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ONE-PARAMETER SEMIGROUPS IN THE CONVOLUTION ALGEBRA OF RAPIDLY DECREASING DISTRIBUTIONS

BY

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Abstract. The paper is devoted to infinitely differentiable one-parameter convolution semigroups in the convolution algebra $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ of matrix valued rapidly decreasing distributions on $\mathbb{R}^n$. It is proved that $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is the generating distribution of an i.d.c.s. if and only if the operator $\partial_t \otimes \mathbb{1}_{m\times m} - G*$ on $\mathbb{R}^{1+n}$ satisfies the Petrovskiĭ condition for forward evolution. Some consequences are discussed.

Introduction. The present article is an improved version of a part of [K2]. It refers to the papers [P] and [S1]. In [P] I. G. Petrovskiĭ revealed the significance of smooth slowly increasing functions for the theory of Cauchy's problem for PDEs with constant coefficients. In [S1] L. Schwartz demonstrated that rapidly decreasing distributions play an important role there.

One-parameter semigroups in the convolution algebra of rapidly decreasing distributions. Let $M_{m\times m}$ be the set of $m \times m$ matrices with complex entries, and $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ the convolution algebra of $M_{m\times m}$-valued distributions on $\mathbb{R}^n$ rapidly decreasing in the sense of L. Schwartz. The Fourier transformation $\mathcal{F}$ is an isomorphism of $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ onto the algebra $\mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ of $M_{m\times m}$-valued infinitely differentiable slowly increasing functions on $\mathbb{R}^n$. We shall prove that *an $M_{m\times m}$-valued distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is the generating distribution of a one-parameter infinitely differentiable convolution semigroup $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ if and only if*

(i) $\quad 0 \vee \max \operatorname{Re} \sigma((\mathcal{F}G)(\xi)) = O(\log |\xi|) \quad$ *as $\xi \in \mathbb{R}^n$ and $|\xi| \to \infty$,*

where, for any $\xi \in \mathbb{R}^n$, $\sigma((\mathcal{F}G)(\xi))$ denotes the spectrum of the square matrix $(\mathcal{F}G)(\xi)$.

Here the capital O is the Landau symbol, so that (i) means that there is $C \in\,]0,\infty[$ such that whenever $\xi \in \mathbb{R}^n$ and $\lambda \in \sigma((\mathcal{F}G)(\xi))$, then $\operatorname{Re}\lambda \leq C(1 + \log(1 + |\xi|))$.

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If $G=\mathcal{G}(\partial_1,\dots,\partial_n)\otimes\delta$ where δ is the Dirac distribution on $\mathbb{R}^n$, $\partial_1,\dots,\partial_n$ denote the partial derivatives with respect to the coordinates of $\mathbb{R}^n$ and $\mathcal{G}(\partial_1,\dots,\partial_n)$ is an $m\times m$ matrix whose entries are scalar partial differential operators with constant coefficients, then $(\mathcal{F}G)(\xi)=\mathcal{G}(i\xi)$ for every $\xi\in\mathbb{R}^n$, and condition (i) takes the form

$$\text{(i)}'\qquad \max\{0\vee \operatorname{Re}\lambda:(\lambda,\xi)\in\mathbb{C}\times\mathbb{R}^n,\ \det P(\lambda,i\xi)=0\}=O(\log|\xi|)\quad\text{as } |\xi|\to\infty$$

where $P(\lambda,\zeta_1,\dots,\zeta_n)=\lambda\mathbb{1}_{m\times m}-\mathcal{G}(\zeta_1,\dots,\zeta_n)$. Thanks to the fact that $\det P(\lambda,\zeta_1,\dots,\zeta_n)$ is a polynomial, L. Gårding was able to prove that (i)′ holds if and only if

$$\text{(ii)}\qquad \sup\{\operatorname{Re}\lambda:\lambda\in\sigma(\mathcal{G}(i\xi)),\ \xi\in\mathbb{R}^n\}<\infty.$$

i.e. *if and only if the real parts of λ-roots of the polynomial* $\det P(\lambda,\xi_1,\dots,\xi_n)$ *remain bounded from above when ξ ranges over* $\mathbb{R}^n$. This last is known as the *Petrovskiĭ condition* for forward evolutionarity in t of the matricial differential operator

$$P(\partial_t,\partial_1,\dots,\partial_n):=\partial_t\otimes\mathbb{1}_{m\times m}-\mathcal{G}(\partial_1,\dots,\partial_n)\quad\text{ on } \mathbb{R}^{1+n}.$$

Notice that if $G\in\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ and $\operatorname{supp}G\neq\{0\}$, then (i) may hold without the set $\{\operatorname{Re}\lambda:\lambda\in\sigma((\mathcal{F}G)(\xi)),\ \xi\in\mathbb{R}^n\}$ being bounded from above. This is so if, for instance, $m=n=1$ and $(\mathcal{F}G)(\xi)=\log(e+\xi^2)$.

Application to the Cauchy problem for convolution equations. If $G\in\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$, then $(G*)|_{\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^m)}\in L(\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^m);\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^m))$. An argument similar to one used in the proof of [K3, Theorem 1(ii)] leads to the conclusion that whenever $G\in\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$, then G *satisfies the condition* (i) *if and only if the operator* $(G*)|_{\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^m)}$ *is equal to the infinitesimal generator of a one-parameter operator semigroup* $(T_t)_{t\geq 0}\subset L(\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^m);\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^m))$ of class (C_0).

In the following we assume that the distribution $G\in\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ satisfies the condition (i), and $(S_t)_{t\geq 0}\subset\mathcal{O}'_C(\mathbb{R}^n;M_{m\times m})$ is the i.d.c.s. whose generating distribution is G. By an $(S.*)$-*invariant l.c.v.s.* we mean a sequentially complete l.c.v.s. E continuously imbedded in $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$ such that: 1° $(S_t*)E\subset E$ for every $t\in[0,\infty[$, 2° the mapping $[0,\infty[\,\times E\ni(t,u)\mapsto S_t*u\in E$ is continuous, 3° $((S_t*)|_E)_{t\geq 0}\subset L(E;E)$ is a one-parameter (C_0)-semigroup with infinitesimal generator G_E such that

$$D(G_E)=\{u\in E:G*u\in E\},\qquad G_Eu=G*u\quad\text{ for } u\in D(G_E).$$

An l.c.v.s. E imbedded in $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ will be called *strongly* $(G*)$-*invariant* if $(G*)E\subset E$ and E is $(S.*)$-invariant.

Each of the function spaces

(a) $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)$ and $\mathcal{D}_{L^p}(\mathbb{R}^n;\mathbb{C}^m)=\{u\in C^\infty(\mathbb{R}^n;\mathbb{C}^m):\partial^\alpha u\in L^p(\mathbb{R}^n;\mathbb{C}^m)$ for every $\alpha\in\mathbb{N}_0^n\}$, $p\in[1,\infty]$, as well as each of the distribution spaces

(b) $\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{O}'_C(\mathbb{R}^n;\mathbb{C}^m)$ and $\mathcal{D}'_{L^q}(\mathbb{R}^n;\mathbb{C}^m) = (\mathcal{D}_{L^p})'(\mathbb{R}^n;\mathbb{C}^m)$, $q \in]1,\infty]$, $p = q/(q-1)$,

is a strongly $(G*)$-invariant l.c.v.s. for every $G \in \mathcal{O}'_C(\mathbb{R}^n;\mathbb{C}^m)$ satisfying the condition (i) (1). If $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ satisfies (i), and E is an $(S.*)$-invariant l.c.v.s., then for every $u_0 \in D(G_E)$ and $f \in C^1([0,\infty[;E)$ the Cauchy problem

$$\frac{du(t)}{dt} = G * u(t) + f(t) \quad \text{for } t \in [0,\infty[,\ u(0) = u_0,$$

has a solution $u(\cdot) \in C^1([0,\infty[;E)$ which is unique in the class $C^1([0,\infty[;\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m))$. Moreover, this solution can be expressed by the formula

$$u(t) = S_t * u_0 + \int_0^t S_\tau * f(t-\tau)\,d\tau \quad \text{for } t \in [0,\infty[.$$

The above assertions will be proved in a separate paper.

Hyperbolic partial differential systems with constant coefficients. The matricial partial differential operator $\mathbb{1}_{m\times m}\otimes\partial_t - \mathcal{G}(\partial_1,\ldots,\partial_n)$ on $\mathbb{R}^{1+n} = \{(t,x_1,\ldots,x_n)\}$ is called *hyperbolic with respect to the coordinate* t if (ii) above is satisfied and the hyperplane $t = 0$ is non-characteristic. This last holds if and only if

(iii) the degree of the polynomial $\det(\lambda\mathbb{1}_{m\times m} - \mathcal{G}(\zeta_1,\ldots,\zeta_n))$ of $1+n$ variables $\lambda,\zeta_1,\ldots,\zeta_n$ is equal to m.

