply as J/ϱ for brevity. Conversely, if J is any right ideal of U then J/ϱ is a subact of U/ϱ.

Theorem 3.7 can be generalised in the following way.

THEOREM 5.1. *Let* U *be a monoid*, ϱ *a right* U*-congruence on* U *and* J/ϱ *a right subact of* U/ϱ.

(1) *Let* $J/\varrho \to U/\varrho$ *be a right weakly stable* U*-monomorphism and let* $u \in J$. *Then there exists* $v \in J$ *such that* $u\varrho vu$.

(2) *If ϱ is both a left and right U-congruence on U and if $J/\varrho \to U/\varrho$ is a right U-monomorphism such that for all $u \in J$ there exists $v \in J$ with $u\varrho vu$ then $J/\varrho \to U/\varrho$ is stable.*

Proof. (1) Since $(J/\varrho \to U/\varrho, Uu \to U)$ is stable and since $[u]_\varrho = [1]_\varrho u$ in U/ϱ we have $[u]_\varrho = [v]_\varrho u$ for some v in J as required.

(2) Let $\lambda : X \to Y$ be a left U-monomorphism, let $u_0 \in J$ and suppose that $[u_0]_\varrho \otimes b = [u]_\varrho \otimes \lambda(a)$ in $U/\varrho \otimes B$. Then we have a U-scheme over U/ϱ and B:

$$\begin{aligned}
[u_0]_\varrho &= [u_1]_\varrho s_1, & s_1 b &= t_1 b_2,\\
[u_1]_\varrho t_1 &= [u_2]_\varrho s_2, & s_2 b_2 &= t_2 b_3,\\
&\cdots & &\cdots\\
[u_{n-1}]_\varrho t_{n-1} &= [u_n]_\varrho s_n, & s_n b_n &= t_n \lambda(a).\\
[u_n]_\varrho t_n &= [u]_\varrho,
\end{aligned}$$

Since $[u_0]_\varrho = [vu_0]_\varrho$ for some $v \in J$, we see that

$$\begin{aligned}
[u_0]_\varrho &= [vu_1]_\varrho s_1, & s_1 b &= t_1 b_2,\\
[vu_1]_\varrho t_1 &= [vu_2]_\varrho s_2, & s_2 b_2 &= t_2 b_3,\\
&\cdots & &\cdots\\
[vu_{n-1}]_\varrho t_{n-1} &= [vu_n]_\varrho s_n, & s_n b_n &= t_n \lambda(a),\\
[vu_n]_\varrho t_n &= [vu]_\varrho,
\end{aligned}$$

is a U-scheme so that $[u_0]_\varrho \otimes b = [vu]_\varrho \otimes \lambda(a)$ in $U/\varrho \otimes B$ as required. Notice that this scheme is in fact over J/ϱ and B. ∎

In particular, if we take $J = uU$ and if $J/\varrho \to U/\varrho$ is weakly stable then there exists $w \in U$ with $u\varrho uwu$.

Corollary 5.2. *If ϱ is both a right and left U-congruence on U then $J/\varrho \to U/\varrho$ is stable if and only if it is weakly stable.*

Theorem 5.3. *Let $f : X \to Y$ be a right pure U-monomorphism with Y indecomposable. Then X is indecomposable.*

Suppose that $x, x' \in X$ so that $f(x) \sim f(x')$ in Y. Then we have a system of equations $f(x) = y_1 u_1$, $y_1 v_1 = y_2 u_2$, $\ldots$, $y_n u_n = f(x')$ in Y. But then $f(x) \otimes z = f(x') \otimes z$ in $Y \otimes \Theta$ where $\Theta = \{z\}$ is the one-element left U-act. Hence $x \otimes z = x' \otimes z$ in $X \otimes \Theta$ and so $x \sim x'$ in X and so X is indecomposable. ∎

Theorem 5.4. *Let U be a monoid. Then the following are equivalent*:

(1) *$J/\varrho \to U/\varrho$ splits*,
(2) *$J/\varrho \to U/\varrho$ is right quasi-unitary*,

(3) *J/ϱ is cyclic with generator $u_0 \in J$, we have $u_0 \varrho u_0^2$ and for all $u, v \in U$, $u\varrho v \Rightarrow u_0 u \varrho u_0 v$.*

Proof. (1)⇒(2). Clear.

