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T-RICKART MODULES

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Abstract. We introduce the notions of T-Rickart and strongly T-Rickart modules. We provide several characterizations and investigate properties of each of these concepts. It is shown that R is right Σ-t-extending if and only if every R-module is T-Rickart. Also, every free R-module is T-Rickart if and only if $R = Z_2(R_R) \oplus R'$, where R' is a hereditary right R-module. Examples illustrating the results are presented.

1. Introduction. The notions of Rickart, Baer and quasi-Baer rings have their roots in functional analysis, with close links to C^*-algebras and von Neumann algebras. In [8], Kaplansky defined abstract W^*-algebras, or AW^*-algebras (C^*-algebras in which the right annihilator of any subset is generated by a projection). Alternatively, AW^*-algebras are C^*-algebras with the Baer property. The Baer property for rings was first considered by Kaplansky [9, 10]. He introduced Baer rings to describe abstract various properties of von Neumann algebras and complete *-regular rings. A number of interesting properties of Baer rings were shown by Kaplansky and further investigated by several other mathematicians. In [6], the notion of quasi-Baer rings was introduced by Clark and used to characterize the case where a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. A ring R is called *Baer* (resp. *quasi-Baer*) if the right annihilator of a left ideal (resp. two-sided ideal) is generated as a right ideal by an idempotent. Baer and quasi-Baer property are left-right symmetric for every ring.

Motivated by Kaplansky's work on Baer rings, the notion of Rickart rings appeared in Maeda [15] and was further studied by Hattori [11], Berberian [2] and other authors. A ring R is said to be *right Rickart* if the right annihilator of any single element of R is generated by an idempotent as a right ideal (equivalently, every principal right ideal of R is projective, i.e. R is a right p.p. ring). Left Rickart rings are defined similarly. The notion of Rickart ring is not left-right symmetric.

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Recently, the notions of Baer, quasi-Baer and Rickart rings were extended and studied in a general module-theoretic setting by Rizvi, Roman and Lee [16], [17], [13], [14].

An R-module M is called *extending* if each submodule is essential in a direct summand of M. In [1], Asgari and Haghany introduced the concept of t-extending and t-Baer modules by using second singular submodules. Motivated by the definition of t-Baer modules and Rickart modules, we define the notion of T-Rickart ring and investigate related results.

In Section 3, we show that a direct summand of a T-Rickart module is T-Rickart. We provide some equivalent conditions for a module M to be T-Rickart. We introduce the notion of relative T-Rickart rings to show that the class of rings R for which every R-module is T-Rickart is precisely the right Σ-t-extending rings. It is also shown that every free R-module is T-Rickart if and only if $R = Z_2(R_R) \oplus R'$ where R' is a hereditary right R-module.

In Section 4, the notion of strongly T-Rickart module is defined and several characterizations of such modules are given. We show that each direct summand of a strongly T-Rickart module is strongly T-Rickart, and give necessary and sufficient conditions for the direct sum of two strongly T-Rickart modules to be strongly T-Rickart.

2. Preliminaries. Throughout, all rings (not necessarily commutative) have identities and all modules are unital right modules. For completeness, we state some definitions and notation used throughout this paper. Let M be a module over a ring R. For submodules N and K of M, $N \leq K$ denotes that N is a submodule of K, and $S = \mathrm{End}(M)$ denotes the ring of right R-module endomorphisms of M. We denote by $\mathrm{r}_M(\cdot)$ the right annihilator of a subset of $\mathrm{End}(M)$ with elements from M. We let $\leq^{\oplus}$, $\leq^{\mathrm{ess}}$ and $E(M)$ denote, respectively, a module direct summand, an essential submodule and the injective hull of M. By $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$ we denote the ring of integers, the ring of residues modulo n and the ring of rational numbers, respectively. We also define

$$\mathrm{t}_M(I) = \{m \in M \mid Im \leq Z_2(M)\} \quad \text{for } \emptyset \neq I \subseteq S = \mathrm{End}(M).$$