Suppose that (ii) is satisfied and $(S_t)_{t\ge0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is the infinitely differentiable convolution semigroup whose generating distribution is $\mathcal{G}(\partial_1,\ldots,\partial_n)\otimes\delta$. Then the question arises about properties of $(S_t)_{t\ge0}$ corresponding to (iii). We shall prove that:

(a) *if* (iii) *holds, then* $(S_t)_{t\ge0}$ *extends to a one-parameter convolution group* $(S_t)_{t\in\mathbb{R}}$ *such that* $\operatorname{supp} S_t$ *is bounded for every* $t \in \mathbb{R}$,

(b) *if* (iii) *does not hold, then* $\operatorname{supp} S_t$ *is unbounded for every* $t \in]0,\infty[$.

1. Preliminaries. Throughout the present paper the symbols $\partial_1,\ldots,\partial_n$ denote partial derivatives of the first order (not multiplied by any constant) of a function or distribution on $\mathbb{R}^n$. For partial derivatives of higher order

(1) Some similar assertions not involving the convolution semigroup $(S_t)_{t\ge0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ were known much ealier. The results of I. G. Petrovskiĭ [P, Sec. I.5], G. Birkhoff [B1], S. D. Eidelman and S. G. Krein [Kr, Sec. I.8], T. Ushijima [U, Theorem 10.1] concern respectively $\mathcal{D}_{L^\infty}(\mathbb{R}^n;\mathbb{C}^m)$, the Banach spaces $\mathcal{B}_{\mathcal{N},p}$, the Hilbert spaces $\mathcal{L}_\mathcal{B}$ and the space $U_\mathcal{G}(\mathbb{R}^n;\mathbb{C}^m) = \{u \in L^2(\mathbb{R}^n;\mathbb{C}^m) : \mathcal{G}(\partial_1,\ldots,\partial_n)^k u \in L^2(\mathbb{R}^n;\mathbb{C}^m)$ for $k = 1,2,\ldots\}$. All these results refer to $\mathcal{G}(\partial_1,\ldots,\partial_n)$.

we use the abbreviation $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a *multiindex* whose *length* is defined as $|\alpha| = \alpha_1 + \dots + \alpha_n$.

1.1. The spaces $\mathcal{O}_M(\mathbb{R}^n)$ and $\mathcal{O}'_C(\mathbb{R}^n)$. Let $\mathcal{O}_M(\mathbb{R}^n)$ be the space of *infinitely differentiable slowly increasing* complex functions on $\mathbb{R}^n$. Recall that $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ if and only if for every multiindex $\alpha \in \mathbb{N}_0^n$ there is $m_\alpha \in \mathbb{N}_0$ such that

$$\sup_{\xi\in\mathbb{R}^n} (1+|\xi|)^{-m_\alpha} |\partial^\alpha \phi(\xi)| < \infty.$$

Obviously $\mathcal{O}_M(\mathbb{R}^n)$ is a function algebra. Furthermore

$$\mathcal{O}_M(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \phi\cdot\varphi \in \mathcal{S}(\mathbb{R}^n) \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^n)\}.$$

For every $k \in \mathbb{N}_0$ denote by $\mathbf{B}_k(\mathbb{R}^n)$ the space of continuous complex functions f on $\mathbb{R}^n$ such that $f(x) = O(|x|^{-k})$ as $|x| \to \infty$. Following L. Schwartz [S3, p. 244] a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is *rapidly decreasing* if and only if for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that $T = \sum_{|\alpha|\le m_k} \partial^\alpha f_{k,\alpha}$ where $f_{k,\alpha} \in \mathbf{B}_k(\mathbb{R}^n)$ for every $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \le m_k$. The space of rapidly decreasing distributions on $\mathbb{R}^n$, denoted by $\mathcal{O}'_C(\mathbb{R}^n)$, is a convolution algebra. One has

$$\mathcal{O}'_C(\mathbb{R}^n) = \{T \in \mathcal{S}'(\mathbb{R}^n) : T * \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^n)\},$$

and

$$\mathcal{F}\mathcal{O}'_C(\mathbb{R}^n) = \mathcal{O}_M(\mathbb{R}^n)$$

where $\mathcal{F}$ denotes the Fourier transformation on $\mathbb{R}^n$. In the following we assume that $\mathcal{F}$ is defined so that $(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{R}^n} e^{-i\xi x}\varphi(x)\,dx$ whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\xi \in \mathbb{R}^n$, and subsequently is extended onto $\mathcal{S}'(\mathbb{R}^n)$ by duality. Then $\mathcal{F}$ is an algebraic isomorphism of the convolution algebra $\mathcal{O}'_C(\mathbb{R}^n)$ onto the function algebra $\mathcal{O}_M(\mathbb{R}^n)$.

The operators $\phi\,\cdot$, $\phi \in \mathcal{O}_M(\mathbb{R}^n)$, and $U*$ for $U \in \mathcal{O}'_C(\mathbb{R}^n)$, are elements of the space $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ of continuous linear operators of $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. Let $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ denote the space $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ equipped with the topology of uniform convergence on bounded subsets of $\mathcal{S}(\mathbb{R}^n)$. Everywhere below, the topology in $\mathcal{O}'_C(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n)$ is induced from $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ via $*$ and $\cdot$ (2). The Fourier transformation is a continuous linear isomorphism of $\mathcal{O}'_C(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n)$.

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(2) The operators $\phi\,\cdot$, $\phi \in \mathcal{O}_M(\mathbb{R}^n)$, and $U*$, $U \in \mathcal{O}'_C(\mathbb{R}^n)$, can also be treated as elements of $L(\mathcal{S}'(\mathbb{R}^n); \mathcal{S}'(\mathbb{R}^n))$, so that $\mathcal{O}'_C(\mathbb{R}^n)$ and $\mathcal{O}_M(\mathbb{R}^n)$ can be equipped with the topology induced from $L_b(\mathcal{S}'(\mathbb{R}^n); \mathcal{S}'(\mathbb{R}^n))$ via $*$ and $\cdot$. Because $\mathcal{S}(\mathbb{R}^n)$ is a reflexive l.c.v.s., the topology in $\mathcal{O}'_C(\mathbb{R}^n)$ induced from $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ is identical with the topology in $\mathcal{O}'_C(\mathbb{R}^n)$ induced from $L_b(\mathcal{S}'(\mathbb{R}^n); \mathcal{S}'(\mathbb{R}^n))$. Indeed, both are determined by the system of seminorms

$$p_{B,C}(U) = \sup_{\varphi\in B,\, T\in C} |\langle T, U*\varphi\rangle| = \sup_{\varphi\in B,\, T\in C} |\langle \check{U} * T, \varphi\rangle|, \qquad U \in \mathcal{O}'_C(\mathbb{R}^n),$$

where B and C range respectively over bounded subsets of $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. The same holds for the topology in $\mathcal{O}_M(\mathbb{R}^n)$.

1.2. The spaces $\mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ and $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$. Let $m, n \in \mathbb{N}$, and let $M_{m\times m}$ be the set of $m \times m$ matrices with complex entries. Denote by $\mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ the space of functions of the form $\phi : \mathbb{R}^n \ni \xi \mapsto (\phi_{j,k}(\xi))_{j,k=1}^m \in M_{m\times m}$ such that $\phi_{j,k} \in \mathcal{O}_M(\mathbb{R}^n)$ for all j, k. This spaces carries the topology of $\mathcal{O}_M(\mathbb{R}^n)^{m^2}$ where each factor is equipped with the topology induced from $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. Multiplication in $\mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ is defined by the rule

$$(\phi \cdot \psi)(\xi) = \Big(\sum_{\iota=1}^{m} \phi_{j,\iota}(\xi)\psi_{\iota,k}(\xi)\Big)_{j,k=1}^m.$$

Denote by $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ the space of $m \times m$ matrices $T = (T_{j,k})_{j,k=1}^m$ such that $T_{j,k} \in \mathcal{O}'_C(\mathbb{R}^n)$ for all j, k. The convolution in $\mathcal{O}'_M(\mathbb{R}^n; M_{m\times m})$ is determined by the rule

$$S * T = \Big(\sum_{\iota=1}^{m} S_{j,\iota} * T_{\iota,k}\Big)_{j,k=1}^m.$$

The space $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ carries the topology of $\mathcal{O}'_C(\mathbb{R}^n)^{m^2}$ where each factor is equipped with the topology induced from $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$.