(2)⇒(3). Let $\phi : U/\varrho \to U/\varrho$ be the morphism associated with the quasi-unitary property. Since $[u]_\varrho = [1]_\varrho u$ for all $u \in J$, we have $\phi([1]_\varrho) \in J/\varrho$ and it is clear that $J/\varrho = \phi([1]_\varrho)U$. If we let $\phi([1]_\varrho) = [u_0]_\varrho$ then we find that $[u_0^2]_\varrho = [u_0]_\varrho u_0 = \phi([1]_\varrho)u_0 = \phi([u_0]_\varrho) = [u_0]_\varrho$ since $\phi|_{J/\varrho} = 1_{J/\varrho}$. The other property holds since ϕ is well defined.

(3)⇒(1). Define $g : U/\varrho \to J/\varrho$ by $g([u]_\varrho) = [u_0 u]_\varrho$. Then g is a well defined right U-map. Finally, $g([u_0 u]_\varrho) = [u_0^2 u]_\varrho = [u_0 u]_\varrho$ as required. ∎

THEOREM 5.5. *Let U be a monoid and let $f : X \to Y$ be a right U-monomorphism. If Y is indecomposable* (*resp. locally cyclic*) *then f splits if and only if f is quasi-unitary. In that case*, *X is also indecomposable* (*resp. locally cyclic*)*.*

Proof. We prove the locally cyclic case, the other being similar. If f splits then f is quasi-unitary. Conversely, if f is quasi-unitary with associated map $\phi : Y \to Y$ then for all $x \in X$, $y \in Y$ there exist $y' \in Y$, $u_1, u_2 \in U$ with $f(x) = y'u_1$, $y = y'u_2$. Then $\phi(y') \in \mathrm{im}(f)$ and so $\phi(y) \in \mathrm{im}(f)$. Hence $\mathrm{im}(\phi) \subseteq \mathrm{im}(f)$ and the result follows. ∎

THEOREM 5.6. *Let U be a monoid and let $f : X \to Y$ be a right U-monomorphism with Y locally cyclic. Then the following are equivalent*:

(1) *f is P-unitary*,
(2) *f is E-unitary*,
(3) $\mathrm{im}(f) = Y$ *or* $|X| = 1$.

Proof. (1)⇒(2). Clear.

(2)⇒(3). Suppose $\mathrm{im}(f) \neq Y$ and $|X| > 1$. Let $y \in Y \setminus \mathrm{im}(f)$, $x \neq x' \in X$. Then there exist $y' \in Y$, $u_1, u_2, u_3 \in U$ such that $y = y'u_1$, $f(x) = y'u_2$, $f(x') = y'u_3$. Since f is E-unitary we have $y' = f(x'')$ for some $x'' \in X$ and so $y \in \mathrm{im}(f)$. This is a contradiction and so $|X| = 1$.

(3)⇒(1). Clear. ∎

It is clear that a right U-monomorphism $f : X \to Y$ with Y indecomposable can only be unitary if $\mathrm{im}(f) = Y$.

6. Pushouts, Rees quotients and unitary monomorphisms. The next result is a variation on Theorem 3.13.

THEOREM 6.1. *Let $f : X \to Y$ be a right U-monomorphism with pushout diagram as in diagram* $(*)$. *Then*

(1) *P is torsion free if and only if Y is torsion free and f is c-unitary.*

(2) *P satisfies condition* (P) *if and only if Y satisfies condition* (P) *and f is unitary.*

(3) *P satisfies condition* (E) *if and only if Y satisfies condition* (E)*.*

(4) *P is strongly flat if and only if Y is strongly flat and f is unitary.*

(5) *P is projective if and only if Y is projective and f is unitary.*

(6) *P is free if and only if Y is free and f is unitary.*

Proof. (1) ($\Rightarrow$) Suppose that $yu = f(x)$ with u right cancellative. Then $\alpha(y)u = \alpha f(x) = \beta f(x) = \beta(y)u$ and so $\alpha(y) = \beta(y)$ and $y \in \mathrm{im}(f)$.

