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MORE EASTON THEOREMS FOR LEVEL BY LEVEL EQUIVALENCE

BY

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Abstract. We establish two new Easton theorems for the least supercompact cardinal that are consistent with the level by level equivalence between strong compactness and supercompactness. These theorems generalize Theorem 1 in our earlier paper [Math. Logic Quart. 51 (2005)]. In both our ground model and the model witnessing the conclusions of our present theorems, there are no restrictions on the structure of the class of supercompact cardinals.

1. Introduction and preliminaries. Say that a model of ZFC witnesses *level by level equivalence between strong compactness and supercompactness* iff for every measurable cardinal κ and every regular cardinal $\lambda > \kappa$, κ is λ strongly compact iff κ is λ supercompact, except if κ is a measurable limit of cardinals δ which are λ supercompact. Models containing supercompact cardinals which also witness the level by level equivalence between strong compactness and supercompactness and satisfy GCH were first constructed in [9]. Note that the results of [9] generalize the fundamental work of Magidor [19] where it is shown, relative to a supercompact cardinal, that it is consistent for the least strongly compact and supercompact cardinals to coincide precisely (1). These results also generalize the later work of Kimchi and Magidor [17] who showed, relative to the existence of a class of supercompact cardinals, that it is consistent for the classes of strongly compact and supercompact cardinals to coincide precisely, except at measurable limit points.

We remark that the exceptions in the previous paragraph are provided by a theorem of Menas [21], who showed that if κ is a measurable limit of cardinals δ which are either λ strongly compact or λ supercompact, then κ is

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(1) Magidor also showed in [19] that it is consistent, relative to the existence of a strongly compact cardinal, for the least strongly compact and measurable cardinals to coincide precisely. In this situation, the least strongly compact cardinal is not the least supercompact cardinal.

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λ strongly compact but need not be λ supercompact. (Menas' results of [21] were also a precursor to the later work of [19], [17], and [9].) When this situation occurs, we will henceforth say that *κ is a witness to the Menas exception at λ*. If κ is measurable and for every regular cardinal $\lambda > \kappa$, κ is λ strongly compact iff κ is λ supercompact, then we will say that κ *is a witness to level by level equivalence between strong compactness and supercompactness.*

We continue now with the main narrative. In [1], the following theorem was proven.

THEOREM 1. *Let $V \models$ "ZFC + GCH + Level by level equivalence between strong compactness and supercompactness holds + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + κ is the least supercompact cardinal". Let $A = \{\delta \le \kappa \mid \delta$ is either a strong cardinal or the regular limit of strong cardinals$\}$. Suppose that $F : A \to \kappa$, $F \in V$ is a function with the following properties:*

(1) *$F(\delta) \in (\delta, \delta^*)$ is a cardinal, where δ^* is the least strong cardinal above δ.*
(2) $\mathrm{cof}(F(\delta)) > \delta$.
(3) *If $\delta \in A$ is λ supercompact for $\lambda > \delta$, then there is an elementary embedding $j : V \to M$ witnessing the λ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\lambda)$ such that either $j(F)(\delta) = F(\delta) = \delta^+$ or $j(F)(\delta) = F(\delta) = \delta^{++}$.*

There is then a cardinal and cofinality preserving partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \models$ "ZFC + $\mathcal{K}$ is the class of supercompact cardinals (so κ is the least supercompact cardinal) + Level by level equivalence between strong compactness and supercompactness holds + For every $\delta \in A$, $2^\delta = F(\delta)$".

In Theorem 1, it is unfortunately the case that the Easton function F is defined only on a restricted set of inaccessible cardinals at and below the least supercompact cardinal κ. Furthermore, because of the restrictions placed on F's range by clause (1), if $\delta \in A$, then $2^\delta \le \delta^{++}$. It therefore becomes desirable to see if it is possible to remove these constraints.

The purpose of this paper is to show that this is indeed the case, and to address further the general question of what GCH patterns are consistent with the level by level equivalence between strong compactness and supercompactness. Specifically, we prove the following two theorems.

THEOREM 2. *Let $V \models$ "ZFC + GCH + Level by level equivalence between strong compactness and supercompactness holds + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals + κ is the least supercompact cardinal". Let $A = \{\delta \le \kappa \mid \delta$ is a regular cardinal which is not the successor of a singular cardinal and $\neg\exists\gamma < \delta$ [γ is α supercompact for every $\alpha < \delta$]$\}$. Suppose that $F : A \to \kappa$, $F \in V$, is a function with the following properties:*

(1) *If* $\delta_1 < \delta_2$, *then* $F(\delta_1) \le F(\delta_2)$.
(2) $F(\delta) \in (\delta, \delta')$ *is a cardinal, where* δ' *is the least Mahlo cardinal above* δ.
(3) $\mathrm{cof}(F(\delta)) > \delta$.
(4) *If* $\delta \in A$ *is* λ *supercompact for* $\lambda > \delta$, *then there is an elementary embedding* $j : V \to M$ *witnessing the* λ *supercompactness of* δ *generated by a supercompact ultrafilter over* $P_\delta(\lambda)$ *such that either* $j(F)(\delta) = F(\delta) = \delta^+$ *or* $j(F)(\delta) = F(\delta) = \delta^{++}$.

There is then a cardinal and cofinality preserving partial ordering $\mathbb{P} \in V$ *such that* $V^{\mathbb{P}} \models$ *"ZFC +* $\mathcal{K}$ *is the class of supercompact cardinals (so* κ *is the least supercompact cardinal) + Level by level equivalence between strong compactness and supercompactness holds + For every* $\delta \in A$, $2^\delta = F(\delta)$*".*

THEOREM 3. *Let* $V \models$ *"ZFC + GCH + Level by level equivalence between strong compactness and supercompactness holds +* $\mathcal{K} \neq \emptyset$ *is the class of supercompact cardinals +* κ *is the least supercompact cardinal". Let* $A = \{\delta \le \kappa \mid \delta$ *is either a strong cardinal or the regular limit of strong cardinals*$\}$. *Suppose that* $F : A \to \kappa$, $F \in V$, *is a function with the following properties:*

(1) $F(\delta) \in (\delta, \delta^*)$ *is a cardinal, where* δ^* *is the least strong cardinal above* δ.
(2) $\mathrm{cof}(F(\delta)) > \delta$.
(3) *If* $\delta \in A$ *is* λ *supercompact for* $\lambda \ge \delta^{+16}$, *then there is an elementary embedding* $j : V \to M$ *witnessing the* λ *supercompactness of* δ *generated by a supercompact ultrafilter over* $P_\delta(\lambda)$ *such that* $j(F)(\delta) = F(\delta) = \delta^{+17}$.

There is then a cardinal and cofinality preserving partial ordering $\mathbb{P} \in V$ *such that* $V^{\mathbb{P}} \models$ *"ZFC +* $\mathcal{K}$ *is the class of supercompact cardinals (so* κ *is the least supercompact cardinal) + Level by level equivalence between strong compactness and supercompactness holds + For every* $\delta \in A$, $2^\delta = F(\delta)$*".*

We take this opportunity to make several remarks concerning Theorems 2 and 3. Theorem 2 is a generalization of Theorem 1 where A, the domain of the Easton function F, has been expanded. It includes not only those $\delta \le \kappa$ which are either strong cardinals or regular limits of strong cardinals, but also both certain successor cardinals and additional regular limit cardinals which are neither strong cardinals nor limits of strong cardinals. More specifically, as the proof of [7, Lemma 2.4] shows, if δ is γ supercompact for every $\gamma < \lambda$ and λ is strong, then δ is supercompact as well. From this, since κ is the least supercompact cardinal, it immediately follows that $\kappa \in A$ and if $\lambda < \kappa$ is either a strong cardinal or a limit of strong cardinals, then for no cardinal $\delta < \lambda$ can it be the case that δ is γ supercompact for every $\gamma < \lambda$. Consequently, the domain of the Easton function in Theorem 1 is a subset of the domain of the Easton function in Theorem 2. We note that it is a proper subset. To see this, suppose $\delta < \kappa$ is a non-measurable limit of strong cardinals. Suppose in addition that $\rho > \delta$ is, e.g., the least measurable cardinal above δ, the least measurable limit of measurable cardinals above δ, the least cardinal above δ which is ρ^{+n} supercompact for every $n \in \omega$, etc. If $\gamma \in (\delta, \rho]$ is an inaccessible cardinal, then γ is a member of the domain of the Easton function in Theorem 2 but is not a member of the domain of the Easton function in Theorem 1. Further, if δ is a non-measurable limit of strong cardinals, ρ is the least measurable cardinal above δ, and $\gamma \in (\delta, \rho)$ is a successor cardinal which is not the successor of a singular cardinal, then γ is a member of the domain of the Easton function in Theorem 2 as well.