1.3. One-parameter convolution semigroups in $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$. By a one-parameter *infinitely differentiable convolution semigroup in* $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$, briefly i.d.c.s., we mean an infinitely differentiable mapping $[0, \infty[\, \ni t \mapsto S_t \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ such that $S_{s+t} = S_s * S_t$ for every $s, t \in [0, \infty[$, and $S_0 = \mathbb{1}_{m\times m} \otimes \delta$ where $\mathbb{1}_{m\times m}$ is the unit $m \times m$ matrix and δ is the Dirac distribution on $\mathbb{R}^n$.

The *generating distribution* of the i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is defined as

$$G := \frac{d}{dt}\Big|_{t=0} S_t.$$

Since $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ is a Fréchet space and the topology in $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is induced from $L_b(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$, the Banach–Steinhaus equicontinuity theorem works for convolution operators belonging to $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$. Therefore an analogue of the argument used in the proof of [H-P, Theorem 23.8.1] shows that every i.d.c.s. in $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is uniquely determined by its generating distribution.

1.4. Fourier transformation with respect to spatial coordinates. Let $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$. The Cauchy problem for the convolution equation $\partial_t u = G * u$ can be reduced by Fourier transformation with respect to the spatial coordinates to the Cauchy problem with parameter for an ODO. In the framework of the spaces $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ and $\mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ this method consists in making use of the following

LEMMA 1.4. *Suppose that $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ and let $A = \mathcal{F}G$, so that $A \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$. Then the following two conditions are equivalent:*

(a) *G is the generating distribution of an i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$,*
(b) *$\exp(tA(\cdot)) \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ for every $t \in [0,\infty[$ and the mapping $[0,\infty[\ni t \mapsto \exp(tA(\cdot)) \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ is infinitely differentiable.*

Furthermore, if $A = \mathcal{F}G$ and (a), (b) *are satisfied, then* $\exp(tA(\cdot)) = \mathcal{F}S_t$ *and*

$$(S_t\,*)|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)} = \mathcal{F}^{-1} \circ [(\exp(tA(\cdot)))\,\cdot] \circ \mathcal{F}|_{\mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)} \quad \textit{for every } t \in [0,\infty[.$$

Lemma 1.4 implies once again that every i.d.c.s. in $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is uniquely determined by its generating distribution.

Proof of Lemma 1.4. The Fourier transformation is a continuous isomorphism of the convolution algebra $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ of $M_{m\times m}$-valued distributions onto the pointwise multiplication algebra $\mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ of $M_{m\times m}$-valued functions. Consequently, condition (a) is equivalent to the condition that if $f_t := \mathcal{F}S_t$, then

(a)′ the mapping $[0,\infty[\ni t \mapsto f_t \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ is infinitely differentiable, $f_t \cdot f_\tau = f_{t+\tau}$ whenever $t,\tau \in [0,\infty[$, and $\frac{d}{dt}\big|_{t=0} f_t = \mathcal{F}G = A$,

where $\frac{d}{dt}\big|_{t=0}$ denotes the right derivative at $t = 0$, in the sense of the topology of $\mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$. From the condition that $f_t := \mathcal{F}S_t$ satisfies (a)′ it follows that whenever $\xi \in \mathbb{R}^n$ is fixed, then for the matrices $f_t(\xi) \in M_{m\times m}$ and $A(\xi) \in M_{m\times m}$ one has $\frac{d}{dt}\big|_{t=0} f_t(\xi) = A(\xi)$ and $f_t(\xi)\cdot f_\tau(\xi) = f_{t+\tau}(\xi)$ whenever $t,\tau \in [0,\infty[$. Consequently, $f_t(\xi) = \exp(tA(\xi))$ for every $t \in [0,\infty[$ and $\xi \in \mathbb{R}^n$, so that $f_t := \mathcal{F}S_t$ satisfies (a)′ if and only if $f_t = \exp(tA(\cdot))$ and (b) holds. Furthermore, whenever $T \in \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ and $t \in [0,\infty[$, then

$$\begin{aligned}[\mathcal{F}^{-1} \circ [\exp(tA(\cdot))\,\cdot] \circ \mathcal{F}](T) &= [\mathcal{F}^{-1} \circ [(\mathcal{F}S_t)\cdot] \circ \mathcal{F}](T)\\ &= \mathcal{F}^{-1}((\mathcal{F}S_t)\cdot(\mathcal{F}T)) = S_t * T\end{aligned}$$

where $(\mathcal{F}S_t)\cdot(\mathcal{F}T)$ makes sense and belongs to $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ because $\mathcal{F}T \in \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ and $\mathcal{F}S_t \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ is an $M_{m\times m}$-valued multiplier of $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$.

2. The results

2.1. Generation and growth of an i.d.c.s. in $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$

THEOREM 2.1. *A distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is the generating distribution of an i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ if and only if*

$$0 \vee \max \operatorname{Re} \sigma((\mathcal{F}G)(\xi)) = O(\log|\xi|) \quad \textit{as } \xi \in \mathbb{R}^n,\ |\xi| \to \infty. \tag{2.1}$$

For any $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ the quantity

(2.2) $$s(G) := \sup\{\operatorname{Re}\lambda : \text{there is } \xi \in \mathbb{R}^n \text{ such that } \lambda \in \sigma((\mathcal{F}G)(\xi))\},$$

finite or equal to $+\infty$, is called the *spectral bound* of G. For any i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ let

(2.3) $$\omega(S.) := \inf\{\omega \in \mathbb{R}^n : \{(e^{-\omega t} S_t *)|_{\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)} : t \in [0,\infty[\} \text{ is an equicontinuous subset of } L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))\}$$

where it is assumed that $\inf\emptyset = +\infty$. We call $\omega(S.)$ the *equicontinuity exponent* of the i.d.c.s. $(S_t)_{t\geq 0}$.

THEOREM 2.2. *For every i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ its equicontinuity exponent is equal to the spectral bound of its generating distribution* (3).

Let $\mathcal{G}(\partial_1, \ldots, \partial_n)$ be an $m \times m$ matrix whose entries are PDOs on $\mathbb{R}^n$ with constant complex coefficients. Let δ be the Dirac distribution on $\mathbb{R}^n$. Then $\mathcal{G}(\partial_1, \ldots, \partial_n) \otimes \delta \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ and

$$[\mathcal{F}(\mathcal{G}(\partial_1, \ldots, \partial_n) \otimes \delta)](\xi) = \mathcal{G}(i\xi_1, \ldots, i\xi_n)$$

for every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. The quantity

$$s_0(\mathcal{G}) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathcal{G}(i\xi_1, \ldots, i\xi_n)) \text{ for some } (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n\}$$

is equal to the spectral bound of the distribution $\mathcal{G}(\partial_1, \ldots, \partial_n) \otimes \delta$. From the conjecture by I. G. Petrovskiĭ [P, footnote on p. 24] proved by L. Gårding [G, Lemma on p. 11] it follows that $s_0(\mathcal{G}) < \infty$ if and only if $G = \mathcal{G}(\partial_1, \ldots, \partial_n) \otimes \delta$ satisfies (2.1). Therefore Theorems 2.1 and 2.2 imply

THEOREM 2.3. *Let $\mathcal{G}(\partial_1, \ldots, \partial_n)$ be an $m \times m$ matrix whose entries are PDOs on $\mathbb{R}^n$ with constant complex coefficients. Then the following two conditions are equivalent:*

(2.4) $s_0(\mathcal{G}) < \infty$,

(2.5) *$\mathcal{G}(\partial_1, \ldots, \partial_n) \otimes \delta$ is the generating distribution of an i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$.*

---

(3) By [K3, Theorem 1], whenever $G = \mathcal{G}(\partial_1, \ldots, \partial_n) \otimes \delta$ and E is one of the spaces $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m), \mathcal{S}'(\mathbb{R}^n;\mathbb{C}^m)$, $\mathcal{D}_{L^\infty}(\mathbb{R}^n;\mathbb{C}^m)$ $\mathcal{D}_{L^2}(\mathbb{R}^n;\mathbb{C}^m)$ or $U_{\mathcal{G}}(\mathbb{R}^n;\mathbb{C}^m)$, then:

1° $s_0(G) < \infty$ if and only if $\mathcal{G}(\partial_1, \ldots, \partial_n)$ is the infinitesimal generator of an operator semigroup in $L(E; E)$ of class (C_0),

2° $s_0(G) = \sup\{s \in \mathbb{R} : \mathcal{G}(\partial_1, \ldots, \partial_n) + s\mathbb{1}_{m\times m}$ is the infinitesimal generator of an equicontinuous operator semigroup in $L(E; E)$ of class $(C_0)\}$.