Recall that the *singular submodule* $Z(M)$ of a module M is the set of $m \in M$ with $\mathrm{r}_R(m) \leq^{\mathrm{ess}} R_R$, or equivalently, $mI = 0$ for some essential right ideal I of R. The *second singular* (or *Goldie torsion*) *submodule* $Z_2(M)$ is the submodule of M which is defined by

$$Z(M/Z(M)) = Z_2(M)/Z(M).$$

If N is a submodule of M, then $Z(N) = Z(M) \cap N$ and so $Z_2(N) = Z_2(M) \cap N$. A module M is called *singular* if $Z(M) = M$ and *nonsingular* if $Z(M) = 0$. A module M is called *Z_2-torsion* if $Z_2(M) = M$. If M_i are *R-modules* ($i \in I$), then $Z(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} Z(M_i)$ and so $Z_2(\bigoplus_{i\in I} M_i) = \bigoplus_{i\in I} Z_2(M_i)$. Let $f : M \to N$ be an R-module homomorphism. Clearly, $f(Z(M)) \leq Z(N)$ and so $f(Z_2(M)) \leq Z_2(N)$.

Definition 2.1.

(a) A submodule N of M is called *t-essential* in M, written $N \leq^{\text{tess}} M$, if for every submodule N' of M, $N \cap N' \leq Z_2(M)$ implies that $N' \leq Z_2(M)$ (see [1]).
(b) A submodule C of M is called *t-closed* if C has no t-essential extension in M (see [1]).
(c) A module M is called *t-extending* if every t-closed submodule of M is a direct summand of M (see [1]).
(d) An R-module M is said to be *Baer* (resp. *Rickart*) if for any left ideal I of $\text{End}(M)$ (resp. $\phi \in \text{End}(M)$), $\text{r}_M(I)$ (resp. $\text{r}_M(\phi)$) is a direct summand of M (see [14], [16]).
(e) An R-module M is called *strongly Rickart* if $\text{r}_M(\phi)$ is a fully invariant direct summand of M for each $\phi \in \text{End}(M)$ (equivalently, M is Rickart and each idempotent of the endomorphism ring of M is central) (see [7]).
(f) An R-module M is said to be *t-Baer* if $\text{t}_M(I)$ is a direct summand of M for each left ideal I of S (see [1]).
(g) A ring R is *right Σ-t-extending* if every free R-module is t-extending (see [1]).
(h) An idempotent $e \in R$ is called *left semicentral* if $re = ere$ for each $r \in R$. Equivalently, eR is an ideal of R. The set of left semicentral idempotents of R will be denoted by $S_l(R)$. It is known that eM (where $e^2 = e \in \text{End}(M)$) is a fully invariant direct summand of module M if and only if $e \in S_l(\text{End}(M))$ (see [5], [3]).
(i) An R-module M is said to have *SIP* (summand intersection property) if the intersection of any two direct summands is a direct summand of M; and M has *SSIP* (*strong summand intersection property*) if the intersection of any family of direct summands is a direct summand of M (see [14]).

We need the following propositions, proved in [1, Proposition 2.2, Proposition 2.6 and Theorem 3.12], respectively.

Proposition 2.2.

(a) *The following statements are equivalent for a submodule N of M:*
   (i) $N \leq^{\text{tess}} M$*;*
   (ii) $N + Z_2(M) \leq^{\text{ess}} M$*;*
   (iii) M/N *is Z_2-torsion.*

(b) *Let C be a submodule of M. The following statements are equivalent:*

(i) *C is t-closed in M;*
(ii) *C contains $Z_2(M)$ and C is closed in M;*
(iii) *M/C is nonsingular.*

(c) *The following statements are equivalent for a ring R:*

(i) *R is right Σ-t-extending;*
(ii) *every R-module is t-Baer;*
(iii) *every R-module is t-extending.*

3. T-Rickart modules. Motivated by the definitions of Rickart modules and t-Baer modules, we introduce the key definition of this paper.

DEFINITION 3.1. A module M is called *T-Rickart* if $\mathrm{t}_M(\phi)$ is a direct summand of M for every $\phi \in \mathrm{End}(M)$.