The restrictions given by δ' on F's range are to allow for the preservation of level by level equivalence between strong compactness and supercompactness while still maintaining a great deal of freedom in the values that F may attain. Other values of δ', e.g., the least weakly compact cardinal above δ, are also possible.

Theorem 3 is a generalization of Theorem 1 which changes the Easton function F so that the power set of any ground model strong cardinal or regular limit of strong cardinals δ can be larger than δ^{++}. In order to achieve this, we employ forcing conditions first defined in [10, Section 4], as ordinary Cohen forcing seems to pose certain technical challenges which are difficult to overcome. Also, the use of δ^{+16} and δ^{+17} in the statement of Theorem 3 is for ease of presentation and comprehensibility. In essence, δ^{+17} should be seen as representing any suitable regular cardinal λ for which it will be the case that $2^\delta = \lambda$. The cardinals δ^{+16} and δ^{+17} may be viewed as a form of "wild card" standing in for the more general possibilities, which are somewhat technical in nature and are explicitly stated in [10, Section 4].

We note that there are many natural functions meeting the requirements of the statements of Theorems 2 and 3. For instance, if we let $B = \{\delta \in A \mid \delta \text{ is a successor cardinal}\}$ and $C = A - B = \{\delta \leq \kappa \mid \delta \text{ is a limit cardinal and } \neg\exists\gamma < \delta\ [\gamma \text{ is } \alpha \text{ supercompact for every } \alpha < \delta]\}$, then

$$F_0(\delta) = \begin{cases} \delta^{+68} & \text{if } \delta \in B, \\ \delta^{++} & \text{if } \delta \in C, \end{cases}$$

and

$$F_1(\delta) = \begin{cases} \text{The least inaccessible cardinal above } \delta & \text{if } \delta \in B, \\ \delta^{+} & \text{if } \delta \in C. \end{cases}$$

are candidates for the function F mentioned in the statement of Theorem 2. If we let $B = \{\delta \in A \mid \delta$ is either a strong cardinal which is not a limit of strong cardinals or a non-measurable limit of strong cardinals$\}$ and $C = A - B = \{\delta \leq \kappa \mid \delta$ is a measurable limit of strong cardinals$\}$, then

$$F_0(\delta) = \begin{cases} \delta^{+95} & \text{if } \delta \in B, \\ \delta^{+17} & \text{if } \delta \in C, \end{cases}$$

and

$$F_1(\delta) = \begin{cases} \text{The least Mahlo cardinal above } \delta & \text{if } \delta \in B, \\ \delta^{+17} & \text{if } \delta \in C, \end{cases}$$

are candidates for the function F mentioned in the statement of Theorem 3. In fact, the Easton functions can essentially take on arbitrary values for either Theorem 2 or Theorem 3 when $\delta \in B$, subject to the restrictions given above in the statements of these theorems.

Before presenting the proofs of our theorems, we briefly state some preliminary information. Our notation and terminology will follow that given in [1] and [6]. We do wish to mention a few things explicitly, however. When forcing, $q \geq p$ means that q is stronger than p. For κ a regular cardinal and α an ordinal, $\mathrm{Add}(\kappa, \alpha)$ is the standard Cohen partial ordering for adding α Cohen subsets of κ. For $\alpha < \beta$ ordinals, $[\alpha, \beta]$, $[\alpha, \beta)$, $(\alpha, \beta]$, and (α, β) are as in standard interval notation. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. We will, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$.

A partial ordering $\mathbb{P}$ is *κ-directed closed* for κ a cardinal if every directed set of conditions of size less than κ has an upper bound. $\mathbb{P}$ is *κ-strategically closed* if in the two-person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \leq \kappa \rangle$, where player I plays odd stages and player II plays even stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is *$<\kappa$-strategically closed* if $\mathbb{P}$ is δ-strategically closed for all cardinals $\delta < \kappa$. Note that if $\mathbb{P}$ is κ-strategically closed and $f : \kappa \to V$ is a function in $V^{\mathbb{P}}$, then $f \in V$. In addition, if $\mathbb{P}$ is κ-directed closed, then $\mathbb{P}$ is $<\kappa$-strategically closed.

Finally, we mention that we are assuming familiarity with the large cardinal notions of measurability, strongness, strong compactness, and supercompactness. Interested readers may consult [13] or [16] for further details. We do note, however, that we will say κ is *supercompact* (*strongly compact*) *up to the cardinal* λ if κ is γ supercompact (γ strongly compact) for every $\gamma < \lambda$. Also, if κ is λ supercompact and λ is a cardinal, then κ is supercompact up to λ^+, i.e., κ is α supercompact for every $\alpha < \lambda^+$.

2. The proofs of Theorems 2 and 3

Proof of Theorem 2. Let V, A, F, and κ be as in the hypotheses for Theorem 2. Let $\langle \zeta_\alpha \mid \alpha < \kappa \rangle \in V$ enumerate in increasing order $\{\zeta < \kappa \mid \zeta$ is either a Mahlo cardinal or a limit of Mahlo cardinals$\}$. We define three partial orderings $\mathbb{P}^0$, $\mathbb{P}^1$, and $\mathbb{P}^2$, where $\mathbb{P}^0 \in V$, $\mathbb{P}^1 \in V^{\mathbb{P}^0}$, and $\mathbb{P}^2 \in V^{\mathbb{P}^0 * \dot{\mathbb{P}}^1}$. The partial ordering $\mathbb{P}$ with which we force to complete the proof of Theorem 2 will then be defined as $\mathbb{P} = (\mathbb{P}^0 * \dot{\mathbb{P}}^1) * \dot{\mathbb{P}}^2 = \overline{\mathbb{P}} * \dot{\mathbb{P}}^2$. Specifically, $\mathbb{P}^0 = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa \rangle \in V$ is the reverse Easton iteration of length κ which begins by forcing with $\mathrm{Add}(\omega, 1)$ (so $\mathbb{P}_0 = \mathrm{Add}(\omega, 1)$). $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is a term for the Easton support product $\prod_{\delta \in A \cap [\zeta_\alpha, \zeta_{\alpha+1})} \mathrm{Add}(\delta, F(\delta))$. $\mathbb{P}^1 \in V^{\mathbb{P}^0}$ is $(\mathrm{Add}(\kappa, F(\kappa)))^{V^{\mathbb{P}^0}}$, and $\mathbb{P}^2 \in V^{\overline{\mathbb{P}}}$ is $\prod_{\delta \in A \cap [\omega, \zeta_0)} \mathrm{Add}(\delta, F(\delta))$.