Assertion 1° was earlier proved for $\mathcal{D}_{L^\infty}(\mathbb{R}^n;\mathbb{C}^m)$ by I. G. Petrovskiĭ [P, Sec. I.5], and for $U_{\mathcal{G}}(\mathbb{R}^n;\mathbb{C}^m)$ by T. Ushijima [U, Theorem 10.1]. The theory of equicontinuous (C_0)-semigroups of linear operators in locally convex spaces is presented in [S2] and [Y, Chap. IX].

Furthermore, if these equivalent conditions are fulfilled, then there is exactly one i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ *satisfying* (2.5), *and the equicontinuity exponent of this i.d.c.s. is equal to* $s_0(\mathcal{G})$.

2.2. Application to PDOs of higher order. Let

$$P(\lambda, \zeta_1, \ldots, \zeta_n) = \sum_{\mu=0}^{m} \lambda^\mu Q_\mu(\zeta_1, \ldots, \zeta_n)$$

be a complex polynomial of $1+n$ variables such that $Q_m(i\xi_1, \ldots, i\xi_n) \neq 0$ for every $(\xi_1, \ldots, \xi_n) \in \mathbb{R}$ and the Petrovskiĭ condition

$$\sup\{\operatorname{Re}\lambda : (\lambda, \xi_1, \ldots, \xi_n) \in \mathbb{C}\times\mathbb{R}^n,\ P(\lambda, i\xi_1, \ldots, i\xi_n) = 0\} < \infty$$

is satisfied. From [H, Example A.2.7] it follows that the equality

$$(\mathcal{F}G)(\xi) = \begin{vmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & & 1 & \\ & & & 0 & 1 \\ -\frac{Q_0(i\xi)}{Q_m(i\xi)} & -\frac{Q_1(i\xi)}{Q_m(i\xi)} & \cdots & -\frac{Q_{m-2}(i\xi)}{Q_m(i\xi)} & -\frac{Q_{m-1}(i\xi)}{Q_m(i\xi)} \end{vmatrix}, \quad \xi \in \mathbb{R}^n,$$

determines a distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ generating an i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$. To see this, it suffices to note that $\det(\lambda \mathbb{1}_{m\times m} - (\mathcal{F}G)(\xi)) = (Q_m(i\xi))^{-1} P(\lambda, i\xi_1, \ldots, i\xi_n)$. The i.d.c.s. $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ defined above satisfies the equality

$$\begin{vmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & Q_m \end{vmatrix} \partial_t S_t$$
$$= \begin{vmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & & 0 & 1 \\ -Q_0 & -Q_1 & \cdots & -Q_{m-2} & -Q_{m-1} \end{vmatrix} S_t, \quad t \in [0, \infty[,$$

where $Q_k = Q_k(\partial_1, \ldots, \partial_n)$ for $k = 0, \ldots, m$. It follows that for every $u_0, u_1, \ldots, u_{m-1} \in \mathcal{S}(\mathbb{R}^n)$ the following two conditions are equivalent:

(a) $u(\cdot)$ is a solution of the Cauchy problem

$$P(\partial_t, \partial_1, \ldots, \partial_n) u(t, x) = 0 \quad \text{for } (t, x) \in [0, \infty[\times \mathbb{R}^n,$$
$$\partial_t^k u(0, x) = u_k(x) \quad \text{for } x \in \mathbb{R}^n \text{ and } k = 0, \ldots, m-1,$$

$$\text{(b)} \quad \begin{vmatrix} u(t,\cdot) \\ \partial_t u(t,\cdot) \\ \vdots \\ \partial_t^{m-1} u(t,\cdot) \end{vmatrix} = S_t * \begin{vmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{vmatrix} \qquad \text{for } t \in [0,\infty[.$$

In this way one obtains the theorem of J. Rauch [R, Sec. 3.10, Theorem 2] about well-posedness of the Cauchy problem for the scalar differential operator $P(\partial_t, \partial_1, \ldots, \partial_n)$ in the space $C^\infty([0,\infty[; \mathcal{S}(\mathbb{R}^n))$ ([4]).

2.3. Comments. I. G. Petrovskiĭ [P] was the first to notice the significance of smooth slowly increasing functions and conditions (i) and (ii) in the theory of those evolutionary PDOs which may be reduced to the form $\partial_t \otimes \mathbb{1}_{m\times m} - \mathcal{G}(\partial_1, \ldots, \partial_n)$. The theory of distributions did not yet exist in 1938 when [P] was published, and only in 1950 did L. Schwartz explain in [S1] how the results of Petrovskiĭ may be elucidated by placing them in the framework of $\mathcal{O}'_C$. However in [S1] conditions (i)$'$ and (ii) do not appear. The role of these conditions in the theory of PDOs is discussed in [H, Secs. 12.3 and 12.8] and [R, Sec. 3.10].

Suppose that the matricial differential operator $\mathcal{G}(\partial_1, \ldots, \partial_n)$ on $\mathbb{R}^n$ satisfies condition (ii). In terms of the operator

$$\text{(iii)} \qquad P(\partial_t, \partial_0, \ldots, \partial_n) := \partial_t \otimes \mathbb{1}_{m\times m} - \mathcal{G}(\partial_1, \ldots, \partial_n)$$

this last is equivalent to the Petrovskiĭ condition

$$\text{(iv)} \quad \sup\{\operatorname{Re}\lambda : (\lambda, \xi_1, \ldots, \xi_n) \in \mathbb{C}\times\mathbb{R}^n,\ \det P(\lambda, i\xi_1, \ldots, i\xi_n) = 0\} < \infty.$$

Let $(S_t)_{t\ge 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ be the i.d.c.s. generated by $\mathcal{G}(\partial_1, \ldots, \partial_n)\otimes\delta$. Then the distribution $\mathcal{N} \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$ defined by the equality

$$\mathcal{N}(\phi) = \int_0^\infty (S_\tau)_{(y)}\phi(\tau, y)\, d\tau \qquad \text{for every } \phi \in \mathcal{D}(\mathbb{R}^{1+n})$$

is a fundamental solution of the operator $P(\partial_t, \partial_0, \ldots, \partial_n)$, and $\operatorname{supp}\mathcal{N} \subset H_+ = \{(t, x_1, \ldots, x_n) \in \mathbb{R}^{1+n} : t \ge 0\}$.

Condition (iv) may also be applied to matricial differential operators

$$\text{(v)} \qquad P(\partial_t, \partial_1, \ldots, \partial_n) = [P_{i,j}(\partial_t, \partial_1, \ldots, \partial_n)]_{i,j=1}^m$$

of a form more general than (iii), still under the assumption that $P_{i,j}(\partial_t, \partial_1, \ldots, \partial_n)$ are scalar differential operators with constant coefficients. The sys-

([4]) The argument presented above depends on our Section 3 while the argument of J. Rauch consists in direct estimation of certain contour integrals representing the Fourier transform of the solution. Suppose conversely that the polynomial $P(\lambda, \zeta_1, \ldots \zeta_n)$ does not satisfy the Petrovskiĭ condition. Then the contour integrals of J. Rauch still work, and their properties lead to Theorem 4 of [R, Sec. 3.10] asserting that the Cauchy problem for the operator $P(\partial_t, \partial_0, \ldots, \partial_n)$ is ill-posed.

tems of PDEs considered in [P], [B1], [F, Sec.7.1] constitute here a particular case. They involve the operators (v) satisfying (iv) and satisfying moreover the condition

(vi) for every $l = 1, \dots, m$ there is $d_l > 0$ such that $\deg_\lambda P_{k,l} < d_l$ for every $k \in \{1, \dots, m\} \setminus \{l\}$ and $P_{l,l}(\lambda, \zeta_1, \dots, \zeta_n) = \lambda^{d_l} + \sum_{d=0}^{d_l-1} \lambda^d \cdot Q_{d,l}(\zeta_1, \dots, \zeta_n)$.

Under assumptions (iv) and (vi) the operator (v) may be replaced by the operator $\tilde{P}(\partial_t, \partial_1, \dots, \partial_n)$ with $\tilde{m} = d_1 + \dots + d_m$ satisfying (iv) and having the form (iii). This implies that the fundamental solution of $\tilde{P}(\partial_t, \partial_1, \dots, \partial_n)$ with support in $\{(t, x_1, \dots, x_n) \in \mathbb{R}^{1+n} : t \geq 0\}$ may be expressed by an i.d.c.s. $\subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$. However from [H, Sec. 12.8] and [K4] it follows that condition (vi) is not necessary for forward evolutionarity of PDOs of the form (v) satisfying (iv). Indeed, if for a PDO of the form (v) condition (iv) holds but (vi) is not satisfied, then the fundamental solutions with support contained H_+ can be constructed by a method independent of convolution semigroups in $\mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$.