If $yu = y'u$ in Y with u right cancellative, then $\alpha(y)u = \alpha(y')u$ in P and so $\alpha(y) = \alpha(y')$, giving $y = y'$ and so Y is torsion free.

($\Leftarrow$) Suppose that $pu = qu$ with u right cancellative and $p, q \in P$. Then there are essentially two cases to consider. Either $p = \alpha(y)$, $q = \alpha(y')$ for some $y, y' \in Y$, in which case $yu = y'u$ in Y and so $y = y'$ and hence $p = q$, or else $p = \alpha(y)$, $q = \beta(y')$ for $y, y' \in Y$, in which case $yu = y'u \in \mathrm{im}(f)$ and so $y = y' \in \mathrm{im}(f)$. But then $p = \alpha(y) = \alpha(y') = \beta(y') = q$ as required.

(2) ($\Rightarrow$) Suppose that $yu = f(x)$ for some $u \in U$, $y \in Y$, $x \in X$. Then $\alpha(y)u = \alpha f(x) = \beta f(x) = \beta(y)u$ in P. Hence there exist $p \in P$, $s, t \in U$ with $\alpha(y) = ps$, $\beta(y) = pt$ and $su = tu$. But then $p \in \mathrm{im}(\alpha) \cup \mathrm{im}(\beta)$ and so $y \in \mathrm{im}(f)$.

If $yu = y'u'$ in Y then $\alpha(y)u = \alpha(y')u'$ in P and so there exist $p \in P$, $s, s' \in U$ with $su = s'u'$ and $\alpha(y) = ps$, $\alpha(y') = pu'$. If $p = \alpha(y'')$ or $p = \beta(y'')$ then $y = y''s$, $y' = y''s'$ and $su = s'u'$ and Y satisfies condition (P).

($\Leftarrow$) Suppose that $pu = p'u'$ in P. Then there are two cases to consider:

(a) $p = \alpha(y)$, $p' = \alpha(y')$ for some $y, y' \in Y$. But then $yu = y'u' \in Y$ and the result follows since Y satisfies condition (P).

(b) $p = \alpha(y)$, $p' = \beta(y')$ for some $y, y' \in Y$. In this case we have $yu = y'u' \in Y$ and $yu = f(x)$ for some $x \in X$. Consequently, $y = f(x')$ for some $x' \in X$. Since Y satisfies condition (P), there exist $y'' \in Y$, $s, s' \in U$ such that $y = y''s$, $y' = y''s'$, $su = s'u'$. Hence $y'' = f(x'')$ for some $x'' \in X$ and so $p = \alpha(y) = \alpha(y''s) = \beta f(x'')s$, $p' = \beta(y') = \beta f(x'')s'$ and $su = s'u'$. Therefore P satisfies condition (P).

(3) Straightforward.

(4) Straightforward.

(5) ($\Rightarrow$) Since P is projective, it is strongly flat and so f is unitary. Hence by Lemma 4.20, $\alpha : Y \to P$ is unitary and hence Y is projective.

($\Leftarrow$) Notice that since f is unitary and Y is projective, X and $Y \backslash X$ are both projective as well. Hence $P = Y \mathbin{\dot\cup} Y \backslash X$ is projective.

(6) Similar to the projective case. ∎

If we use Rees quotients rather than pushouts we can deduce:

THEOREM 6.2. *Let $f : X \to Y$ be a right U-monomorphism.*

(1) *If Y/X is torsion free then f is c-unitary.*

(2) *If Y is torsion free and if f is c-unitary then Y/X is torsion free.*

(3) *If Y/X satisfies condition* (P) *then U is right reversible and f is P-unitary.*

(4) *If Y satisfies condition* (P), *U is right reversible and f is P-unitary then Y/X satisfies condition* (P).

(5) *If Y/X satisfies condition* (E) *then U is left collapsible and f is E-unitary.*

(6) *If Y satisfies condition* (E), *U is left collapsible and f is E-unitary then Y/X satisfies condition* (E).