A few explanatory remarks are perhaps now in order concerning the above definition of $\mathbb{P}$. Note that it is possible to write $\overline{\mathbb{P}} = \mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| = \omega$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is $\aleph_2$-directed closed". In Hamkins' terminology of [11, 12], "$\overline{\mathbb{P}}$ admits a gap at $\aleph_1$". $\overline{\mathbb{P}}$ has been defined in this manner so that the results of [11, 12] may be applied and allow us to infer that the model $V^{\overline{\mathbb{P}}}$ satisfies level by level equivalence between strong compactness and supercompactness. In particular, the gap at $\aleph_1$ ensures that any cardinal δ which is λ supercompact in $V^{\overline{\mathbb{P}}}$ had to have been λ supercompact in V. We use an iteration of products in the definition of $\overline{\mathbb{P}}$ in order to allow the usual supercompactness lifting arguments to be applied (2). In addition, as readers may verify for themselves, the standard Easton arguments for products and iterations (see [13]) show that in $V^{\mathbb{P}}$, cardinals and cofinalities are preserved and $2^\delta = F(\delta)$ for every $\delta \in A$. Since in $V^{\overline{\mathbb{P}}}$, $\mathbb{P}^2$ has cardinality the least Mahlo cardinal, the Lévy–Solovay results [18] ensure that forcing with $\mathbb{P}^2$ over $V^{\overline{\mathbb{P}}}$ will not destroy any relevant properties true in $V^{\overline{\mathbb{P}}}$, e.g., level by level equivalence between strong compactness and supercompactness. Finally, the forcing $\mathbb{P}^2$ is performed at the end of the construction, and not the beginning, so that a gap at $\aleph_1$ may be introduced.

Lemma 2.1. *$V^{\overline{\mathbb{P}}} \models$ "κ is the least supercompact cardinal".*

Proof. The proof of Lemma 2.1 is a somewhat more complicated version of the proof of [1, Lemma 2.1]. By our remarks in the preceding paragraph, since forcing with $\overline{\mathbb{P}}$ creates no new supercompact cardinals, it suffices to show that $V^{\overline{\mathbb{P}}} \models$ "κ is supercompact". Towards this end, let $\lambda > \kappa^+ = 2^\kappa$ be the successor of a regular cardinal, and let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_\kappa(\lambda)$ with $F(\kappa) = j(F)(\kappa)$. Let $\theta = F(\kappa) = j(F)(\kappa)$ (so θ is either κ^+ or κ^{++}). Since $V \models \mathrm{GCH}$, $M \models$ "κ is α supercompact

(2) The author wishes to thank Brent Cody for suggesting this approach.

for every $\alpha < \lambda$". In addition, since $V \models$ "No cardinal is supercompact up to κ", $M \models$ "No cardinal is supercompact up to κ", i.e., $M \models$ "$\kappa \in j(A)$". Further, $M \models$ "$\zeta_\kappa = \kappa$". The definition of $\overline{\mathbb{P}}$ therefore implies that we can write $j(\overline{\mathbb{P}}) = \mathbb{P}_\kappa * ((\dot{\mathrm{Add}}(\kappa, \theta) \times \dot{\mathbb{Q}}^0) * \dot{\mathbb{Q}}^1) * \dot{\mathrm{Add}}(j(\kappa), j(\theta)) = \mathbb{P}_\kappa * \dot{\mathbb{Q}} * \dot{\mathrm{Add}}(j(\kappa), j(\theta))$, where $\dot{\mathrm{Add}}(\kappa, \theta) \times \dot{\mathbb{Q}}^0$ is a term for the Easton support product $\prod_{\delta \in j(A) \cap [\zeta_\kappa, \zeta_{\kappa+1})} \mathrm{Add}(\delta, F(\delta))$, $\dot{\mathbb{Q}}^1$ is a term for the portion of $j(\overline{\mathbb{P}})$ defined between $\zeta_{\kappa+1}$ and $j(\kappa)$, and the first ordinal at which $\dot{\mathbb{Q}}^0$ is forced to do nontrivial forcing is greater than or equal to λ^+.

Let G be V-generic over $\mathbb{P}_\kappa$, and let H be $V[G]$-generic over $(\mathrm{Add}(\kappa, \theta))^{V[G]}$. Since $\mathbb{P}_\kappa$ is κ-c.c., standard arguments show that $M[G]$ remains λ closed with respect to $V[G]$, and that $\mathbb{Q}^0$ is λ^+-directed closed in both $M[G]$ and $V[G]$. Since $M[G] \models$ "$|\mathbb{Q}^0| < j(\kappa)$" and $j(\kappa)$ is inaccessible in both M and $M[G]$, there are $j(\kappa)$ dense open subsets of $\mathbb{Q}^0$ present in $M[G]$. However, since $|j(\kappa)| = |\{f \mid f : P_\kappa(\lambda) \to \kappa \text{ is a function}\}| = \lambda^+$ by GCH, we can use the usual diagonalization arguments (as given, e.g., in the construction of the generic object G_1 in [7, Lemma 2.4]) to construct in $V[G]$ an $M[G]$-generic object H' over $\mathbb{Q}^0$. (An outline of this argument is as follows. Let $\langle D_\alpha \mid \alpha < \lambda^+ \rangle$ enumerate in $V[G]$ the dense open subsets of $\mathbb{Q}^0$ present in $M[G]$. Because $M[G]$ remains λ closed with respect to $V[G]$, by the λ^+-directed closure of $\mathbb{Q}^0$ in both $M[G]$ and $V[G]$, we may work in $V[G]$ and meet each D_α in order to construct H'.) Since $\mathbb{Q}^0$ is λ^+-directed closed in both $M[G]$ and $V[G]$, standard arguments again show that $M[G][H']$ remains λ closed with respect to $V[G][H'] = V[G]$. In addition, since H is $V[G]$-generic over $\mathrm{Add}(\kappa, \theta)$ and $M[G][H'] \subseteq V[G]$, H is $M[G][H']$-generic over $\mathrm{Add}(\kappa, \theta)$. Consequently, since $\mathrm{Add}(\kappa, \theta)$ is κ^+-c.c. in both $V[G]$ and $M[G][H']$, $M[G][H'][H] = M[G][H][H']$ remains λ closed with respect to $V[G][H]$. Further, since $M[G][H][H'] \models$ "$|\mathbb{Q}^1| = j(\kappa)$" and GCH holds in both V and M, there are $2^{j(\kappa)} = j(\kappa^+)$ dense open subsets of $\mathbb{Q}^1$ present in $M[G][H][H']$. However, since $|j(\kappa^+)| = |\{f \mid f : P_\kappa(\lambda) \to \kappa^+ \text{ is a function}\}| = \lambda^+$ by GCH, we can use the same diagonalization arguments as in the construction of H' to construct in $V[G][H]$ an $M[G][H][H']$-generic object H'' over $\mathbb{Q}$ and lift j to $j : V[G] \to M[G][H][H'][H'']$ in $V[G][H]$. Note that $M[G][H][H'][H'']$ remains λ closed with respect to $V[G][H][H'][H''] = V[G][H]$, since $\mathbb{Q}^1$ is λ^+-directed closed in $M[G][H][H']$ and $M[G][H][H']$ is λ closed with respect to $V[G][H]$. Then, as the number of dense open subsets of $\mathrm{Add}(j(\kappa), j(\theta))$ in $M[G][H][H'][H'']$ is either $j(\kappa^{++})$ (if $F(\kappa) = \kappa^+$) or $j(\kappa^{+++})$ (if $F(\kappa) = \kappa^{++}$), which by GCH and the fact that $\lambda \geq \kappa^{++} \geq \theta$ has size λ^+ in $V[G][H]$, and as $\mathrm{Add}(j(\kappa), j(\theta))$ is λ^+-directed closed in both $M[G][H][H'][H'']$ and $V[G][H]$ and $V[G][H] \models$ "$|j''H| \leq \lambda$", we can once again use the standard diagonalization arguments to construct in $V[G][H]$ an $M[G][H][H'][H'']$-generic object H''' containing a master condition for $j''H$. We can now fully lift j to $j : V[G][H] \to M[G][H][H'][H''][H''']$ in $V[G][H]$, thereby showing that $V[G][H] \models$ "κ is λ supercompact". Since $\lambda \geq \kappa^{++}$ was an arbitrary successor of a regular cardinal, this completes the proof of Lemma 2.1. ∎

LEMMA 2.2. *If $\delta \in A$ and $\lambda > \delta$ is a regular cardinal such that $V \models$ "δ is λ supercompact", then $V^{\overline{\mathbb{P}}} \models$ "δ is λ supercompact".*

