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FACTORWISE RIGIDITY OF EMBEDDINGS OF PRODUCTS OF PSEUDO-ARCS

BY

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Abstract. An embedding from a Cartesian product of two spaces into the Cartesian product of two spaces is said to be factorwise rigid provided that it is the product of embeddings on the individual factors composed with a permutation of the coordinates. We prove that each embedding of a product of two pseudo-arcs into itself is factorwise rigid. As a consequence, if X and Y are metric continua with the property that each of their nondegenerate proper subcontinua is homeomorphic to the pseudo-arc, then $X \times Y$ is factorwise rigid. This extends results of D. P. Bellamy and J. M. Lysko (for the case that X and Y are pseudo-arcs) and of K. B. Gammon (for the case that X is a pseudo-arc and Y is either a pseudo-circle or a pseudo-solenoid).

1. Introduction. A *continuum* is a nondegenerate compact connected metric space. Given continua X, Y, X_2, Y_2, an embedding $e : X\times Y \to X_2\times Y_2$ is said to be *factorwise rigid* provided that there exist continua X_1, Y_1 and embeddings $e_X : X \to X_1$ and $e_Y : Y \to Y_1$ such that $\{X_1, Y_1\} = \{X_2, Y_2\}$ and either for each $(x, y) \in X \times Y$, $e(x, y) = (e_X(x), e_Y(y))$, or for each $(x, y) \in X \times Y$, $e(x, y) = (e_Y(y), e_X(x))$. The product $X \times Y$ is said to be *factorwise rigid* provided that every self-homeomorphism of $X \times Y$ can be written as a composition of a product of homeomorphisms on individual coordinates with a permutation of the coordinates.

In 1983, D. P. Bellamy and J. M. Lysko [BL] proved that the product of two pseudo-arcs is factorwise rigid. This result was extended by D. P. Bellamy and J. A. Kennedy [BK] to an arbitrary product of pseudo-arcs. In 2010, K. B. Gammon [G1] showed that the product of a pseudo-arc and a pseudo-circle is factorwise rigid, and very recently, K. B. Gammon [G2] has also proved that the product of a pseudo-arc and a pseudo-solenoid is factorwise rigid.

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In this paper we prove that every embedding of the product of two pseudo-arcs into itself is factorwise rigid. The above mentioned results by Bellamy, Lysko and Gammon are obtained as corollaries.

2. Results. The letter P will denote the pseudo-arc. For a very complete information about the pseudo-arc, the reader is referred to [L]. A *map* is a continuous function. An *ε-map* between continua is a map $f : X \to Y$ such that $\mathrm{diam}(f^{-1}(y)) < \varepsilon$ for each $y \in Y$. A continuum X is said to be *chainable* provided that for each $\varepsilon > 0$, there exists an ε-map from X into $[0, 1]$. For a continuum X, we denote by $C(X)$ the hyperspace of subcontinua of X, endowed with the Hausdorff metric [IN, Definition 2.1]. Given subcontinua A and B of a continuum X such that $A \subsetneq B$, an *order arc* from A to B is a map $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and, if $s < t$, then $\alpha(s) \subsetneq \alpha(t)$. The existence of order arcs is proved in [IN, Theorem 14.6].

A continuum X is said to be *hereditarily indecomposable* provided that if $A, B \in C(X)$, then either $A \cap B = \emptyset$, or $A \subset B$, or $B \subset A$. It is well known that a continuum is indecomposable if and only if each of its subcontinua has empty interior. Using this fact and order arcs it is possible to show that, if X is a continuum such that each of its proper nondegenerate subcontinua is indecomposable, then X is hereditarily indecomposable. Given continua X and Y, let π_1 and π_2 be the respective projections from $X \times Y$ onto the first and second coordinates.

We will use that P is chainable and hereditarily indecomposable. We also need the following result [BL, Corollary 3].

LEMMA 2.1. *Let X, Y be chainable continua. Suppose that M and N are subcontinua of $X \times Y$ such that $\pi_1(M) \subset \pi_1(N)$ and $\pi_2(N) \subset \pi_2(M)$. Then $M \cap N \neq \emptyset$.*

LEMMA 2.2. *Suppose that X and Y are hereditarily indecomposable. Let $e : X \times Y \to X \times Y$ be an embedding. Suppose that for each $(p, q) \in X \times Y$, $\pi_i(e(\{p\} \times Y))$ is degenerate for some $i \in \{1, 2\}$ and $\pi_j(e(X \times \{q\}))$ is degenerate for some $j \in \{1, 2\}$. Then e is factorwise rigid.*

Proof. First, we will show the following claim.