2.4. Hyperbolic systems of PDOs with constant coefficients as a subclass of systems satisfying the Petrovskiĭ condition. Let $\mathcal{E}'(\mathbb{R}^n)$ be the space of distributions on $\mathbb{R}^n$ with compact support equipped with the topology of uniform convergence on bounded subsets of $C^\infty(\mathbb{R}^n)$. L. Ehrenpreis [E, Sec. V.5] proved $\mathcal{E}'(\mathbb{R}^n) = \{T \in \mathcal{D}'(\mathbb{R}^n) : T* \in L(\mathcal{D}(\mathbb{R}^n); \mathcal{D}(\mathbb{R}^n))\}$ and that the topology induced in $\mathcal{E}'(\mathbb{R}^n)$ from $L_b(\mathcal{D}(\mathbb{R}^n); \mathcal{D}(\mathbb{R}^n))$ via the mapping $T \mapsto T*$ coincides with the original topology of $\mathcal{E}'(\mathbb{R}^n)$. This topology is stronger than the one induced in $\mathcal{E}'(\mathbb{R}^n)$ from $\mathcal{O}'_C(\mathbb{R}^n)$. See [S3, Sec. III.7], [E, Sec. V.5, Lemma 5.17]. Let $\mathcal{E}'(\mathbb{R}^n; M_{m\times m})$ be the space of $M_{m\times m}$-valued distributions on $\mathbb{R}^n$ with compact support, i.e. the space of $m \times m$ matrices whose entries belong to $\mathcal{E}'(\mathbb{R}^n)$.

As in Theorem 2.3, let $\mathcal{G}(\partial_1, \dots, \partial_n)$ be an $m \times m$ matrix whose entries are PDOs on $\mathbb{R}^n$ with constant complex coefficients. Put

$$P(\lambda, \zeta_1, \dots, \zeta_n) = \det(\lambda \mathbb{1}_{m\times m} - \mathcal{G}(\zeta_1, \dots, \zeta_n)) \tag{2.6}$$

where $(\lambda, \zeta_1, \dots, \zeta_n) \in \mathbb{C}^{1+n}$.

THEOREM 2.4. *Assume that $\mathcal{G}(\partial_1, \dots, \partial_n)$ satisfies condition* (2.4). *Let $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ be the i.d.c.s. whose generating distribution is equal to $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$. Then the following three conditions are equivalent:*

(2.7) *there is $t_0 \in \,]0, \infty[$ such that $S_{t_0} \in \mathcal{E}'(\mathbb{R}^n; M_{m\times m})$,*

(2.8) *the polynomial $P(\lambda, \zeta_1, \dots, \zeta_n)$ of $1 + n$ variables defined by* (2.6) *has degree m,* (2.9) $(S_t)_{t\geq 0}$ *is an i.d.c.s. in the topological convolution algebra* $\mathcal{E}'(\mathbb{R}^n; M_{m\times m})$, *and it may be uniquely extended to a one-parameter infinitely differentiable subgroup of* $\mathcal{E}'(\mathbb{R}^n; M_{m\times m})$.

Let

$$\mathcal{N} = \{(\lambda, \zeta_1, \ldots, \zeta_n) \in \mathbb{C}^{1+n} : P(\lambda, \zeta_1, \ldots, \zeta_n) = 0\}. \tag{2.10}$$

The matricial PDO

$$\partial_t \otimes \mathbb{1}_{m\times m} - \mathcal{G}(\partial_1, \ldots, \partial_n) \tag{2.11}$$

on $\mathbb{R}^{1+n} = \{(t, x_1, \ldots, x_n) : t \in \mathbb{R}, (x_1, \ldots, x_n) \in \mathbb{R}^n\}$ is said to be *hyperbolic in the sense of Ehrenpreis with respect to the vector* $(1, 0, \ldots, 0) \in \mathbb{R}^{1+n}$ whenever there is $C \in \,]0, \infty[$ such that

$$\text{if } (\lambda, \zeta_1, \ldots, \zeta_n) \in \mathcal{N} \text{ then } |\mathrm{Re}\,\lambda| \leq C(1 + |\mathrm{Re}\,\zeta_1| + \cdots + |\mathrm{Re}\,\zeta_n|). \tag{2.12}$$

Condition (2.12), formulated in [E, Sec.VIII.3], is stronger than (2.4) which may be written in the following form: there is $C \in \,]0, \infty[$ such that

$$\text{if } (\lambda, \zeta_1, \ldots, \zeta_n) \in \mathcal{N} \text{ and } \mathrm{Re}\,\zeta_1 = \cdots = \mathrm{Re}\,\zeta_n = 0, \text{ then } \mathrm{Re}\,\lambda \leq C. \tag{2.4$'$}$$

The matricial PDO of the form (2.11) is said to be *hyperbolic in the sense of Gårding with respect to the vector* $(1, 0, \ldots, 0) \in \mathbb{R}^{1+n}$ if the polynomial (2.6) satisfies (2.4)$'$ and (2.8). In the proof of Theorem 2.4 it will be shown that for the matricial PDO of the form (2.11) the above two notions of hyperbolicity are equivalent. Therefore Theorem 2.4 may be formulated as follows:

If $\mathcal{G}(\partial_1, \ldots, \partial_n)$ *satisfies condition* (2.4), *then for the semigroup* $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ *with generating distribution* $\mathcal{G}(\partial_1, \ldots, \partial_n) \otimes \delta$ *the properties* (2.7) *and* (2.9) *are equivalent and they both hold if and only if the matricial PDO* (2.11) *is hyperbolic with respect to the vector* $(1, 0, \ldots, 0) \in \mathbb{R}^{1+n}$.

Suppose that the matricial differential operator of the form (2.11) is hyperbolic with respect to the vector $(1, 0, \ldots, 0) \in \mathbb{R}^{1+n}$. Let P_m be the principal homogeneous part of the polynomial (2.6), and let Γ be the connected component of the set $\{(\sigma, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{1+n} : P_m(\sigma, \xi_1, \ldots, \xi_n) \neq 0\}$ that contains $(1, 0, \ldots, 0)$. By [H, Lemma 8.7.3], Γ is a convex cone. Let Γ^0 be the closed cone dual to Γ. Making use of [H, Theorem 12.5.1] it is proved in [K2] that

$$\Gamma^0 = \{(t, x_1, \ldots, x_n) \in \mathbb{R}^{1+n} : t \geq 0,\ (x_1, \ldots, x_n) \in \mathrm{conv\,supp}\, S_t\}$$

where $(S_t)_{t\geq 0}$ is the i.d.c.s. occurring in Theorem 2.4. It follows that the distribution $N \in \mathcal{D}'(\mathbb{R}^{1+n}; M_{m\times m})$ such that $N(\phi) = \int_0^\infty S_t(\varphi(t, \cdot))\, dt$ for every $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ is a fundamental solution of the operator (2.11) with support contained in Γ^0.

3. A link between properties of $M_{m\times m}$-valued functions $\xi \mapsto A(\xi)$ and $(t,\xi) \mapsto \exp(tA(\xi))$

THEOREM 3.1 (The Shilov inequality). *Let $A \in M_{m\times m}$. Then for every $t \in [0,\infty[$ one has*

$$\|\exp(tA)\|_{M_{m\times m}} \le \rho(\exp(tA))\Big(1 + \sum_{k=1}^{m-1} \frac{(2t)^k}{k!}\|A\|^k_{M_{m\times m}}\Big) \tag{3.1}$$

and

$$\rho(\exp(tA)) = e^{t\max\operatorname{Re}\sigma(A)} \tag{3.2}$$

where ρ stands for the spectral radius, and $\sigma(A)$ denotes the spectrum of A.