(7) *If Y/X is strongly flat then U is left collapsible and f is P-unitary.*

(8) *If Y is strongly flat, U is left collapsible and f is P-unitary then Y/X is strongly flat.*

(9) *If Y/X is projective then U contains a left zero and f is P-unitary.*

(10) *If Y is projective, U contains a left zero and f is P-unitary then Y/X is projective.*

(11) *If Y/X is free then U contains a left zero and f is P-unitary but not unitary if $|U| > 1$.*

(12) *If Y is free, U contains a left zero and f is P-unitary but not unitary if $|U| > 1$ then Y/X is free.*

Proof. (1) If $yu = f(x)$ with u right cancellative then $\overline{y}u = \overline{f(x)} = \overline{f(x)}u$ in Y/X and so $\overline{y} = \overline{f(x)}$ in Y/X and hence $y \in \operatorname{im}(f)$.

(2) Suppose that $\overline{y}u = \overline{y'}u$ in Y/X with u right cancellative. Then either $yu = y'u$ in Y, in which case $y = y'$ since Y is torsion free and hence $\overline{y} = \overline{y'}$ in Y/X; or else $yu = f(x)$, $y'u = f(x')$ for some $x, x' \in X$. But then $y, y' \in \operatorname{im}(f)$ and so $\overline{y} = \overline{y'}$ in Y/X as required.

(3) If Y/X satisfies condition (P) then Y/X is flat and so U is right reversible. Suppose that $yu \neq y'u'$ but $yu, y'u' \in \operatorname{im}(f)$ and suppose also that $y' \notin \operatorname{im}(f)$. Then $\overline{y}u = \overline{yu} = \overline{y'u'} = \overline{y'}u'$ in Y/X and so there exist $z \in Y$, $s, s' \in U$ with $\overline{y} = \overline{z}s$, $\overline{y'} = \overline{z}s'$ and $su = s'u'$. Now $y' = zs'$ since $y' \notin \operatorname{im}(f)$. But if $y = zs$ then $yu = zsu = zs'u' = y'u'$, a contradiction, and so it follows that $y \in \operatorname{im}(f)$ as required.

(4) Suppose that $\overline{y}u = \overline{y'}u'$ in Y/X. Then there are two possibilities:

(a) $yu = y'u'$ in Y. In this case there exist $z \in Y$, $s, s' \in U$ with $y = zs$, $y' = zs'$ and $su = s'u'$ and the result then follows.

(b) $yu = f(x)$, $y'u' = f(x')$ for some $x, x' \in X$. Suppose then that $y \in \operatorname{im}(f)$. If $y' \in \operatorname{im}(f)$ also then $\overline{y} = \overline{y'}$ and the result easily follows since U is right reversible. Otherwise $y' \notin \operatorname{im}(f)$. Notice then that $y'u'$ is a fixed point in Y. Hence $y'u' = y'u'u = f(x')u$ and so there exist $z \in Y$, $s, s' \in U$

with $y' = zs'$, $f(x') = zs$ and $su = s'u'$. Hence $\overline{y} = \overline{f(x)} = \overline{f(x')} = \overline{z}s$, $\overline{y'} = \overline{z}s'$ and $su = s'u'$ as required.

(5) Let $u \neq v \in U$ and $x \in X$. Then $\overline{f(x)}u = \overline{f(x)}v$ in Y/X and so there exist $\overline{y} \in Y/X$, $s \in U$ with $\overline{f(x)} = \overline{y}s$ and $su = sv$. Hence U is left collapsible. Suppose then that $y \in Y$, $u, v \in U$ are such that $yu \neq yv$ but $yu, yv \in \mathrm{im}(f)$. Then $\overline{y}u = \overline{y}v$ and so there exist $\overline{z} \in Y/X$, $s \in U$ with $\overline{y} = \overline{z}s$ and $su = sv$. Now $y \neq zs$ since $yu \neq yv$ and so $y \in \mathrm{im}(f)$.