Proof. The proof of Lemma 2.2 is once again along the lines of the proof of [1, Lemma 2.2], only somewhat more complicated. In analogy to Lemma 2.1, let $\theta = F(\delta)$, i.e., θ is either δ^+ or δ^{++}. Write $\overline{\mathbb{P}} = \mathbb{P}_{\delta+1} * \dot{\mathbb{Q}}$. By the definition of $\overline{\mathbb{P}}$, since $V \models$ "δ is α supercompact for every $\alpha < \lambda^+$", $\Vdash_{\mathbb{P}_{\delta+1}}$ "$\dot{\mathbb{Q}}$ is γ-directed closed where $\gamma > \lambda^+$ is Mahlo". Hence, to show that $V^{\overline{\mathbb{P}}} \models$ "δ is λ supercompact", it suffices to show that $V^{\mathbb{P}_{\delta+1}} \models$ "δ is λ supercompact". In addition, the definition of $\overline{\mathbb{P}}$ implies that $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$, where $\dot{\mathbb{Q}}_\delta$ is a term for a partial ordering having the form $\mathrm{Add}(\delta, \theta) \times \mathbb{R}$. Here, since $V \models$ "δ is α supercompact for every $\alpha < \lambda^+$", $\mathbb{R}$ is an Easton product of the Cohen forcings $\mathrm{Add}(\gamma, \eta)$, where by GCH at and above δ in V and $V^{\mathbb{P}_\delta}$, $\gamma \geq \lambda^{++} = (2^{[\lambda]^{<\delta}})^+ = (2^\lambda)^+$. Consequently, in $V^{\mathbb{P}_\delta}$, $\mathbb{R}$ is $(2^{[\lambda]^{<\delta}})^+$-directed closed. Therefore, since $\theta \leq \delta^{++} < \lambda^{++}$, to show that $V^{\mathbb{P}_{\delta+1}} = V^{\mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta} = V^{\mathbb{P}_\delta * (\dot{\mathrm{A}}\mathrm{dd}(\delta,\theta) \times \dot{\mathbb{R}})} \models$ "δ is λ supercompact", it suffices to show that $V^{\mathbb{P}_\delta * \dot{\mathrm{A}}\mathrm{dd}(\delta,\theta)} \models$ "δ is λ supercompact". With a severe abuse of notation, for the remainder of the proof of Lemma 2.2, we denote $\mathbb{P}_\delta * \dot{\mathrm{A}}\mathrm{dd}(\delta, \theta)$ by $\mathbb{P}_{\delta+1}$.

Suppose now that $\lambda \geq \theta$. By the definition of F, we may choose $j : V \to M$ as an elementary embedding witnessing the λ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\lambda)$ such that $j(F)(\delta) = F(\delta) = \theta$. If $V \models$ "λ is the successor of a singular cardinal", then since $M^\lambda \subseteq M$, $M \models$ "λ is the successor of a singular cardinal" as well. If not, i.e., if $V \models$ "λ is either a regular limit cardinal or the successor of a regular cardinal", then by GCH in V and M, $M \models$ "δ is α supercompact for every $\alpha < \lambda$". In either case, as in Lemma 2.1, $j(\mathbb{P}_{\delta+1}) = \mathbb{P}_\delta * ((\dot{\mathrm{A}}\mathrm{dd}(\delta,\theta) \times \dot{\mathbb{Q}}^0) * \dot{\mathbb{Q}}^1) * \dot{\mathrm{A}}\mathrm{dd}(j(\delta), j(\theta))$, where the first ordinal at which $\dot{\mathbb{Q}}^0$ is forced to do nontrivial forcing is greater than or equal to λ^+. The argument given in the proof of Lemma 2.1 may therefore now be used to show that $V^{\mathbb{P}_{\delta+1}} \models$ "δ is λ supercompact". Thus, we assume that $\lambda < \theta$. Since $\lambda > \delta$, this means that $\lambda = \delta^+$.

The remainder of the proof of Lemma 2.2 is along the lines of [1, Lemma 2.2], from which for the relevant portions we quote almost verbatim. If $j(F)(\delta) = F(\delta) = \delta^+$, then the argument given in the preceding paragraph shows that $V^{\mathbb{P}_{\delta+1}} \models$ "δ is λ supercompact". We consequently assume that $\lambda = \delta^+$ and $j(F)(\delta) = F(\delta) = \delta^{++}$. Under these circumstances, the argument given in [9, pp. 119–120], [3, pp. 88–90], [8, pp. 832–833], or

[2, pp. 591–592] (which is originally due to Magidor and is also found earlier in [14], [15], and [20]), suitably modified to take into account the definition of $\overline{\mathbb{P}}$, can be used to show that $V^{\mathbb{P}_{\delta+1}} \models$ "δ is λ supercompact".

Getting specific, write $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathrm{A}}\mathrm{dd}(\delta, \delta^{++})$. Let G be V-generic over $\mathbb{P}_\delta$, and let H be $V[G]$-generic over $(\mathrm{Add}(\delta, \delta^{++}))^{V[G]}$. Fix $j : V \to M$ an elementary embedding witnessing the $\lambda = \delta^+$ supercompactness of δ generated by a supercompact ultrafilter $\mathcal{U}$ over $P_\delta(\lambda)$ such that $j(F)(\delta) = F(\delta) = \delta^{++}$. We then have as above that $j(\mathbb{P}_{\delta+1}) = \mathbb{P}_\delta * ((\dot{\mathrm{A}}\mathrm{dd}(\delta, \theta) \times \dot{\mathbb{Q}}^0) * \dot{\mathbb{Q}}^1) * \dot{\mathrm{A}}\mathrm{dd}(j(\delta), j(\theta))$, where the first ordinal at which $\dot{\mathbb{Q}}^0$ is forced to do nontrivial forcing is greater than or equal to $\lambda^+ = \delta^{++}$. Therefore, the arguments used in the proof of Lemma 2.1 allow us to construct in $V[G][H]$ generic objects H' and H'' over $\mathbb{Q}^0$ and $\mathbb{Q}^1$ respectively and lift j in $V[G][H]$ to $j : V[G] \to M[G][H][H'][H'']$, where $M[G][H][H'][H'']$ remains δ^+ closed with respect to $V[G][H]$.

We construct now in $V[G][H]$ an $M[G][H][H'][H'']$-generic object over $\mathrm{Add}(j(\delta), j(\delta^{++}))$. For $\alpha \in (\delta, \delta^{++})$ and $p \in \mathrm{Add}(\delta, \delta^{++})$, let $p \restriction \alpha = \{\langle\langle\rho, \sigma\rangle, \eta\rangle \in p \mid \sigma < \alpha\}$ and $H \restriction \alpha = \{p \restriction \alpha \mid p \in H\}$. Clearly, $V[G][H] \models$ "$|H \restriction \alpha| \le \delta^+$ for all $\alpha \in (\delta, \delta^{++})$". Thus, since $\mathrm{Add}(j(\delta), j(\delta^{++}))^{M[G][H][H'][H'']}$ is $j(\delta)$-directed closed and $j(\delta) > \delta^{++}$, $q_\alpha = \bigcup\{j(p) \mid p \in H \restriction \alpha\}$ is well-defined and is an element of $\mathrm{Add}(j(\delta), j(\delta^{++}))^{M[G][H][H'][H'']}$. Further, if $\langle\rho, \sigma\rangle \in \mathrm{dom}(q_\alpha) - \mathrm{dom}(\bigcup_{\beta<\alpha} q_\beta)$ ($\bigcup_{\beta<\alpha} q_\beta$ is well-defined by closure), then $\sigma \in [\bigcup_{\beta<\alpha} j(\beta), j(\alpha))$. To see this, assume to the contrary that $\sigma < \bigcup_{\beta<\alpha} j(\beta)$. Let β be minimal such that $\sigma < j(\beta)$. It must thus be the case that for some $p \in H \restriction \alpha$, $\langle\rho, \sigma\rangle \in \mathrm{dom}(j(p))$. Since by elementarity and the definitions of $H \restriction \beta$ and $H \restriction \alpha$, for $p \restriction \beta = q \in H \restriction \beta$, $j(q) = j(p) \restriction j(\beta) = j(p \restriction \beta)$, it must be the case that $\langle\rho, \sigma\rangle \in \mathrm{dom}(j(q))$. This means $\langle\rho, \sigma\rangle \in \mathrm{dom}(q_\beta)$, a contradiction.