CLAIM 1. *There exists $i_0 \in \{1, 2\}$ such that $\pi_{i_0}(e(\{p\} \times Y))$ is degenerate for every $p \in X$.*

To prove Claim 1, let $A = \{p \in X : \pi_1(e(\{p\} \times Y))$ is degenerate$\}$ and $B = \{p \in X : \pi_2(e(\{p\} \times Y))$ is degenerate$\}$. By hypothesis, $X = A \cup B$. Since e is one-to-one, $A \cap B = \emptyset$. It is easy to show that A and B are closed. By the connectedness of X, either $A = \emptyset$ or $B = \emptyset$. Thus, either $X = A$ or $X = B$. This ends the proof of Claim 1.

CLAIM 2. *Let* $j_0 \in \{1,2\} - \{i_0\}$. *Then* $\pi_{j_0}(e(X \times \{q\}))$ *is degenerate for every* $q \in Y$.

A similar argument as in Claim 1 implies that there exists $j \in \{1,2\}$ such that $\pi_j(e(X \times \{q\}))$ is degenerate for every $q \in Y$. If $j = i_0$, then for all points $(x_1, y_1), (x_2, y_2) \in X \times Y$, the sets $\pi_{i_0}(e(\{x_1\} \times Y))$ and $\pi_{i_0}(e(X \times \{y_2\}))$ are degenerate. Since their intersection is nonempty, $\pi_{i_0}(e(\{x_1\} \times Y)) \cup \pi_{i_0}(e(X \times \{y_2\}))$ is degenerate. Thus $\pi_{i_0}(e(x_1, y_1)) = \pi_{i_0}(e(x_2, y_2))$. This proves that $\pi_{i_0}(e(X \times Y)) = \{z\}$ for some z. This proves that $e(X \times Y)$ is contained in a slice of $X \times Y$. Since X and Y are hereditarily indecomposable, we conclude that $e(X \times Y)$ and $X \times Y$ are indecomposable. This is a contradiction since the product of two nondegenerate continua is not indecomposable. We have shown that $j \neq i_0$ and $j = j_0$. Claim 2 is proved.

Define $X_{i_0} = \pi_{i_0}(X \times Y)$ and $Y_{j_0} = \pi_{j_0}(X \times Y)$. Then $\{X_{i_0}, Y_{j_0}\} = \{X, Y\}$. Fix $x_0 \in X$ and $y_0 \in Y$. Define $e_{i_0} : X \to X_{i_0}$ and $e_{j_0} : Y \to Y_{j_0}$ by $e_{i_0}(p) = \pi_{i_0}(e(p, y_0))$ and $e_{j_0}(q) = \pi_{j_0}(e(x_0, q))$. Then e_{i_0} and e_{j_0} are continuous. By the choice of i_0 and j_0, for each $(p,q) \in X \times Y$, $\pi_{i_0}(e(p,q)) = \pi_{i_0}(e(p, y_0)) = e_{i_0}(p)$ and $\pi_{j_0}(e(p,q)) = \pi_{j_0}(e(x_0, q)) = e_{j_0}(q)$. This implies that $e(p,q) = (e_{i_0}(p), e_{j_0}(q))$ or $e(p,q) = (e_{j_0}(q), e_{i_0}(p))$, depending on whether $i_0 = 1$ or $i_0 = 2$.

In order to see that e_{i_0} is one-to-one, let $p, x \in X$ be such that $e_{i_0}(p) = e_{i_0}(x)$. By definition, $\pi_{i_0}(e(p, y_0)) = e_{i_0}(p) = e_{i_0}(x) = \pi_{i_0}(e(x, y_0))$. By the choice of j_0, $\pi_{j_0}(e(p, y_0)) = \pi_{j_0}(e(x, y_0))$. Hence, $e(p, y_0) = e(x, y_0)$. Since e is one-to-one, $p = x$. Thus e_{i_0} is an embedding. Similarly, e_{j_0} is an embedding. Therefore, e is factorwise rigid. ∎

THEOREM 2.3. *Every embedding of* $P \times P$ *into itself is factorwise rigid.*

Proof. Let d be a metric for P. Let $e : P \times P \to P \times P$ be an embedding.