Clearly, Z_2-torsion modules and t-Baer modules are T-Rickart. One can easily show that the notions of Rickart module and T-Rickart module coincide for every nonsingular module. In particular, every Rickart ring is a T-Rickart ring. In the next proposition, for a module M, equivalent conditions for $\mathrm{t}_M(\phi)$, with $\phi \in \mathrm{End}(M)$, to be a t-essential submodule in M are given.

PROPOSITION 3.2. *Let M be a module and $\phi \in S = \mathrm{End}(M)$. The following are equivalent:*

(1) $\mathrm{t}_M(\phi) \leq^{\mathrm{tess}} M$;
(2) $\mathrm{t}_M(\phi) = M$;
(3) $\mathrm{Ker}(\phi) \leq^{\mathrm{tess}} M$.

Proof. (1)⇒(2). Let $\mathrm{t}_M(\phi) \leq^{\mathrm{tess}} M$. Since $Z_2(M) \subseteq \mathrm{t}_M(\phi)$, by Proposition 2.2(a) we have $\mathrm{t}_M(\phi) \leq^{\mathrm{ess}} M$. If $x \in \mathrm{Im}(\phi)$, then there exists $m \in M$ such that $\phi(m) = x$. Since $\mathrm{t}_M(\phi) \leq^{\mathrm{ess}} M$, it follows that $mI \subseteq \mathrm{t}_M(\phi)$ for some $I \leq^{\mathrm{ess}} R_R$. Hence $xI = \phi(mI) \subseteq Z_2(M)$ and this implies that $x + Z_2(M) \in Z(M/Z_2(M)) = 0$; so $x \in Z_2(M)$. Therefore $\mathrm{Im}(\phi) \subseteq Z_2(M)$, and so $\mathrm{t}_M(\phi) = M$.

(2)⇒(3). If $\mathrm{t}_M(\phi) = M$, then $\phi(M) \subseteq Z_2(M)$. Thus $\phi(M)$ is Z_2-torsion, and so $M/\mathrm{Ker}(\phi) \cong \phi(M)$ is Z_2-torsion. By using Proposition 2.2(a), we obtain $\mathrm{Ker}(\phi) \leq^{\mathrm{tess}} M$.

(3)⇒(1) is clear. ∎

THEOREM 3.3. *Let M be a T-Rickart module. Then every direct summand of M is T-Rickart.*

Proof. Let N be a direct summand of M. Suppose that $M = N \oplus N'$ for some submodule N' of M. If $\phi \in \mathrm{End}(N)$, then $\phi \oplus 1_{\mathrm{End}(N')} \in \mathrm{End}(M)$. Since

M is T-Rickart, $\mathrm{t}_M(\phi \oplus 1_{\mathrm{End}(N')})$ is a direct summand of M. An inspection shows that $\mathrm{t}_M(\phi \oplus 1_{\mathrm{End}(N')}) = \mathrm{t}_N(\phi) \oplus Z_2(N')$. Let

$$M = \mathrm{t}_M(\phi \oplus 1_{\mathrm{End}(N')}) \oplus K = \mathrm{t}_N(\phi) \oplus Z_2(N') \oplus K$$

for some $K \leq M$. Then by the modular law, $\mathrm{t}_N(\phi)$ is a direct summand of N. ■

We next give four characterizations of T-Rickart modules.

THEOREM 3.4. *Let M be a module. Then the following are equivalent:*

(1) *M is T-Rickart;*
(2) *$M = Z_2(M) \oplus K$, where K is a Rickart module;*
(3) *$\phi^{-1}(Z_2(M))$ is a direct summand of M for all $\phi \in S$;*
(4) *for each $\phi \in S$, there exists $N \leq^{\oplus} M$ such that $\mathrm{t}_M(\phi) \leq^{\mathrm{tess}} N$;*
(5) *for each $\phi \in S$, there exists $N \leq^{\oplus} M$ such that $\mathrm{t}_M(\phi) \leq^{\mathrm{ess}} N$.*

Proof. (1)⇒(2). Clearly, $\mathrm{t}_M(1_S) = Z_2(M)$. Since M is T-Rickart, $\mathrm{t}_M(1_S) = Z_2(M)$ is a direct summand of M; thus $M = Z_2(M) \oplus K$ for some submodule K of M. By Theorem 3.3, K is T-Rickart. Since K is nonsingular, it is Rickart.