The equality (3.2) follows from the spectral mapping theorem. See [E-N, Sec. I.3, Lemma 3.19]. The Shilov inequality (3.1) is an elaborate result of the theory of functions of matrices. See [Sh], [Ge, Sec. I.4], [G-S, Sec. II.6], [F, Sec. 7.2]. We say that $\varPhi \subset \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ is a *set of uniformly slowly increasing infinitely differentiable functions* if for every $\alpha \in \mathbb{N}_0^n$ there exists $k_\alpha \in \mathbb{N}_0$ such that

$$\sup\Big\{(1+|\xi|)^{-k_\alpha}\Big\|\Big(\frac{\partial}{\partial\xi}\Big)^\alpha \phi(\xi)\Big\|_{M_{m\times m}} : \phi \in \varPhi,\ \xi \in \mathbb{R}^n\Big\} < \infty.$$

PROPOSITION 3.2. *For any $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ the following three conditions are equivalent:*

(3.3) $0 \vee \max\operatorname{Re}\sigma(A(\xi)) = O(\log|\xi|)$ *as $\xi \in \mathbb{R}^n$ and $|\xi| \to \infty$,*

(3.4) *for every $T \in\,]0,\infty[$ there are $C \in\,]0,\infty[$ and $k \in \mathbb{N}_0$ such that*

$$\|\exp(tA(\xi))\|_{M_{m\times m}} \le C(1+|\xi|)^k \quad \text{whenever } t \in [0,T] \text{ and } \xi \in \mathbb{R}^n,$$

(3.5) *whenever $T \in\,]0,\infty[$, then $\{\exp(tA(\cdot)) : t \in [0,T]\}$ is a set of uniformly slowly increasing infinitely differentiable $M_{m\times m}$-valued functions on $\mathbb{R}^n$.*

PROPOSITION 3.3. *For every $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ and $s_0 \in \mathbb{R}$ the following five conditions are equivalent:*

(3.6) $\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A(\xi)),\ \xi \in \mathbb{R}^n\} \le s_0$,

(3.7) *there is $k \in \mathbb{N}_0$ such that for every $\varepsilon > 0$,*

$$\sup\{e^{-(s_0+\varepsilon)t}(1+|\xi|)^{-k}\|\exp(tA(\xi))\|_{M_{m\times m}} : t \in [0,\infty[,\ \xi \in \mathbb{R}^n\} < \infty,$$

(3.7)* *for every $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that*

$$\sup\{e^{-(s_0+\varepsilon)t}(1+|\xi|)^{-k}\|\exp(tA(\xi))\|_{M_{m\times m}} : t \in [0,\infty[,\ \xi \in \mathbb{R}^n\} < \infty,$$

(3.8) *for every $\alpha \in \mathbb{N}_0^n$ there is $k_\alpha \in \mathbb{N}_0$ such that for every $\varepsilon > 0$,*

$$\sup\Big\{e^{-(s_0+\varepsilon)t}(1+|\xi|)^{-k_\alpha}\Big\|\Big(\frac{\partial}{\partial\xi}\Big)^\alpha \exp(tA(\xi))\Big\|_{M_{m\times m}} : t \in \{0,\infty[,\ \xi \in \mathbb{R}^n\Big\} < \infty,$$

(3.8)* *whenever* $\varepsilon \in]0,\infty[$, *then* $\{e^{-(s+\varepsilon)t}\exp(tA(\cdot)) : t \in [0,\infty[\}$ *is a set of uniformly slowly increasing infinitely differentiable* $M_{m\times m}$*-valued functions on* $\mathbb{R}^n$.

Our proofs of Propositions 3.2 and 3.3 are based on the Shilov inequality. In this connection notice that the arguments of I. G. Petrovskiĭ from [P, Sec. I.5] prove the implication (3.3)⇒(3.5) without referring to the Shilov inequality. We shall prove Propositions 3.2 and 3.3 according to the schemes (3.3)⇒(3.4)⇒(3.5)⇒(3.4)⇒(3.3) and (3.6)⇒(3.7)⇒(3.8)⇒(3.8)*⇒(3.7)*⇒(3.6) where the implications (3.5)⇒(3.4) and (3.8)⇒(3.8)*⇒(3.7)* are trivial.

Proof of (3.3)⇔(3.4). If $A(\cdot)\in\mathcal{O}_M(\mathbb{R}^n;M_{m\times m})$ and (3.3) holds, then, by (3.1) and (3.2), for any fixed $T\in]0,\infty[$ there are $C,D\in]0,\infty[$ and $l\in\mathbb{N}_0$ such that for every $(t,\xi)\in[0,T]\times\mathbb{R}^n$ one has

$$\begin{aligned}\|\exp(tA(\xi))\|_{M_{m\times m}} &\le e^{t\max\operatorname{Re}\sigma(A(\xi))}\Big(1+\sum_{k=1}^{m-1}\frac{(2t)^k}{k!}\|A(\xi)\|^k_{M_{m\times m}}\Big)\\ &\le e^{TC(1+\log(1+|\xi|))}D(1+\|A(\xi)\|_{M_{m\times m}})^{m-1}\\ &\le e^{TC}D(1+|\xi|)^{TC+l(m-1)},\end{aligned}$$

so that (3.4) is satisfied. Conversely, if (3.4) holds, then there are $C\in]0,\infty[$ and $k\in\mathbb{N}_0$ such that $\|\exp A(\xi)\|_{M_{m\times m}}\le C(1+|\xi|)^k$ for every $\xi\in\mathbb{R}^n$, whence, by (3.2),

$$\begin{aligned}\max\operatorname{Re}\sigma(A(\xi)) &= \log\rho(\exp A(\xi))\\ &\le \log\|\exp A(\xi)\|_{M_{m\times m}}\le \log C + k\log(1+|\xi|),\end{aligned}$$

so that (3.3) holds.

Proof of (3.6)⇒(3.7). If (3.6) holds, then, by (3.1) and (3.2), for every $t\in[0,\infty[$ and $\xi\in\mathbb{R}^n$ one has

$$\begin{aligned}\|\exp(tA(\xi))\|_{M_{m\times m}} &\le e^{s_0t}\Big(1+\sum_{k=1}^{m-1}\frac{(2t)^k}{k!}\|A(\xi)\|^k_{M_{m\times m}}\Big)\\ &\le e^{s_0t}(1+2t)^{m-1}(1+\|A(\xi)\|_{M_{m\times m}})^{m-1}.\end{aligned}$$

Furthermore, since $A(\cdot)\in\mathcal{O}_M(\mathbb{R}^n;M_{m\times m})$, there are $C\in]0,\infty[$ and $l\in\mathbb{N}_0$ such that $\|A(\xi)\|_{M_{m\times m}}\le C(1+|\xi|)^l$ for every $\xi\in\mathbb{R}^n$. The above inequalities imply (3.7).

Proof of (3.7)*⇒(3.6). By (3.2),

$$\max\operatorname{Re}\sigma(A(\xi)) = \frac{1}{t}\log\rho(\exp(tA(\xi)))\le\frac{1}{t}\log\|\exp(tA(\xi))\|_{M_{m\times m}}$$

for every $t\in]0,\infty[$ and $\xi\in\mathbb{R}^n$, so that, if (3.7)* holds, then for every $\varepsilon>0$ there are $C\in]0,\infty[$ and $k\in\mathbb{N}$ such that

$$\max \operatorname{Re}\sigma(A(\xi)) \le s_0 + \varepsilon + \frac{1}{t}\log(C(1+|\xi|)^k)$$

for every $t \in \,]0,\infty[$ and $\xi \in \mathbb{R}^n$, whence (3.6) follows.

Proof of (3.4)⇒(3.5) *and* (3.7)⇒(3.8). These implications can be proved by an argument resembling [P, Sec. I.2, proof of Lemma 2], [S1, Sec. 5]. We shall limit ourselves to (3.7)⇒(3.8).

For every $\alpha \in \mathbb{N}_0^n$ let

$$U_{\alpha,t}(\xi) = \left(\frac{\partial}{\partial\xi}\right)^{\alpha} \exp(tA(\xi)).$$

Consider the condition

$(3.9)_\alpha$ there is $k_\alpha \in \mathbb{N}_0$ such that for every $\varepsilon > 0$ there is $C_{\alpha,\varepsilon} \in \,]0,\infty[$ such that whenever $(t,\xi) \in [0,\infty[\, \times \mathbb{R}^n$, then

$$\|U_{\alpha,t}(\xi)\|_{M_{m\times m}} \le C_{\alpha,\varepsilon} e^{(s_0+\varepsilon)t}(1+|\xi|)^{k_\alpha}.$$

Then (3.7) means that $(3.9)_0$ holds, and (3.8) means that $(3.9)_\alpha$ holds for all $\alpha \in \mathbb{N}_0^n$. So, still assuming that $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$, we have to prove that $(3.9)_0$ implies $(3.9)_\alpha$ for every $\alpha \in \mathbb{N}_0^n$. We proceed by induction on the length of α. Suppose that $(3.9)_\beta$ is satisfied whenever $|\beta| \le l$, and take $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| = l+1$. To prove that $(3.9)_\alpha$ is satisfied, put

$$V_{\alpha,t}(\xi) = \sum_{\beta\le\alpha,\, |\beta|\le l} \binom{\alpha}{\beta}\left(\left(\frac{\partial}{\partial\xi}\right)^{\alpha-\beta} A(\xi)\right) U_{\beta,t}(\xi).$$

Since $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ and $(3.9)_\beta$ holds whenever $|\beta| \le l$, it follows that