CLAIM 3. *For each* $p \in P$, $\pi_1(e(\{p\} \times P))$ *or* $\pi_2(e(\{p\} \times P))$ *is degenerate.*

In order to prove Claim 3, suppose to the contrary there exists $p \in P$ such that $\pi_1(e(\{p\} \times P))$ and $\pi_2(e(\{p\} \times P))$ are nondegenerate. Let $\varepsilon > 0$ be such that $\varepsilon < \min\{\mathrm{diam}(\pi_i(e(\{p\} \times P))) : i \in \{1,2\}\}$. Let $\delta > 0$ be such that if $a, b, x, y \in P$, $i \in \{1,2\}$ and $\max\{d(a,b), d(x,y)\} < \delta$, then $d(\pi_i(e(a,x)), \pi_i(e(b,y))) < \varepsilon/3$. Let $\alpha : [0,1] \to C(P)$ be an order arc from $\{p\}$ to P. Then there exists $t > 0$ such that $\mathrm{diam}(\alpha(t)) < \delta$. Let $A = \alpha(t)$. Then $p \in A$, A is a nondegenerate subcontinuum of P and $\mathrm{diam}(A) < \delta$.

Let $i \in \{1,2\}$. Let $x_0, y_0 \in P$ be such that $\varepsilon < d(\pi_i(e(p, x_0)), \pi_i(e(p, y_0)))$. Given $a, b \in A$, $d(\pi_i(e(a,p)), \pi_i(e(b,p))) < \varepsilon/3$. Thus, $\mathrm{diam}(\pi_i(e(A \times \{p\}))) < \varepsilon/3$. Given $a \in A$, notice that $d(\pi_i(e(a, x_0)), \pi_i(e(p, x_0))) < \varepsilon/3$ and $d(\pi_i(e(a, y_0)), \pi_i(e(p, y_0))) < \varepsilon/3$. This implies $d(\pi_i(e(a, x_0)), \pi_i(e(a, y_0))) > \varepsilon/3$. We have shown that $\mathrm{diam}(\pi_i(e(\{a\} \times P))) > \varepsilon/3$.

Let $G(a,i) = \pi_i(e(\{a\} \times P))$. Then for all $a \in A$ and $i \in \{1,2\}$, $G(a,i)$ is a subcontinuum of P such that $\operatorname{diam}(G(a,i)) > \varepsilon/3$. Moreover, $\pi_i(e(a,p)) \in \pi_i(e(A \times \{p\})) \cap G(a,i)$ and $\operatorname{diam}(\pi_i(e(A \times \{p\}))) < \varepsilon/3$. This implies that $\pi_i(e(A \times \{p\})) \subset G(a,i)$. Therefore, for all $a, b \in A$ and $i \in \{1,2\}$, $G(a,i) \cap G(b,i) \neq \emptyset$, so either $G(a,i) \subset G(b,i)$ or $G(b,i) \subset G(a,i)$.

We claim that if $a \neq b$, then $G(a,i) \neq G(b,i)$. Suppose to the contrary that $G(a,i) = G(b,i)$. Let $j \in \{1,2\}$ be such that $j \neq i$. Then either $G(a,j) \subset G(b,j)$ or $G(b,j) \subset G(a,j)$. Applying Lemma 2.1 to the continua $e(\{a\} \times P)$ and $e(\{b\} \times P)$, we obtain $e(\{a\} \times P) \cap e(\{b\} \times P) \neq \emptyset$. This contradicts the fact that e is one-to-one. We have shown that, if $i \in \{1,2\}$ and $a, b \in A$ are such that $a \neq b$, then either $G(a,i) \subsetneq G(b,i)$ or $G(b,i) \subsetneq G(a,i)$.

Consider the map $\varphi : A \to C(P)$ given by $\varphi(a) = G(a,1)$. Notice that φ is continuous. By the previous paragraph, φ is one-to-one. Let $\mu : C(X) \to [0,1]$ be a Whitney map [IN, Definition 13.1]. Then $\mu \circ \varphi : A \to [0,1]$ is continuous and one-to-one, and so $\mu \circ \varphi$ is an embedding. This implies that A is an arc, which contradicts the fact that P is hereditarily indecomposable. This completes the proof of Claim 3.

With similar arguments, the following claim can be proved.

Claim 4. *For each $q \in P$, either $\pi_1(e(P \times \{q\}))$ or $\pi_2(e(P \times \{q\}))$ is degenerate.*

Hence, by Lemma 2.2 we conclude that e is factorwise rigid. ∎

The proof of the following theorem is straightforward.