Since $M[G][H][H'][H''] \models$ "GCH holds for all cardinals at or above $j(\delta)$", $M[G][H][H'][H''] \models$ "$\mathrm{Add}(j(\delta), j(\delta^{++}))$ is $j(\delta^+)$-c.c. and has $j(\delta^{++})$ many maximal antichains". This means that if $\mathcal{A} \in M[G][H][H'][H'']$ is a maximal antichain of $\mathrm{Add}(j(\delta), j(\delta^{++}))$, then $\mathcal{A} \subseteq \mathrm{Add}(j(\delta), \beta)$ for some $\beta \in (j(\delta), j(\delta^{++}))$. Thus, since GCH in V and the fact j is generated by a supercompact ultrafilter over $P_\delta(\delta^+)$ imply that $V \models$ "$|j(\delta^{++})| = \delta^{++}$", we can let $\langle\mathcal{A}_\alpha \mid \alpha \in (\delta, \delta^{++})\rangle \in V[G][H]$ be an enumeration of all of the maximal antichains of $\mathrm{Add}(j(\delta), j(\delta^{++}))$ present in $M[G][H][H'][H'']$.

Working in $V[G][H]$, we define now an increasing sequence $\langle r_\alpha \mid \alpha \in (\delta, \delta^{++})\rangle$ of elements of $\mathrm{Add}(j(\delta), j(\delta^{++}))$ such that $\forall\alpha \in (\delta, \delta^{++})$ $[r_\alpha \ge q_\alpha$ and $r_\alpha \in \mathrm{Add}(j(\delta), j(\alpha))]$ and such that $\forall\mathcal{A} \in \langle\mathcal{A}_\alpha \mid \alpha \in (\delta, \delta^{++})\rangle$ $\exists\beta \in (\delta, \delta^{++})$ $\exists r \in \mathcal{A}$ $[r_\beta \ge r]$. Assuming we have such a sequence, $H''' = \{p \in \mathrm{Add}(j(\delta), j(\delta^{++})) \mid \exists r \in \langle r_\alpha \mid \alpha \in (\delta, \delta^{++})\rangle$ $[r \ge p]\}$ is an $M[G][H][H'][H'']$-generic object over $\mathrm{Add}(j(\delta), j(\delta^{++}))$. To define $\langle r_\alpha \mid \alpha \in (\delta, \delta^{++})\rangle$, if α is a limit, we let $r_\alpha = \bigcup_{\beta \in (\delta, \alpha)} r_\beta$. By the facts $\langle r_\beta \mid \beta \in (\delta, \alpha) \rangle$ is (strictly) increasing and $M[G][H][H'][H'']$ is δ^+ closed with respect to $V[G][H]$, this definition is valid. Assuming now r_α has been defined and we wish to define $r_{\alpha+1}$, let $\langle \mathcal{B}_\beta \mid \beta < \eta \leq \delta^+ \rangle$ be the subsequence of $\langle \mathcal{A}_\beta \mid \beta \leq \alpha + 1 \rangle$ containing each antichain $\mathcal{A}$ such that $\mathcal{A} \subseteq \mathrm{Add}(j(\delta), j(\alpha + 1))$. Since $q_\alpha, r_\alpha \in \mathrm{Add}(j(\delta), j(\alpha))$, $q_{\alpha+1} \in \mathrm{Add}(j(\delta), j(\alpha + 1))$, and $j(\alpha) < j(\alpha + 1)$, the condition $r'_{\alpha+1} = r_\alpha \cup q_{\alpha+1}$ is well-defined, since by our earlier observations, any new elements of $\mathrm{dom}(q_{\alpha+1})$ will not be present in either $\mathrm{dom}(q_\alpha)$ or $\mathrm{dom}(r_\alpha)$. We can thus, using the fact $M[G][H][H'][H'']$ is closed under δ^+ sequences with respect to $V[G][H]$, define by induction an increasing sequence $\langle s_\beta \mid \beta < \eta \rangle$ such that $s_0 \geq r'_{\alpha+1}$, $s_\rho = \bigcup_{\beta<\rho} s_\beta$ if ρ is a limit ordinal, and $s_{\beta+1} \geq s_\beta$ is such that $s_{\beta+1}$ extends some element of $\mathcal{B}_\beta$. The just mentioned closure fact implies $r_{\alpha+1} = \bigcup_{\beta<\eta} s_\beta$ is a well-defined condition.

In order to show that H''' is $M[G][H][H'][H'']$-generic over $\mathrm{Add}(j(\delta), j(\delta^{++}))$, we must show that $\forall \mathcal{A} \in \langle \mathcal{A}_\alpha \mid \alpha \in (\delta, \delta^{++}) \rangle\ \exists \beta \in (\delta, \delta^{++})\ \exists r \in \mathcal{A}\ [r_\beta \geq r]$. To do this, we first note that $\langle j(\alpha) \mid \alpha < \delta^{++} \rangle$ is unbounded in $j(\delta^{++})$. To see this, if $\beta < j(\delta^{++})$ is an ordinal, then for some $f : P_\delta(\delta^+) \to M$ representing β, we can assume that for $p \in P_\delta(\delta^+)$, $f(p) < \delta^{++}$. Thus, by the regularity of δ^{++} in V, $\beta_0 = \bigcup_{p \in P_\delta(\delta^+)} f(p) < \delta^{++}$, and $j(\beta_0) \geq \beta$. This means by our earlier remarks that if $\mathcal{A} \in \langle \mathcal{A}_\alpha \mid \alpha < \delta^{++} \rangle$, $\mathcal{A} = \mathcal{A}_\rho$, then we can let $\beta \in (\delta, \delta^{++})$ be such that $\mathcal{A} \subseteq \mathrm{Add}(j(\delta), j(\beta))$. By construction, for $\eta > \max(\beta, \rho)$, there is some $r \in \mathcal{A}$ such that $r_\eta \geq r$. And, as any $p \in \mathrm{Add}(\delta, \delta^{++})$ is such that for some $\alpha \in (\delta, \delta^{++})$, $p = p \restriction \alpha$, H''' is such that if $p \in H$, then $j(p) \in H'''$. Thus, working in $V[G][H]$, we have shown that j lifts to $j : V[G][H] \to M[G][H][H'][H''][H''']$, i.e., $V[G][H] \vDash$ "δ is $\lambda = \delta^+$ supercompact". This completes the proof of Lemma 2.2. ∎

Lemma 2.3. *$V^{\overline{\mathbb{P}}} \vDash$ "Every measurable cardinal δ either is a witness to level by level equivalence between strong compactness and supercompactness or is a witness to the Menas exception at λ for some $\lambda > \delta$", i.e., $V^{\overline{\mathbb{P}}} \vDash$ "Level by level equivalence between strong compactness and supercompactness holds".*

Proof. We modify the proof of [1, Lemma 2.3]. Since $\overline{\mathbb{P}}$ may be defined so that $|\overline{\mathbb{P}}| \leq \kappa^{++}$, by the results of [18], Lemma 2.3 is true for any measurable cardinal $\delta > \kappa$. By Lemma 2.1, Lemma 2.3 is true for $\delta = \kappa$. It thus suffices to show that Lemma 2.3 holds for any measurable cardinal $\delta < \kappa$. To establish this last fact, we consider the following two cases.

Case 1: $\delta \in A$. Suppose $\lambda > \delta$ is such that $V^{\overline{\mathbb{P}}} \vDash$ "δ is λ strongly compact". Recall from the second paragraph of the proof of Theorem 2 that $\overline{\mathbb{P}}$ may be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| = \omega$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is

$\aleph_2$-directed closed". Further, it is easily seen that any subset x of δ in $V^{\overline{\mathbb{P}}}$ of size below δ has a "nice" name τ of size below δ in V, i.e., there is a set y in V, $|y| < \delta$, such that any ordinal forced by a condition in $\overline{\mathbb{P}}$ to be in τ is an element of y. Therefore, in the terminology of [11, 12], $\overline{\mathbb{P}}$ is a "mild forcing with respect to δ admitting a gap at $\aleph_1$", so by the results of [11, 12], $V \vDash$ "δ is λ strongly compact".