$(3.10)_\alpha$ there is $h_\alpha \in \mathbb{N}_0$ such that for every $\varepsilon > 0$ there is $D_{\alpha,\varepsilon} \in \,]0,\infty[$ such that whenever $(t,\xi) \in [0,\infty[\, \times \mathbb{R}^n$, then

$$\|V_{\alpha,t}(\xi)\|_{M_{m\times m}} \le D_{\alpha,\varepsilon} e^{(s_0+\varepsilon)t}(1+|\xi|)^{h_\alpha}.$$

One has

$$\begin{aligned}\frac{\partial}{\partial t}U_{\alpha,t}(\xi) &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\exp(tA(\xi)) = \left(\frac{\partial}{\partial\xi}\right)^{\alpha}[A(\xi)\exp(tA(\xi))]\\ &= A(\xi)U_{\alpha,t}(\xi) + V_{\alpha,t}(\xi),\end{aligned}$$

and $U_{\alpha,0}(\xi) = 0$ because $|\alpha| = l+1 \ge 1$. Hence

$$U_{\alpha,t}(\xi) = \int_0^t [\exp((t-\tau)A(\xi))]V_{\alpha,t}(\xi)\,d\tau. \tag{3.11}$$

From $(3.9)_0$, $(3.10)_\alpha$ and (3.11) it follows that

$$
\begin{aligned}
&\|U_{\alpha,t}(\xi)\|_{M_{m\times m}} \\
&\qquad \le \int_0^t C_{0,\varepsilon/2} e^{(s_0+\varepsilon/2)(t-\tau)} (1+|\xi|)^{k_0} D_{\alpha,\varepsilon/2} e^{(s_0+\varepsilon/2)\tau} (1+|\xi|)^{h_\alpha}\, d\tau \\
&\qquad = C_{0,\varepsilon/2} D_{\alpha,\varepsilon/2} t e^{(s_0+\varepsilon/2)t} (1+|\xi|)^{k_0+h_\alpha} \le \tilde{C}_{\alpha,\varepsilon} e^{(s_0+\varepsilon)t} (1+|\xi|)^{k_\alpha}
\end{aligned}
$$

for $k_\alpha = k_0 + h_\alpha$ and $\tilde{C}_{\alpha,\varepsilon} = C_{0,\varepsilon/2} D_{\alpha,\varepsilon/2} \max_{t\in[0,\infty[} te^{-(\varepsilon/2)t}$.

4. Proof of Theorem 2.1

Necessity of (2.1). Suppose that $(S_t)_{t\ge 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ is an i.d.c.s. with generating distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$. Let $A = \mathcal{F}G$. Then we have $A, \mathcal{F}S_t \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$ and $(\mathcal{F}S_t)(\xi) = \exp(tA(\xi))$ for every $t \in [0,\infty[$ and $\xi \in \mathbb{R}^n$. Since the mapping

$$[0,\infty[\ \ni t \mapsto [\exp(tA(\cdot))]\cdot = (\mathcal{F}S_t)\cdot \in L_b(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$$

is continuous, the Banach–Steinhaus theorem implies that whenever $T \in\ ]0,\infty[$, then the set of multiplication operators $\{[\exp(tA(\cdot))]\cdot : t \in [0,T]\}$ is an equicontinuous subset of $L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$. By [K1, Theorem 3.1], this is equivalent to (3.5). From Proposition 3.2 it follows that (3.5) is equivalent to (3.3). Since, $A = \mathcal{F}G$, (3.3) means that (2.1) is satisfied.

Sufficiency of (2.1). Suppose that $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ satisfies (2.1). Let $A = \mathcal{F}G$. Then $A \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$, and A satisfies (3.3). Hence, by Proposition 3.2 and [K1, Theorem 3.1], whenever $T \in\ ]0,\infty[$, then $\{[\exp(tA(\cdot))]\cdot : t \in [0,T]\}$ is an equicontinuous subset of $L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);$ $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$. By the theorem on differentiating a solution of an ODE with respect to a parameter ([Ha, Sec.V.4, Corollary 4.1]), the mapping $\mathbb{R}^{1+n} \ni (t,\xi) \mapsto \exp(tA(\xi)) \in M_{m\times m}$ is infinitely differentiable. By [K1, Theorem 3.2], so is the mapping $[0,\infty[\ \ni t \mapsto [\exp(tA(\cdot))]\cdot \in L_b(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);$ $\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$ whose right-side derivative at zero (computed in the topology of $L_b(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)))$ is $A\cdot \in L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$. By Lemma 1.4 it follows that

$$G* = (\mathcal{F}^{-1}A)* = \mathcal{F}^{-1}\circ(A\cdot)\circ\mathcal{F} \in L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$$

is the infinitesimal generator of the infinitely differentiable operator semigroup

$$
\begin{aligned}
(\mathcal{F}^{-1}\exp(tA(\cdot))\,*)_{t\ge 0} &= (\mathcal{F}^{-1}\circ([\exp(tA(\cdot))]\,\cdot)\circ\mathcal{F})_{t\ge 0} \\
&\subset L_b(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m)).
\end{aligned}
$$

Hence, G is the generating distribution of the i.d.c.s. $(\mathcal{F}^{-1}\exp(tA(\cdot)))_{t\ge 0}$ $\subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$.

5. Proof of Theorem 2.2. Let $(S_t)_{t\geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ be an i.d.c.s. with generating distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$. Put $A = \mathcal{F}G$. Then $A, \mathcal{F}S_t \in \mathcal{O}_M(\mathbb{R}^n; M_{m\times m})$, condition (b) from Lemma 1.4 is satisfied, and $(\exp(tA(\cdot)))\cdot = \mathcal{F}\circ(S_t\,*)\circ\mathcal{F}^{-1}$ for all $t\in[0,\infty[$. As $\mathcal{F}, \mathcal{F}^{-1} \in L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m); \mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))$, it follows that for $\omega(S.)$ defined by (2.3) one has

$$\omega(S.) = \inf\{\omega\in\mathbb{R} : \{[e^{-\omega t}\exp(tA(\cdot))]\cdot : t\in[0,\infty[\} \text{ is an equicontinuous subset of } L(\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m);\mathcal{S}(\mathbb{R}^n;\mathbb{C}^m))\}.$$

From [K1, Theorem 3.1] it follows that whenever $s_0\in\mathbb{R}$, then the condition

$$\omega(S.) < s_0 + \varepsilon \quad \text{for every } \varepsilon > 0 \tag{5.1}$$

is equivalent to (3.8)*. Hence, by Proposition 3.3, the condition (5.1) is equivalent to (3.6). In other words, whenever $s_0\in\mathbb{R}$, then (5.1) holds if and only if $s_0(G)\leq s_0$, where $s_0(G) = \sup\{\operatorname{Re}\lambda : \lambda\in\sigma(A(\xi)),\ \xi\in\mathbb{R}^n\}$ is the spectral bound of G. This last equivalence implies that $\omega(S.) = s_0(G)$.

6. Proof of Theorem 2.4

6.1. Condition (2.7) implies Gårding hyperbolicity. Let $(S_t)_{t\geq 0}\subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ be an i.d.c.s. with generating distribution equal to $\mathcal{G}(\partial_1,\ldots,\partial_n)\otimes\delta$. Then the condition (2.4) is satisfied and $\mathcal{F}S_t\in\mathcal{O}_M(\mathbb{R}^n;M_{m\times m})$, $(\mathcal{F}S_t)(\xi) = \exp(t\mathcal{G}(i\xi))$ for every $t\in[0,\infty[$ and $\xi\in\mathbb{R}^n$. Suppose that (2.7) holds, i.e. $S_{t_0}\in\mathcal{E}'(\mathbb{R}^n;M_{m\times m})$ for some $t_0\in\,]0,\infty[$. Then, by the Paley–Wiener–Schwartz theorem, i.e. by [H, Theorem 7.3.1], there are $C,k,l\in\,]0,\infty[$ such that whenever $\zeta\in\mathbb{C}^n$, then

$$\|\exp(t_0\mathcal{G}(i\zeta))\|_{M_{m\times m}} = \|(\mathcal{F}S_{t_0})(\zeta)\|_{M_{m\times m}} \leq C(1+|\zeta|)^l e^{k|\operatorname{Im}\zeta|}. \tag{6.1}$$