Theorem 2.4. *Let R, S, T and Y be continua homeomorphic to the pseudo-arc. Then each embedding from $R \times S$ into $T \times Y$ is factorwise rigid.*

Theorem 2.5. *Let X and Y be continua all of whose nondegenerate proper subcontinua are pseudo-arcs. Then $X \times Y$ is factorwise rigid.*

Proof. Let $f : X \times Y \to X \times Y$ be a homeomorphism.

Claim 5. *If $p \in X$ and $f(\{p\} \times Y)$ is contained in a slice Z of $X \times Y$, then $f(\{p\} \times Y) = Z$.*

In order to prove Claim 5, suppose to the contrary that $f(\{p\} \times Y) \neq Z$. Suppose, for example, that $Z = X \times \{y_0\}$ for some $y_0 \in Y$; the case when Z is a slice with Y as a factor is similar.

Let $W = \pi_1(f(\{p\} \times Y))$. Then $f(\{p\} \times Y) = W \times \{y_0\} \subsetneq X \times \{y_0\}$. Thus, W is a nondegenerate proper subcontinuum of X. Hence, W is a pseudo-arc. Since f is a homeomorphism, we conclude that $\{p\} \times Y$ and Y are homeomorphic to the pseudo-arc. Since $\pi_1(f^{-1}(W \times \{y_0\}) = \{p\} \subsetneq X$, using order arcs it is possible to find nondegenerate proper subcontinua R and S of

X and Y, respectively, such that $W \subsetneq R$, $y_0 \in S$ and $\pi_1(f^{-1}(R \times S)) \neq X$. Let $T = \pi_1(f^{-1}(R \times S))$. Then T is either a one-point set or a pseudo-arc.

Notice that R and S are pseudo-arcs and $f^{-1}|R\times S : R\times S \to T\times Y$ is an embedding. Since Y is hereditarily indecomposable, T cannot be degenerate. Thus, R, S, T and Y are pseudo-arcs.

By Theorem 2.4, the embedding $e = f^{-1}|R \times S$ is factorwise rigid. Thus, there exist continua R_1 and S_1 and embeddings $e_R : R \to R_1$ and $e_S : S \to S_1$ such that $\{R, S\} = \{R_1, S_1\}$, and either for each $(x, y) \in R \times S$, $e(x, y) = (e_R(x), e_S(y))$, or for each $(x, y) \in R \times S$, $e(x, y) = (e_S(y), e_R(x))$. In the first case, for each $w \in W$, $p = \pi_1(e(w, y_0)) = \pi_1(e_R(w), e_S(y_0))$, that is, $p = e_R(w)$. This is absurd since e_R is one-to-one. Hence, only the second case is possible, that is, for each $(x, y) \in R \times S$, $e(x, y) = (e_S(y), e_R(x))$.

Given $w \in W$, $p = \pi_1(e(w, y_0)) = e_S(y_0)$. Thus, $e_S(y_0) = p$. Given $x \in R$, $e(x, y_0) = (e_S(y_0), e_R(x)) = (p, e_R(x))$. Hence, $(x, y_0) = f(f^{-1}(x, y_0)) = f(e(x, y_0)) = f(p, e_R(x)) \in W \times \{y_0\}$. This implies that $x \in W$. We have shown that $R \subset W$, contrary to the choice of R. This ends the proof of Claim 5.

Claim 6. *If $p \in X$, then $f(\{p\} \times Y)$ contains a slice of the product $X \times Y$.*

In order to prove Claim 6, fix $q_0 \in Y$. Let $\alpha : [0, 1] \to C(X)$ and $\beta : [0, 1] \to C(Y)$ be order arcs from $\{p\}$ to X and from $\{q_0\}$ to Y, respectively. Given $(s, t) \in [0, 1]^2$, let $A(s, t) = \alpha(s) \times \beta(t)$.

Let

$$E = \{(s, t) \in (0, 1) \times (0, 1) : \pi_1(f(A(s, t))) \neq X \text{ and } \pi_2(f(A(s, t))) \neq Y\}.$$

Notice that $\pi_1(f(A(0, 0))) = \{\pi_1(f(p, q_0))\} \neq X$ and $\pi_2(f(A(0, 0))) = \{\pi_2(f(p, q_0))\} \neq Y$, so the continuity of α, β, f, π_1 and π_2 implies that E is nonempty. Given $(s, t) \in E$, the map $f|A(s, t) : A(s, t) \to \pi_1(f(A(s, t))) \times \pi_2(f(A(s, t)))$ is an embedding from the product of two pseudo-arcs into the product of two pseudo-arcs. By Theorem 2.4, $f|A(s, t)$ is factorwise rigid.