Note now that δ cannot be a witness in V to the Menas exception at λ, i.e., δ is not in V a limit of cardinals which are λ supercompact. This follows since otherwise, there are $\gamma < \delta$ such that γ is α supercompact for every $\alpha < \delta$, an immediate contradiction to the fact that $\delta \in A$. Hence, by level by level equivalence between strong compactness and supercompactness in V, δ is λ supercompact in V. We may therefore apply Lemma 2.2 to infer that $V^{\overline{\mathbb{P}}} \vDash$ "δ is λ supercompact".

Case 2: $\delta \notin A$. As before, suppose $\lambda > \delta$ is such that $V^{\overline{\mathbb{P}}} \vDash$ "δ is λ strongly compact". Let $S = \{\rho \in A \mid \rho < \delta\}$, with γ either the largest member of S (if it exists), or the supremum of the members of S otherwise. In the former situation, it must be true that $F(\gamma) < \delta$, since $F(\gamma)$ has size below the least V-Mahlo cardinal above it. Write $\overline{\mathbb{P}} = \mathbb{P}_S * \dot{\mathbb{P}}^S$, where $\mathbb{P}_S$ is the portion of $\overline{\mathbb{P}}$ acting nontrivially on members of S, and $\dot{\mathbb{P}}^S$ is a term for the rest of $\overline{\mathbb{P}}$. It is therefore also true by the definition of $\overline{\mathbb{P}}$ that $|\mathbb{P}_S| < \delta$. The factorizations of $\overline{\mathbb{P}}$ given in Case 1 and the one just presented consequently imply that we once more have in the terminology of [11, 12] that $\overline{\mathbb{P}}$ is a "mild forcing with respect to δ admitting a gap below δ". Hence, again by the results of [11, 12], $V \vDash$ "δ is λ strongly compact". By the level by level equivalence between strong compactness and supercompactness in V, this means that δ is either λ supercompact in V or is a witness to the Menas exception at λ in V. Regardless of which of these situations holds, it must be the case that there is some cardinal $\rho \leq \delta$ such that ρ is λ supercompact in V. Note that since GCH holds in V, $|\mathbb{P}_S| < \delta$, and $V \vDash$ "ρ is α supercompact for every $\alpha < \lambda^+$", we have $\Vdash_{\mathbb{P}_S}$ "$\dot{\mathbb{P}}^S$ is $\lambda^{++} = (2^\lambda)^+ = (2^{[\lambda]^{<\rho}})^+$-directed closed". This and the results of [18] then show that in both $V^{\mathbb{P}_S}$ and $V^{\mathbb{P}_S * \dot{\mathbb{P}}^S} = V^{\overline{\mathbb{P}}}$, δ is either λ supercompact or is a witness to the Menas exception at λ.

If $\gamma = \sup(S)$, then by the definition of $\overline{\mathbb{P}}$, γ has to be singular. Hence, it must be possible as before to write $\overline{\mathbb{P}} = \mathbb{P}_S * \dot{\mathbb{P}}^S$, where $|\mathbb{P}_S| < \delta$. The analysis given in the preceding paragraph thus once again applies to show that in $V^{\overline{\mathbb{P}}}$, δ is either λ supercompact or is a witness to the Menas exception at λ. Therefore, regardless if we are in Case 1 or Case 2, if $V^{\overline{\mathbb{P}}} \vDash$ "δ is λ strongly compact", δ is either λ supercompact in $V^{\overline{\mathbb{P}}}$ or is a witness to the Menas exception at λ in $V^{\overline{\mathbb{P}}}$. This just means that in $V^{\overline{\mathbb{P}}}$, level by level equivalence between strong compactness and supercompactness holds. This completes the proof of Lemma 2.3. ∎

Lemma 2.4. $V^{\overline{\mathbb{P}}} \vDash$ "*$\mathcal{K}$ is the class of supercompact cardinals*".

Proof. By Lemma 2.1, $V^{\overline{\mathbb{P}}} \vDash$ "κ is the least supercompact cardinal". Thus, to prove Lemma 2.4, it suffices to show that in $V^{\overline{\mathbb{P}}}$, the class of supercompact cardinals above κ is the same as in V. However, since as we observed in the proof of Lemma 2.3, $\overline{\mathbb{P}}$ may be defined so that $|\overline{\mathbb{P}}| \le \kappa^{++}$, this follows by the results of [18]. This completes the proof of Lemma 2.4. ∎

Since $\mathbb{P}^2$ has cardinality the least V-Mahlo cardinal, the results of [18] imply that the conclusions of Lemmas 2.1–2.4 remain true in $V^{\overline{\mathbb{P}} * \dot{\mathbb{P}}^2} = V^{\mathbb{P}}$. This observation, together with Lemmas 2.1–2.4 and the remarks made prior to the proof of Lemma 2.1, complete the proof of Theorem 2. ∎

Having completed the proof of Theorem 2, we turn now to the proof of Theorem 3.

Proof of Theorem 3. Let V, A, F, and κ be as in the hypotheses for Theorem 3. In order to present in a meaningful way the iteration to be used in the proof, we first recall the definitions and properties of the fundamental building blocks of this partial ordering. In particular, we describe now a specific form of the partial orderings $\mathbb{P}^0_{\delta,\lambda}$, $\mathbb{P}^1_{\delta,\lambda}[S]$, and $\mathbb{P}^2_{\delta,\lambda}[S]$ of [10, Section 4]. So that readers are not overly burdened, we abbreviate our definitions and descriptions somewhat. Full details may be found by consulting [10], along with the relevant portions of [9]. Note that our presentation is excerpted almost verbatim from [6, Section 2].

Fix regular cardinals $\delta < \lambda$, $\lambda > \delta^+$ in our ground model V, with δ inaccessible and λ either inaccessible or the successor of a cardinal of cofinality greater than δ. We assume GCH holds for all cardinals $\eta \ge \delta$. The first notion of forcing $\mathbb{P}^0_{\delta,\lambda}$ is just the standard notion of forcing for adding a non-reflecting stationary set of ordinals S of cofinality ω to λ. Next, work in $V_1 = V^{\mathbb{P}^0_{\delta,\lambda}}$, letting $\dot{S}$ be a term always forced to denote S. $\mathbb{P}^2_{\delta,\lambda}[S]$ is the standard notion of forcing for introducing a club set C which is disjoint from S (and therefore makes S nonstationary).

We fix now in V_1 a $\clubsuit(S)$ sequence $X = \langle x_\alpha \mid \alpha \in S\rangle$, the existence of which is given by Lemma 1 of [9] and [10]. We are ready to define in V_1 the partial ordering $\mathbb{P}^1_{\delta,\lambda}[S]$. First, since each element of S has cofinality ω, the proof of Lemma 1 of [9] and [10] shows each $x \in X$ can be assumed to be such that the order type of x is ω. Then $\mathbb{P}^1_{\delta,\lambda}[S]$ is defined as the set of all 4-tuples $\langle w, \alpha, \bar{r}, Z\rangle$ with the following properties:

(1) $w \in [\lambda]^{<\delta}$.
(2) $\alpha < \delta$.

(3) $\bar{r} = \langle r_i \mid i \in w \rangle$ is a sequence of functions from α to $\{0,1\}$, i.e., a sequence of subsets of α.
(4) $Z \subseteq \{x_\beta \mid \beta \in S\}$ is a set such that if $z \in Z$, then for some $y \in [w]^\omega$, $y \subseteq z$ and $z - y$ is bounded in the β such that $z = x_\beta$.

The ordering on $\mathbb{P}^1_{\delta,\lambda}[S]$ is given by $\langle w^1, \alpha^1, \bar{r}^1, Z^1 \rangle \le \langle w^2, \alpha^2, \bar{r}^2, Z^2 \rangle$ iff the following hold:

(i) $w^1 \subseteq w^2$.
(ii) $\alpha^1 \le \alpha^2$.
(iii) If $i \in w^1$, then $r^1_i \subseteq r^2_i$.
(iv) $Z^1 \subseteq Z^2$.
(v) If $z \in Z^1 \cap [w^1]^\omega$ and $\alpha^1 \le \alpha < \alpha^2$, then $|\{i \in z \mid r^2_i(\alpha) = 0\}| = |\{i \in z \mid r^2_i(\alpha) = 1\}| = \omega$.