(6) Suppose that $\overline{y}u = \overline{y}v$ in Y/X. Then either $yu = yv$ in Y, in which case $y = zs$ and $su = sv$ for some $z \in Y$, $s \in U$ and consequently $\overline{y} = \overline{z}s, su = sv$ as required, or else $yu \neq yv$ but both terms belong to $\mathrm{im}(f)$. Hence $y = f(x)$ for some $x \in X$. But U left collapsible means that $su = sv$ for some $s \in U$ and so $\overline{y} = \overline{f(x)} = \overline{f(x)s} = \overline{f(x)}s$ and $su = sv$ as required.

(7) Straightforward.

(8) Straightforward.

(9) Since there exists an isomorphism $\phi : Y/X \to \dot{\bigcup} e_iU$ for some family $\{e_i \in E(U)\}$, we see that for any $x \in X$ and any $u \in U$ we have $\phi(\overline{f(x)})u = \phi(\overline{f(xu)}) = \phi(\overline{f(x)})$ and so $\phi(\overline{f(x)})$ is a left zero in U.

Since Y/X is projective, it is strongly flat and so f is P-unitary.

(10) Let $\beta : A \to B$ be a right U-epimorphism and $\alpha : Y/X \to B$ a right U-map. Since Y is projective, there exists a right U-map $\gamma : Y \to A$ such that

$$\begin{tikzcd}
& Y \arrow[d] \arrow[ddl, "\gamma"'] \\
& Y/X \arrow[d, "\alpha"] \\
A \arrow[r, two heads, "\beta"'] & B
\end{tikzcd}$$

commutes. If there exist $y \in Y \backslash \mathrm{im}(f)$, $u \in U$ with $yu \in \mathrm{im}(f)$ then let $a_0 = \gamma(yu)$. Notice that in this case, such a product yu is unique and is a fixed point in Y and so a_0 is a fixed point in A. Otherwise, if there is no such $y \in Y$, choose and fix any $a_0 \in A$ such that for all $x \in X$, $\beta(a_0) = \alpha(\overline{f(x)})$ and $a_0v = a_0$ for all $v \in U$. Notice that this last condition is possible since U has a left zero. Now define $\delta : Y/X \to A$ by

$$\delta(\overline{y}) = \begin{cases} \gamma(y) & \text{if } y \notin \mathrm{im}(f), \\ a_0 & \text{otherwise} \end{cases}$$

and note that by the P-unitary property, δ is well defined and $\beta \circ \delta = \alpha$. We need only check now that δ is a right U-map. Suppose then that $y \in Y$, $u \in U$ and consider $\delta(\overline{y}u)$. If $y \in \mathrm{im}(f)$ then $yu \in \mathrm{im}(f)$ and so $\delta(\overline{y}u) = \delta(\overline{yu}) = a_0 = a_0u = \delta(\overline{y})u$ as required. Otherwise $y \notin \mathrm{im}(f)$ and so there are two possibilities: (i) $yu \notin \mathrm{im}(f)$, in which case $\delta(\overline{y}u) = \gamma(yu) = \gamma(y)u = \delta(\overline{y})u$;

(ii) $yu \in \mathrm{im}(f)$, in which case $\delta(\overline{y}u) = a_0 = \gamma(yu) = \gamma(y)u = \delta(\overline{y})u$ as required.

(11) First notice that if Y/X is free and $|U| > 1$ then f cannot be unitary, otherwise the class containing the elements of $\mathrm{im}(f)$ would be a singleton subact of Y/X.

If Y/X is free then it is projective and so U contains a left zero and f is P-unitary.

(12) Notice that X is isomorphic to a disjoint union of copies of U together with a single proper ideal containing only a left zero element of U. It is then straightforward to see that Y/X is free. ∎

7. Some consequences. Many of the results in this section, in particular those concerning homological classification, are already known (see [8]). We provide alternative proofs of some of these results using some of the previous work on stable and unitary morphisms.

THEOREM 7.1. *Let $f : X \to Y$ be a right U-monomorphism and suppose that Y and Y/X satisfy condition* (P). *Then X also satisfies condition* (P).