This implies that there exist continua $X(s, t), Y(s, t)$ and embeddings $e_{(\alpha,s,t)} : \alpha(s) \to X(s, t)$ and $e_{(\beta,s,t)} : \beta(t) \to Y(s, t)$ satisfying

$$\{X(s, t), Y(s, t)\} = \{\pi_1(f(A(s, t))), \pi_2(f(A(s, t)))\},$$

and either for each $(x, y) \in A(s, t)$, $f(x, y) = (e_{(\alpha,s,t)}(x), e_{(\beta,s,t)}(y))$, or for each $(x, y) \in A(s, t)$, $f(x, y) = (e_{(\beta,s,t)}(y), e_{(\alpha,s,t)}(x))$. In the first case define $i(s, t) = 1$ and in the second case define $i(s, t) = 2$. Let $j(s, t)$ be the only element in $\{1, 2\} - \{i(s, t)\}$. Notice that for each $(x, y) \in A(s, t)$, the continua $\pi_{i(s,t)}(f(\{x\} \times \beta(t))) = \{e_{(\alpha,s,t)}(x)\}$ and $\pi_{j(s,t)}(f(\alpha(s) \times \{y\})) = \{e_{(\beta,s,t)}(y)\}$ are degenerate, and hence $\pi_{j(s,t)}(f(\{x\} \times \beta(t))) = e_{(\beta,s,t)}(\beta(t))$ and $\pi_{i(s,t)}(f(\alpha(s) \times \{y\})) = e_{(\alpha,s,t)}(\alpha(s))$ are nondegenerate.

CLAIM 6.1. *Suppose that* $s_1, s_2 \in (0,1)$ *and* $t_1, t_2 \in (0,1)$ *are such that* $(s_1,t_1),(s_2,t_2) \in E$. *Then* $i(s_1,t_1) = i(s_2,t_2)$ *and* $j(s_1,t_1) = j(s_2,t_2)$.

In order to prove Claim 6.1, let $s = \min\{s_1, s_2\}$ and $t = \min\{t_1, t_2\}$. Notice that $A(s,t) \subset A(s_1,t_1) \cap A(s_2,t_2)$. Then for each $(x,y) \in A(s,t)$, the continua $\pi_{i(s,t)}(f(\{x\} \times \beta(t)))$ and $\pi_{i(s_1,t_1)}(f(\{x\} \times \beta(t)))$ are degenerate, and $\pi_{j(s,t)}(f(\{x\} \times \beta(t)))$ and $\pi_{j(s_1,t_1)}(f(\{x\} \times \beta(t)))$ are nondegenerate. This implies that $i(s,t) = i(s_1,t_1)$, and hence $j(s,t) = j(s_1,t_1)$. Similarly, $i(s,t) = i(s_2,t_2)$ and $j(s,t) = j(s_2,t_2)$. Thus, the proof of Claim 6.1 is complete.

By Claim 6.1, there exist $i_0, j_0 \in \{1,2\}$ such that $\{i_0, j_0\} = \{1,2\}$ and $i_0 = i(s,t)$, $j_0 = j(s,t)$ for each $(s,t) \in E$. By definition $\pi_{i_0}(f(\{p\} \times \beta(t)))$ is degenerate for each $(s,t) \in E$, so we can define $v_0 = \max\{t \in [0,1] : \pi_{i_0}(f(\{p\} \times \beta(t)))$ is degenerate$\}$. Define $W = X$ if $j_0 = 1$, and $W = Y$ if $j_0 = 2$. Notice that $\pi_{j_0}(f(\{p\} \times \beta(v_0))) \subset W$. Let Z be the only element of the set $\{X,Y\} - \{W\}$. Then $W = \pi_{j_0}(X \times Y)$ and $Z = \pi_{i_0}(X \times Y)$.

CLAIM 6.2. $\pi_{j_0}(f(\{p\} \times \beta(v_0))) = W$.

Suppose, contrary to the claim, that $\pi_{j_0}(f(\{p\} \times \beta(v_0))) \neq W$. Since $\pi_{i_0}(f(\{p\} \times \beta(v_0)))$ is degenerate, $f(\{p\} \times \beta(v_0))$ is properly contained in a slice of $X \times Y$, and by Claim 5, $v_0 < 1$. Since $\pi_{i_0}(f(\{p\} \times \beta(v_0)))$ is degenerate and $f(\{p\} \times \beta(v_0)) = f(\alpha(0) \times \beta(v_0))$, by continuity of α, β and f there exist $s \in (0,1]$ and $t \in (v_0, 1]$ such that $(s,t) \in E$. By the choice of $i(s,t)$ and $j(s,t)$, $\pi_{i(s,t)}(f(\{p\} \times \beta(t)))$ and $\pi_{j(s,t)}(f(\alpha(s) \times \{q_0\}))$ are degenerate. Thus, $\pi_{i_0}(f(\{p\} \times \beta(t)))$ is degenerate. This contradicts the choice of v_0 and proves Claim 6.2.