The intuition behind the above definition of $\mathbb{P}^1_{\delta,\lambda}[S]$ is described in the first complete paragraph on [10, p. 2033], from which we quote. We wish to be able simultaneously to make $2^\delta = \lambda$, destroy the measurability of δ, and be able to resurrect the $<\lambda$ supercompactness of δ if necessary. By its design, $\mathbb{P}^1_{\delta,\lambda}[S]$ allows us to accomplish these tasks. Specifically, the proof of [9, Lemma 4] shows that $\mathbb{P}^0_{\delta,\lambda} * (\mathbb{P}^1_{\delta,\lambda}[\dot{S}] \times \mathbb{P}^2_{\delta,\lambda}[\dot{S}])$ is forcing equivalent to $\mathrm{Add}(\lambda, 1) * \dot{\mathrm{Add}}(\delta, \lambda)$. The proofs of [9, Lemmas 3 and 5] and [10, Lemma 6] show that $\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^1_{\delta,\lambda}[\dot{S}]$ preserves cardinals and cofinalities, is λ^+-c.c., is $<\delta$-strategically closed, and is such that $V^{\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^1_{\delta,\lambda}[\dot{S}]} \models$ "$2^\eta = \lambda$ for every cardinal $\eta \in [\delta, \lambda)$ and δ is non-measurable".

We return now to the definition of the partial ordering $\mathbb{P}$ used in the proof of Theorem 3. $\mathbb{P}$ will be the reverse Easton iteration $\langle \langle \mathbb{P}_\delta, \dot{\mathbb{Q}}_\delta \rangle \mid \delta \le \kappa \rangle$ of length $\kappa + 1$ which begins by adding a Cohen subset of ω and then does nontrivial forcing only at members of A. If $\delta \in A$ is γ supercompact for $\gamma \ge \delta^{+16}$, then $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$, where $\dot{\mathbb{Q}}_\delta$ is a term for $\mathbb{P}^0_{\delta,F(\delta)} * (\mathbb{P}^1_{\delta,F(\delta)}[\dot{S}] \times \mathbb{P}^2_{\delta,F(\delta)}[\dot{S}])$ as described above. For all other $\delta \in A$, $\mathbb{P}_{\delta+1} = \mathbb{P}_\delta * \dot{\mathbb{Q}}_\delta$, where $\dot{\mathbb{Q}}_\delta$ is a term for $\mathbb{P}^0_{\delta,F(\delta)} * \mathbb{P}^1_{\delta,F(\delta)}[\dot{S}]$. We explicitly note that if $\delta \in A$ is γ supercompact for $\gamma \ge \delta^{+16}$, then by our restrictions on F, $F(\delta) = \delta^{+17}$.

The standard Easton arguments (see, e.g., [13]) in combination with the properties of $\mathbb{P}^0_{\delta,\lambda} * (\mathbb{P}^1_{\delta,\lambda}[\dot{S}] \times \mathbb{P}^2_{\delta,\lambda}[\dot{S}])$ and $\mathbb{P}^0_{\delta,\lambda} * \mathbb{P}^1_{\delta,\lambda}[\dot{S}]$ described in the next to last paragraph then show that forcing with $\mathbb{P}$ preserves all cardinals and cofinalities and that in $V^{\mathbb{P}}$, $2^\delta = F(\delta)$ for every $\delta \in A$. In addition, note that it is possible to write $\mathbb{P} = \mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| = \omega$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is $\aleph_1$-strategically closed". Therefore, by the results of [11, 12], any cardinal δ which is λ supercompact in $V^{\mathbb{P}}$ had to have been λ supercompact in V, and any cardinal δ which is either a strong cardinal or an inaccessible limit of strong cardinals in $V^{\mathbb{P}}$ had to have been in V either a strong cardinal or an inaccessible limit of strong cardinals.

LEMMA 2.5. *$V^{\mathbb{P}} \models$ "κ is the least supercompact cardinal".*

Proof. The proof of Lemma 2.5 is a simplified version of the proof of Lemma 2.1. By our remarks in the preceding paragraph, since forcing with $\mathbb{P}$ creates no new supercompact cardinals, it suffices to show that $V^{\mathbb{P}} \models$ "κ is supercompact". Towards this end, let $\lambda \geq \kappa^{+17} > 2^{\kappa} = \kappa^{+}$ be a regular cardinal, and let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of κ generated by a supercompact ultrafilter over $P_{\kappa}(\lambda)$ with $F(\kappa) = j(F)(\kappa) = \kappa^{+17}$. Let $\theta = \kappa^{+17}$. Note that since $V \models$ "κ is the least supercompact cardinal", $M \models$ "$j(\kappa) > \kappa$ is the least supercompact cardinal", i.e., $M \models$ "κ is not supercompact". By [7, Lemma 2.1], $M \models$ "κ is a regular cardinal which is a limit of strong cardinals". Also, $M \models$ "No cardinal $\delta \in (\kappa, \lambda]$ is strong", since otherwise κ is supercompact up to a strong cardinal and hence is fully supercompact in M. By the forcing equivalence of $\mathbb{P}^0_{\kappa,\kappa^{+17}} * (\mathbb{P}^1_{\kappa,\kappa^{+17}}[\dot{S}] \times \mathbb{P}^2_{\kappa,\kappa^{+17}}[\dot{S}])$ with $\mathrm{Add}(\kappa^{+17}, 1) * \dot{\mathrm{Add}}(\kappa, \kappa^{+17})$ and the fact that $M \models$ "κ is κ^{+16} supercompact", $j(\mathbb{P})$ is forcing equivalent to $\mathbb{P}_{\kappa} * \dot{\mathrm{Add}}(\kappa^{+17}, 1) * \dot{\mathrm{Add}}(\kappa, \kappa^{+17}) * \dot{\mathbb{Q}} * \dot{\mathrm{Add}}(j(\kappa)^{+17}, 1) * \dot{\mathrm{Add}}(j(\kappa), j(\kappa^{+17}))$. This in turn is forcing equivalent to $\mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathrm{Add}}(j(\kappa)^{+17}, 1) * \dot{\mathrm{Add}}(j(\kappa), j(\kappa^{+17}))$.

Let G be V-generic over $\mathbb{P}_{\kappa}$, and let H be $V[G]$-generic over $(\mathrm{Add}(\kappa^{+17}, 1) * \dot{\mathrm{Add}}(\kappa, \kappa^{+17}))^{V[G]}$. The arguments mentioned in the proof of Lemma 2.1 for the construction of the generic objects H' and H'' may now be used to construct an $M[G][H]$-generic object H' over $\mathbb{Q}$. This allows us to lift j in $V[G][H]$ to $j : V[G] \to M[G][H][H']$, construct in $V[G][H]$ an $M[G][H][H']$-generic object H'' containing a master condition for $j''H$, and then fully lift j to $j : V[G][H] \to M[G][H][H'][H'']$. This shows that $V[G][H] \models$ "κ is λ supercompact". Since $\lambda \geq \kappa^{+17}$ was an arbitrary regular cardinal, this completes the proof of Lemma 2.5. ∎

LEMMA 2.6. *$V^{\mathbb{P}} \models$ "For every $\delta \leq \kappa$ which is a nontrivial stage of forcing, δ is δ^{+16} supercompact if δ is a measurable cardinal".*

Proof. Let $\delta \leq \kappa$ be a nontrivial stage of forcing. By the definition of $\mathbb{P}$, it must be the case that in V, δ is either a strong cardinal or a regular limit of strong cardinals.