For every $\zeta\in\mathbb{C}^n$ put

$$\Lambda(\zeta) = \max\operatorname{Re}\sigma(\mathcal{G}(i\zeta)).$$

Then

$$\Lambda(\zeta) = \max\{\operatorname{Re}\lambda : \lambda\in\mathbb{C},\ P(\lambda, i\zeta_1,\ldots,i\zeta_n) = 0\}$$

where

$$\begin{aligned} P(\lambda,\zeta_1,\ldots,\zeta_n) &= \det(\lambda\mathbb{1}_{m\times m} - \mathcal{G}(\zeta_1,\ldots,\zeta_n)) \\ &= \lambda^m + Q_{m-1}(\zeta_1,\ldots,\zeta_n)\lambda^{m-1} + \cdots + Q_1(\zeta_1,\ldots,\zeta_n)\lambda + Q_0(\zeta_1,\ldots,\zeta_n). \end{aligned}$$

Let

$$p_0 = \inf\Big\{p\in\,]0,\infty[\, : \sup_{\zeta\in\mathbb{C}^n}(1+|\zeta|)^{-p}\Lambda(\zeta) < \infty\Big\}.$$

By (3.2) and (6.1) there is $K\in\,]0,\infty[$ such that

$$\Lambda(\zeta) \leq t_0^{-1}\log\|\exp(t_0\mathcal{G}(i\zeta))\|_{M_{m\times m}} \leq K(1+|\zeta|)$$

for every $\zeta \in \mathbb{C}^n$. Consequently,

$$p_0 \leq 1. \tag{6.2}$$

By the theorem on reduced order [B1], [G-S, Sec. II.6.2], [F, Sec. 7.2, Theorem 4],

$$p_0 = \max_{k=0,\ldots,m-1} (m-k)^{-1} \deg[Q_k(i\zeta_1,\ldots,\zeta_n)],$$

so that, by (6.2), $\deg Q_k \leq m-k$ for every $k = 0,\ldots,m-1$, and hence $\deg P = m$, proving (2.8).

6.2. Gårding hyperbolicity implies Ehrenpreis hyperbolicity. Suppose that the differential operator (2.11) is hyperbolic in the sense of Gårding with respect to $v_0 = (1,0,\ldots,0) \in \mathbb{R}^{1+n}$. Let $\mathcal{N} \subset \mathbb{C}^{1+n}$ be defined by (2.6), (2.10). Then, in view of the results of L. Gårding [G, Lemmas 2.2, 2.6–2.8] or by [H, Theorems 12.4.1 and 12.4.4],

$$\mathcal{N} \subset \mathbb{C}^{1+n} \setminus (i\mathbb{R}^{1+n} + ((\Gamma + s_+v_0) \cup (-\Gamma - s_-v_0))) \tag{6.3}$$

where

$$s_\pm = \sup\{\pm \operatorname{Re}\lambda : (\lambda,\xi) \in \mathbb{C}\times\mathbb{R}^n,\ P(\lambda,i\xi) = 0\}$$

are finite, and Γ contains v_0 and is the open connected component of the set $\{\xi \in \mathbb{R}^{1+n} : P_m(\xi) = 0\}$ where P_m is the principal homogeneous part of P. By [G, Lemma 2.8] or [H, Lemma 8.7.3], Γ is an open convex cone in $\mathbb{R}^{1+n}$ with vertex at 0. The inclusion (6.3) is equivalent to

$$\begin{aligned}\mathcal{N} \subset \{(\lambda,\zeta_1,\ldots,\zeta_n) \in \mathbb{C}^{1+n} :&\\ (\operatorname{Re}\lambda, \operatorname{Re}\zeta_1,\ldots,\operatorname{Re}\zeta_n) \notin (\Gamma + s_+v_0) \cup (-\Gamma - s_-v_0)\}.&\end{aligned} \tag{6.4}$$

Since Γ contains the halfline $\{\sigma v_0 : \sigma > 0\}$, the set on the right side of (6.4) is contained in the set

$$\Big\{(\lambda,\zeta_1,\ldots,\zeta_n) \in \mathbb{C}^{1+n} : |\operatorname{Re}\lambda| \leq C\Big(1 + \sum_{\nu=1}^n |\operatorname{Re}\zeta_\nu|\Big)\Big\}$$

if only $C \in]0,\infty[$ is sufficiently large. Hence the operator (2.11) is hyperbolic in the sense of Ehrenpreis with respect to the vector v_0.

6.3. The Ehrenpreis hyperbolicity implies (2.9). Suppose that the operator (2.11) is hyperbolic in the sense of Ehrenpreis with respect to the vector $(1,0,\ldots,0) \in \mathbb{R}^{1+n}$. This means that if $(\lambda,\zeta_1,\ldots,\zeta_n) \in \mathbb{C}^{1+n}$ and

$$P(\lambda,\zeta) = \det(\lambda \mathbb{1}_{m\times m} - \mathcal{G}(\zeta)) = 0,$$

then

$$|\operatorname{Re}\lambda| \leq C(1 + |\operatorname{Re}\zeta|)$$

for some $C \in]0,\infty[$ independent of (λ,ζ). Since

$$\sigma(\mathcal{G}(i\zeta)) = \{\lambda \in \mathbb{C} : P(\lambda, i\zeta) = 0\},$$

it follows that whenever $\zeta \in \mathbb{C}^n$, then

$$\max |\mathrm{Re}\, \sigma(\mathcal{G}(i\zeta))| \le C(1 + |\mathrm{Im}\, \zeta|).$$

By (3.1) and (3.2), this implies that

$$\begin{aligned}(6.5)\quad \|\exp(t\mathcal{G}(i\zeta))\|_{M_{m\times m}} &\le e^{C|t|}\biggl(1 + \sum_{k=1}^{m-1} \frac{(2|t|)^k}{k!} \|\mathcal{G}(i\zeta)\|^k_{M_{m\times m}}\biggr) e^{C|t|\,|\mathrm{Im}\,\zeta|} \\ &\le e^{C|t|}(1+2|t|)^{m-1} D(1+|\zeta|)^{(m-1)d} e^{C|t|\,|\mathrm{Im}\,\zeta|}\end{aligned}$$

for every $(t,\zeta) \in \mathbb{R} \times \mathbb{C}$ where $C, D \in]0,\infty[$ are independent of (t,ζ), and $d \in \mathbb{N}_0$ is the maximum of the orders of the scalar PDOs which are the entries of $\mathcal{G}(\partial_1,\ldots,\partial_n)$. By the Paley–Wiener–Schwartz theorem, i.e. by [H, Theorem 7.3.1], (6.5) implies that there is a one-parameter convolution group $(\tilde{S}_t)_{t\in\mathbb{R}} \subset \mathcal{E}'(\mathbb{R}^n; M_{m\times m})$ such that

$$(\mathcal{F}\tilde{S}_t)(\zeta) = \exp(t\mathcal{G}(i\zeta)) \quad \text{for every } (t,\zeta) \in \mathbb{R} \times \mathbb{C}^n$$

and

$$\max\{|x| : x \in \operatorname{supp} \tilde{S}_t\} \le C|t| \quad \text{for every } t \in \mathbb{R}.$$

The convolution group $(\tilde{S}_t)_{t\in\mathbb{R}}$ is an extension of the i.d.c.s. $(S_t)_{t\ge 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m\times m})$ with generating distribution $\mathcal{G}(\partial_1,\ldots,\partial_n) \otimes \delta$. This last exists by Theorem 2.3 because (6.5)$\Rightarrow$(2.4) by Proposition 3.3. Furthermore, by (6.5), one has

$$\begin{aligned}(6.6)\quad &\|\mathcal{G}(i\zeta)^k \exp(t\mathcal{G}(i\zeta))\|_{M_{m\times m}} \\ &\qquad \le e^{C|t|}(1+2|t|)^{m-1} D_k(1+|\zeta|)^{(m+k-1)d} e^{C|t|\,|\mathrm{Im}\,\zeta|}\end{aligned}$$

for every $(t,\zeta) \in \mathbb{R} \times \mathbb{C}^n$ and $k \in \mathbb{N}_0$. By the theorem on differentiating a solution of an ODE with respect to a parameter ([Ha, Sec. V.4, Corollary 4.1]), the mapping $\mathbb{R} \times \mathbb{C}^n \ni (t,\zeta) \mapsto \exp(t\mathcal{G}(i\zeta)) \in M_{m\times m}$ is infinitely differentiable. Since

$$\left(\frac{\partial}{\partial t}\right)^k \exp(t\mathcal{G}(i\zeta)) = \mathcal{G}(i\zeta)^k \exp(t\mathcal{G}(i\zeta)),$$

from (6.6) and [E, Sec. V.5, Lemma 5.17] it follows that the mapping $\mathbb{R} \ni t \mapsto \tilde{S}_t \in \mathcal{E}'(\mathbb{R}^n; M_{m\times m})$ is infinitely differentiable in the topology of $\mathcal{E}'(\mathbb{R}^n; M_{m\times m})$.

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