Proof. Suppose that $xu = x'u'$ in X. Then $f(x)u = f(x')u'$ in Y and so there exist $y \in Y$, $s, s' \in S$ with $f(x) = ys$, $f(x') = ys'$ and $su = s'u'$. Now if $y = f(x'')$, say, then $x = x''s$, $x' = x''s'$ and $su = s'u'$ as required. Otherwise $y \notin \mathrm{im}(f)$ and so since f is P-unitary we see that $ys = ys'$ in Y and so $x = x'$ in X. Moreover, x is a fixed point in X and so we have $x = xs$, $x' = xs'$ and $su = s'u'$ as required. ∎

In a similar way we can also show

THEOREM 7.2. *Let $f : X \to Y$ be a right U-monomorphism and suppose that Y and Y/X satisfy condition* (E). *Then X also satisfies condition* (E).

And hence we can deduce

THEOREM 7.3. *Let $f : X \to Y$ be a right U-monomorphism and suppose that Y and Y/X are strongly flat. Then X is also strongly flat.*

In [11, Theorem 6.1], the author proved that if $[U; S_1, S_2]$ is a monoid amalgam such that $S_i, S_i/U$ are flat (both right and left) then the amalgam is strongly embeddable. This is similar to the situation for ring theory except that in that case, we only need one-sided flatness of S_i/U. In [13, Theorem 4.3] we strengthened this result slightly but still needed double-sided flatness of S_i. Recall [5] that if U is a submonoid of a monoid S then we say that U is (*right*) *perfect* in S if S is right flat and $U \to S$ is right pure as U-acts. This is equivalent to Hall's representation extension property [3]. It was proved in [3, Theorem 4] and [5, Lemma 2.6] that if $[U; S, T]$ is an amalgam of monoids such that $U \to S$, $U \to T$ are (right) perfect then the amalgam is strongly embeddable. Let U be a submonoid of a monoid S and suppose that as right U-acts, S and S/U satisfy condition (P). Then we see that S is right flat and $U \to S$ is right pure and so U is right perfect in S and thus we can deduce

THEOREM 7.4. *Let $[U; S_i]$ be a monoid amalgam such that as right U-acts, S_i and S_i/U satisfy condition* (P)*. Then the amalgam is strongly embeddable.*

Suppose now that all flat right U-acts satisfy condition (P). If Y is flat and $f : X \to Y$ is stable then the pushout, P, of diagram $(*)$ is flat and so satisfies condition (P). Hence f is unitary and so we have

THEOREM 7.5. *Let U be a monoid such that all flat right U-acts satisfy condition* (P)*. Suppose that $f : X \to Y$ is a right U-monomorphism with Y flat. Then f is stable if and only if f is unitary.*

In the same way we can also show

THEOREM 7.6. *Let U be a monoid such that all weakly flat right U-acts satisfy condition* (P)*. Suppose that $f : X \to Y$ is a right U-map with Y weakly flat. Then f is weakly stable if and only if f is unitary.*

THEOREM 7.7. *Let U be a monoid and Y a weakly flat right U-act. Then all principally weakly flat right Rees factor acts of Y are weakly flat if and only if U is right reversible.*

From Theorems 3.2, 3.4 and 3.6, we deduce the following interesting results.

COROLLARY 7.8 ([7]). *Let U be a monoid. All right Rees factor acts of U are principally weakly flat if and only if U is regular.*

COROLLARY 7.9 ([2, Theorem 6]). *Let U be a monoid. All right Rees factor acts of U are weakly flat if and only if U is right reversible and regular.*

It is clear that every inclusion $J \to U$ of right ideals is right pure if and only if U is left absolutely weakly flat. Hence, from Theorems 3.6 and 4.13 and Corollary 4.17 we see that we can recover Fleischer's Theorem:

COROLLARY 7.10 ([2, Theorem 4]). *A monoid U is left absolutely weakly flat if and only if U is regular and for every $x, y \in U$ there exists $z \in xU \cap yU$ such that $z\lambda(x, y)x$.*

We also have

COROLLARY 7.11 ([8, Theorem 4.6.6]). *A monoid U is left absolutely principally weakly flat if and only if it is regular.*

LEMMA 7.12. *If all torsion free right U-acts satisfy condition* (P) *then U is right cancellative and right reversible.*

Proof. Suppose U is not right cancellative. Let J be the proper right ideal of U consisting of the non-right cancellative elements of U. Then the inclusion $J \to U$ is c-unitary and so since U is torsion free, the pushout $A(J)$ satisfies condition (P). This is impossible since then $J \to U$ would be unitary. Hence U is right cancellative.