We follow now the proof of [6, Lemma 3.2]. As we have already observed, any $\delta \leq \kappa$ which is in $V^{\mathbb{P}}$ either a strong cardinal or an inaccessible limit of strong cardinals had to have been in V either a strong cardinal or an inaccessible limit of strong cardinals. Further, suppose $\delta < \kappa$ is such that $V \models$ "δ is a strong cardinal which is not a limit of strong cardinals". Since $V \models \mathrm{GCH}$, [7, Lemma 2.1] tells us that $V \models$ "δ is not $2^{\delta} = \delta^{+}$ supercompact", since otherwise it would have to be the case that $V \models$ "δ is a limit of strong cardinals". Consequently, by the definition of $\mathbb{P}$, $V^{\mathbb{P}} \models$ "δ is not a measurable cardinal". Further, by the definition of $\mathbb{P}$, if $V \models$ "δ is a regular limit of strong cardinals which is not δ^{+16} supercompact", then $V^{\mathbb{P}} \models$ "δ is not a measurable cardinal". We may therefore infer that any cardinal $\delta < \kappa$ which is a strong cardinal in $V^{\mathbb{P}}$ had to have been in V a strong cardinal which is both (at least) δ^{+16} supercompact and a limit of strong cardinals. These last two sentences now tell us that the argument of [5, Lemma 4.2] goes through unchanged to show that $V^{\mathbb{P}} \models$ "For every $\delta \leq \kappa$ which is a nontrivial stage of forcing, δ is δ^{+16} supercompact if δ is a measurable cardinal". This completes the proof of Lemma 2.6. ■

Lemma 2.7. $V^{\mathbb{P}} \models$ *"Level by level equivalence between strong compactness and supercompactness holds".*

Proof. We follow the proof of [6, Lemma 3.3]. Suppose $V^{\mathbb{P}} \models$ "$\delta < \lambda$ are regular cardinals such that δ is λ strongly compact and δ is not a measurable limit of cardinals γ which are λ supercompact". Assume first $\delta > \kappa$. Since $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa^{+17}$, by the results of [18], Lemma 2.7 is true if $\delta > \kappa$. By Lemma 2.5, Lemma 2.7 is true if $\delta = \kappa$. It therefore suffices to prove Lemma 2.7 when $\delta < \kappa$, which we assume for the duration of the proof.

Let $A = \{\gamma \leq \delta \mid \gamma$ is a strong cardinal or an inaccessible limit of strong cardinals$\}$. Write $\mathbb{P} = \mathbb{P}_A * \dot{\mathbb{Q}}$, where $\mathbb{P}_A$ is the portion of $\mathbb{P}$ which does nontrivial forcing at ordinals at most δ, and $\dot{\mathbb{Q}}$ is a term for the rest of $\mathbb{P}$, i.e., the portion of $\mathbb{P}$ doing nontrivial forcing at ordinals above δ. We claim that since $\delta < \kappa$, it follows that λ is below the least V-strong cardinal ρ above δ. This is because otherwise $V \models$ "δ is strongly compact up to ρ and ρ is a strong cardinal", so by [4, Lemma 1.1], $V \models$ "δ is strongly compact", a contradiction to the fact that $\delta < \kappa$ and κ is both the least V-supercompact and least V-strongly compact cardinal. As $\Vdash_{\mathbb{P}_A}$ "$\dot{\mathbb{Q}}$ is ρ-strategically closed and ρ is inaccessible", $V^{\mathbb{P}_A} \models$ "δ is λ strongly compact and δ is not a measurable limit of cardinals γ which are λ supercompact". Further, to show that $V^{\mathbb{P}} \models$ "δ is λ supercompact", it hence suffices to show that $V^{\mathbb{P}_A} \models$ "δ is λ supercompact".

Consider now the following two cases.

Case 1: $\sup(A) = \sigma < \delta$. If this is true, then by the definition of $\mathbb{P}$, $|\mathbb{P}_A| < \delta$. Thus, by the results of [18], $V \models$ "δ is λ strongly compact and δ is not a measurable limit of cardinals γ which are λ supercompact". Hence, by the level by level equivalence between strong compactness and supercompactness in V, $V \models$ "δ is λ supercompact", so again by the results of [18], $V^{\mathbb{P}_A} \models$ "δ is λ supercompact" as well.

Case 2: $\sup(A) = \delta$. As before, we can write $\mathbb{P}_A = \mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| = \omega$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is $\aleph_1$-strategically closed". Further, it is easily seen that any subset of δ in $V^{\mathbb{P}_A}$ of size below δ has a name of size below δ in V. Therefore, as in Lemma 2.3, by the results of [11, 12], $V \vDash$ "δ is λ strongly compact". In addition, as in Case 1 above, it is the case that $V \vDash$ "δ is not a measurable limit of cardinals γ which are λ supercompact". This is since otherwise, as $V \vDash$ "δ is a limit of strong cardinals", some cardinal $\gamma < \delta < \kappa$ must be supercompact up to a strong cardinal in V. As we have already observed, γ is then supercompact in V, which contradicts that $V \vDash$ "κ is the least supercompact cardinal". Thus, by the level by level equivalence between strong compactness and supercompactness in V, $V \vDash$ "δ is λ supercompact".

Note that since δ is in V a regular limit of strong cardinals, δ is a nontrivial stage of forcing. It must therefore be the case that $V \vDash$ "δ is δ^{+16} supercompact", because otherwise, by the definition of $\mathbb{P}$, $V^{\mathbb{P}_A} \vDash$ "δ is not measurable". Since by Lemma 2.6, $V^{\mathbb{P}} \vDash$ "δ is δ^{+16} supercompact", which means by the definition of $\mathbb{P}$ that $V^{\mathbb{P}_A} \vDash$ "δ is δ^{+16} supercompact" as well, we may assume without loss of generality that $\lambda \geq \delta^{+17}$. Consequently, let $j : V \to M$ be an elementary embedding witnessing the λ supercompactness of δ generated by a supercompact ultrafilter over $P_\delta(\lambda)$ such that $j(F)(\delta) = F(\delta) = \delta^{+17}$. Since $\lambda \geq \delta^{+17}$, $\mathbb{P}_A$ is forcing equivalent to $\mathbb{P}_\delta * \dot{\mathbb{Q}}^*$, where $\Vdash_{\mathbb{P}_\delta}$ "$|\dot{\mathbb{Q}}^*| = \delta^{+17} \leq \lambda$ and $\dot{\mathbb{Q}}^*$ is δ-directed closed". ($\mathbb{Q}^*$ is forcing equivalent to $\mathrm{Add}(\delta^{+17}, 1) * \dot{\mathrm{Add}}(\delta, \delta^{+17})$.) In addition, the same reasoning as found in the proof of Lemma 2.5 shows that $M \vDash$ "No cardinal in the half-open interval $(\delta, \lambda]$ is strong". Thus, $j(\mathbb{P}_\delta * \dot{\mathbb{Q}}^*)$ is forcing equivalent to $\mathbb{P}_\delta * \dot{\mathbb{Q}}^* * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}}^*)$, where the first ordinal at which $\dot{\mathbb{R}}$ is forced to do nontrivial forcing is well above λ. The same argument as given in the proof of Lemma 2.5 then shows that $V^{\mathbb{P}_A} \vDash$ "δ is λ supercompact".

Cases 1 and 2 now complete the proof of Lemma 2.7. ■

Since $\mathbb{P}$ may be defined so that $|\mathbb{P}| = \kappa^{+17}$, the proof of Lemma 2.4 shows that $V^{\mathbb{P}} \vDash$ "$\mathcal{K}$ is the class of supercompact cardinals". Lemmas 2.5–2.7 and the remarks made prior to the proof of Lemma 2.5 therefore complete the proof of Theorem 3. ■

As in [1], we ask what other types of Easton theorems are consistent with the level by level equivalence between strong compactness and supercompactness. In particular, is it possible to include regular cardinals in the domain of the Easton function F which are above the least supercompact cardinal κ? This is precluded by the restrictions on A given in Theorem 2. Is it possible for F's domain to be all regular cardinals, as in Easton's original result? Which cardinals may be included in F's range? Finally, as the referee has asked, are there any GCH patterns which are incompatible with the level by level equivalence between strong compactness and supercompactness?

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