(2)⇒(1). Assume that $M = Z_2(M) \oplus K$, where K is a Rickart module. Since K is a direct summand of M, we have $K = eM$ for some $e^2 = e \in S$. Let $\phi \in S$. We claim that

$$\mathrm{t}_M(\phi) = Z_2(M) \oplus \mathrm{r}_K(e\phi e).$$

Indeed, let $m = m_1 + m_2 \in \mathrm{t}_M(\phi)$, where $m_1 \in Z_2(M)$ and $m_2 \in K$. Then $\phi(m) = \phi(m_1) + \phi(m_2) \in Z_2(M)$. As $m_1 \in Z_2(M)$, we have $\phi(m_1) \in Z_2(M)$. Hence $\phi(m_2) = \phi(m) - \phi(m_1) \in Z_2(M)$. Thus $0 = e\phi(m_2) = e\phi e(m_2)$, and so $m_2 \in \mathrm{r}_K(e\phi e)$. Therefore $\mathrm{t}_M(\phi) \subseteq Z_2(M) \oplus \mathrm{r}_K(e\phi e)$. For the reverse inclusion, let $m = m_1 + m_2 \in Z_2(M) \oplus \mathrm{r}_K(e\phi e)$, where $m_1 \in Z_2(M)$ and $m_2 \in K$. Since $m_2 \in K$, we have $em_2 = m_2$. Also $\phi(m_1) \in Z_2(M)$ because $m_1 \in Z_2(M)$, and so $e\phi(m_1) = 0$. Hence $e\phi(m) = e\phi(m_1) + e\phi e(m_2) = 0$. Thus $\phi(m) \in \mathrm{Ker}(e) = Z_2(M)$, proving the claim.

As K is Rickart and $e\phi e \in \mathrm{End}(K)$, $\mathrm{r}_K(e\phi e)$ is a direct summand of K; so $\mathrm{t}_M(\phi)$ is a direct summand of M and hence M is T-Rickart.

(1)⇔(3) is clear from $\mathrm{t}_M(\phi) = \phi^{-1}(Z_2(M))$.

(1)⇒(4) is clear.

(4)⇒(5). Let $\mathrm{t}_M(\phi) \leq^{\mathrm{tess}} N$ for some $N \leq^{\oplus} M$. Since $Z_2(M) \subseteq \mathrm{t}_M(\phi)$, Proposition 2.2(a) implies that $\mathrm{t}_M(\phi) \leq^{\mathrm{ess}} N$.

(5)⇒(1) is similar to the proof of Proposition 3.2. ■

The next example shows that the class of T-Rickart modules properly contains the class of t-Baer modules.

EXAMPLE 3.5. (1) Let R be a ring and M be a nonsingular Rickart module which is not a Baer module (see [14, Examples 2.18 and 2.19]) and N be another R-module. Then by Theorem 3.4, $M \oplus Z_2(N)$ is a T-Rickart module which is not t-Baer.

(2) Consider $\mathbb{Z}$ and $\mathbb{Z}_2$ as $\mathbb{Z}$-modules. By [14, Example 2.5], $\mathbb{Z} \oplus \mathbb{Z}_2$ is not a Rickart $\mathbb{Z}$-module; however, it is T-Rickart by Theorem 3.4.

The following example shows that the direct sum of two T-Rickart modules need not be T-Rickart.

EXAMPLE 3.6. [14, Example 2.9] Let

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \quad \text{and} \quad M = R_R.$$

Then

$$M = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix}.$$

Since

$$M_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$$

are nonsingular and Rickart, M_1 and M_2 are T-Rickart. But it can be seen that M_R is not Rickart. Indeed, consider $\left(\begin{smallmatrix} 2 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{End}(M) \cong R$. Then

$$\mathrm{r}_M\left(\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -1 \\ 0 & 2 \end{pmatrix}\mathbb{Z},$$

which is not a direct summand of M. Since M is nonsingular, M is not T-Rickart.