CLAIM 6.3. $\pi_{i_0}(f(\{p\} \times \beta(v_0))) = \{\pi_{i_0}(f(p, q_0))\}$.

To prove Claim 6.3, notice that Claim 6.2 implies that $v_0 > 0$. Take $(s,t) \in E$ such that $t < v_0$. By the choice of $i(s,t)$, $\pi_{i(s,t)}(f(\{p\} \times \beta(t)))$ is degenerate and contains $\{\pi_{i_0}(f(p, q_0))\}$. Since $\pi_{i_0}(f(\{p\} \times \beta(v_0)))$ is degenerate and contains $\pi_{i(s,t)}(f(\{p\} \times \beta(t)))$, we conclude that $\pi_{i_0}(f(\{p\} \times \beta(v_0))) = \{\pi_{i_0}(f(p, q_0))\}$.

Claims 6.2 and 6.3 clearly imply that $f(\{p\} \times \beta(v_0))$ is a slice of the product. This completes the proof of Claim 6.

By symmetry, we can deduce the following.

CLAIM 7. *For each slice* S *in* $X \times Y$, $f(S)$ *contains a slice of* $X \times Y$ *and* $f^{-1}(S)$ *contains a slice of* $X \times Y$.

CLAIM 8. *For each slice* S *in* $X \times Y$, $f(S)$ *is a slice in* $X \times Y$.

In order to prove Claim 8, notice that, by Claim 7, there exists a slice T of $X \times Y$ such that $T \subset f(S)$. Applying again Claim 7, we find a slice

S_1 of $X \times Y$ such that $S_1 \subset f^{-1}(T) \subset S$. This implies that $S_1 = S$; hence $S = f^{-1}(T)$ and $f(S) = T$. This proves Claim 8.

By Lemma 2.2, there exist continua X_1, Y_1 and embeddings $e_X : X \to X_1$ and $e_Y : Y \to Y_1$ such that $\{X, Y\} = \{X_1, Y_1\}$ and either for every $(x, y) \in X \times Y$, $f(x, y) = (e_X(x), e_Y(y))$, or for every $(x, y) \in X \times Y$, $f(x, y) = (e_Y(y), e_X(x))$. Since f is a homeomorphism, it follows that e_X and e_Y are onto. This completes the proof of the theorem. ■

The following corollary extends the main results of [BL], [G1], [G2].

Corollary 2.6. *Suppose X is homeomorphic to either* (a) *the pseudo-arc, or* (b) *the pseudo-circle, or* (c) *some pseudo-solenoid, and Y is also homeomorphic to one of these continua. Then $X \times Y$ is factorwise rigid.*

Problem 2.7. *Can Theorem* 2.3 *be extended to a finite product of more than two factors?*

For infinite products, there is no a direct natural generalization of Theorem 2.3 since we can consider embeddings like $e : P \times P \times \cdots \to P \times P \times \cdots$ given by $e(x_1, x_2, \ldots) = (x_1, x_1, x_1, x_2, x_3, \ldots)$.

Two maps $h, g : P \to P$ are said to be *pseudo-homotopic* provided that there exist a continuum C, points $s_0, t_0 \in C$ and a map $H : P \times C \to P$ such that $H(p, s_0) = g(p)$ and $H(p, t_0) = h(p)$ for each $p \in P$. In this case, we say that H is a *pseudo-homotopy* between g and h.

There are only two known types of pseudo-homotopies for maps into the pseudo-arc, namely either $H(P \times \{c\})$ is degenerate for each $c \in C$, or $H(x, c) = f(x)$ for each $(x, c) \in X \times C$, where $f : P \to P$ is a map.

Problem 2.8. *Do there exist pseudo-homotopies on the pseudo-arc different from the ones described in the paragraph above?*

A negative answer to Problem 2.8 would help to solve Problem 2.7 and other problems about maps defined on products that have some pseudo-arcs as factors (see [I]).

Very recently, the second named author has shown [I] that if H is a pseudo-homotopy between a one-to-one map g and a map h, then $g = h$.

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