Since the 1-element right U-act is torsion free, it satisfies condition (P) and so U is right reversible. ∎

LEMMA 7.13. *A monoid U is either right cancellative or right* 0*-cancellative if and only if no proper right ideal J of U with $|J| > 1$ is such that $J \to U$ is c-unitary.*

Proof. (⇒) Suppose that J is a proper right ideal of U such that $J \to U$ is c-unitary. Note that U cannot be right cancellative as $J \to U$ cannot be unitary and so U is 0-cancellative. But if $|J| > 1$ then there exists a right cancellative $u \in J$ with $1.u \in J$ and so by the c-unitary property it follows that $J = U$, a contradiction.

(⇐) Suppose that U is not right cancellative and let J be the proper right ideal of U consisting of the non-right cancellative elements of U. Then $J \to U$ is c-unitary and so $|J| = 1$, from which it easily follows that U is 0-cancellative. ∎

THEOREM 7.14 ([8, Theorem 4.9.8]). *Let U be a monoid. All torsion free right Rees quotients of U satisfy condition* (P) *if and only if U is right reversible and either right cancellative or right* 0*-cancellative.*

Proof. (⇒) U is right reversible since the 1-element right U-act is torsion free. Suppose then that J is a proper right ideal of U with $J \to U$ c-unitary. It follows by the assumption that $J \to U$ is P-unitary. Consequently $|J| = 1$, from which we deduce that S is right cancellative or right 0-cancellative from Lemma 7.13.

(⇐) Suppose that J is a right ideal of U such that U/J is torsion free. If $J = U$ then right reversibility of U means that U/J has property (P). Otherwise, by Lemma 7.13 we see that $|J| = 1$ and hence $J \to U$ is P-unitary and so U/J satisfies condition (P). ∎

Notice that U/J torsion free means that either $J = U$ or $|J| = 1$. In a similar way we have

THEOREM 7.15 ([8, Theorem 4.10.7]). *Let U be a monoid. All torsion free right Rees quotients of U are strongly flat if and only if U is left collapsible and either right cancellative or right* 0*-cancellative.*

We can also deduce

Theorem 7.16 ([8, Theorem 4.11.12]). *Let U be a monoid. Then all torsion free right Rees quotients of U are projective if and only if U contains a left zero and either U is right cancellative or right* 0*-cancellative.*

Proof. ($\Rightarrow$) Since the 1-element right U-act is torsion free, it is projective and so there exists $e^2 = e \in U$ with $|eU| = 1$ and so U contains a left zero. Suppose then that J is a proper right ideal of U with U/J torsion free. Then U/J is projective and so it follows that $|J| = 1$ and hence U is right cancellative or right 0-cancellative.

($\Leftarrow$) Suppose that J is a right ideal of U such that U/J is torsion free. If $J = U$ then the existence of a left zero means that U/J is projective. Otherwise, by Lemma 7.13 we see that $|J| = 1$ and hence $U/J \cong U$ and so is projective. ∎

8. Problems. 1. Determine which monoids U have the property that all right U-monomorphisms are stable, pure or quasi-unitary.

2. Determine when $I \to U$ is c-unitary for I a right ideal of a monoid U.

3. Determine which monoids U have the property that for any containing monoid S, any U-act X embeds in an S-act, Y, with $X \to Y$ either P-unitary, E-unitary, quasi-unitary, stable or weakly stable. Study the connections between these and embeddability of semigroup amalgams.

4. If $f : X \to Y$ is a (weakly) stable right U-monomorphism with Y indecomposable, is X indecomposable?

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Received 28 March 2001;
revised 6 November 2001 (4101)