The following reformulated proposition characterizes t-Baer modules in terms of SSIP and T-Rickart modules.

PROPOSITION 3.7. *An R-module M is t-Baer if and only if M is a T-Rickart module and M has the strong summand intersection property for direct summands which contain $Z_2(M)$.*

Proof. See [1, Theorem 3.2]. ∎

In the following proposition, we prove that the notions of T-Rickart module and t-Baer module coincide for the modules whose endomorphism ring has no infinite set of nonzero orthogonal idempotents (cf. [12, Theorem 4.5]).

PROPOSITION 3.8. *Let M be a module, and suppose $S = \mathrm{End}(M)$ has no infinite set of nonzero orthogonal idempotents. Then M is a T-Rickart module if and only if M is a t-Baer module.*

Proof. If M is a T-Rickart module, then by Theorem 3.4, $M = Z_2(M) \oplus M'$ for some Rickart module M'. Since M' is nonsingular, we have $\mathrm{Hom}(Z_2(M), M') = 0$. Hence

$$\mathrm{End}(M) = \begin{pmatrix} \mathrm{End}(Z_2(M)) & \mathrm{Hom}(M', Z_2(M)) \\ 0 & \mathrm{End}(M') \end{pmatrix}.$$

Since S has no infinite set of nonzero orthogonal idempotents, $\mathrm{End}(M')$ has no infinite set of nonzero orthogonal idempotents, so by [14, Theorem 4.5], M' is Baer. Hence M is t-Baer by [1, Theorem 3.2]. ∎

The following proposition gives a relation between Rickart and T-Rickart modules.

Proposition 3.9. *Let M be a module. Then M is Rickart such that $Z_2(M)$ is a direct summand of M if and only if M is a T-Rickart module such that $\mathrm{r}_M(\phi)$ is a direct summand of $\mathrm{t}_M(\phi)$ for all $\phi \in S$.*

Proof. Let M be a Rickart module such that $M = Z_2(M) \oplus K$ for some $K \le M$. Since each direct summand of a Rickart module is Rickart (see [14, Theorem 2.7]), K is Rickart. Hence Theorem 3.4 shows that M is a T-Rickart module. Since M is a Rickart module, for each $\phi \in S$, $\mathrm{r}_M(\phi)$ is a direct summand of M. As $\mathrm{r}_M(\phi) \le \mathrm{t}_M(\phi)$, the modular law shows that $\mathrm{r}_M(\phi)$ is a direct summand of $\mathrm{t}_M(\phi)$.

Conversely, suppose M is a T-Rickart module such that $\mathrm{r}_M(\phi)$ is a direct summand of $\mathrm{t}_M(\phi)$ for each $\phi \in S$. Then, first, $Z_2(M)$ is a direct summand of M by Theorem 3.4. Next, as M is T-Rickart, $\mathrm{t}_M(\phi)$ is a direct summand of M. Hence $\mathrm{r}_M(\phi)$ is a direct summand of M, as desired. ∎

Definition 3.10. An R-module M is called *T-Rickart relative to N* (or *N-T-Rickart*) if $\mathrm{t}_M(\phi) \le^{\oplus} M$ for every homomorphism $\phi : M \to N$, where $\mathrm{t}_M(\phi) = \{m \in M \mid \phi(m) \in Z_2(N)\}$.

In view of the above definition, a right R-module M is T-Rickart if and only if M is T-Rickart relative to M. Clearly, If N or M is Z_2-torsion, then M is T-Rickart relative to N. Similarly to [14, Proposition 2.24], we have the following proposition that will be used to prove our main theorems.

Proposition 3.11. *Let M be an R-module. The following are equivalent:*

(1) *M is T-Rickart;*
(2) *every direct summand L of M is T-Rickart relative to N for each submodule N of M;*
(3) *if L and N are direct summands of M, then for each $\phi \in \mathrm{Hom}(M, N)$, $\mathrm{t}_L(\phi|_L)$ is a direct summand of L.*

Proof. (1)⇒(2). Let N be a submodule of M and L be a direct summand of M, where $L = eM$ for some $e^2 = e \in S$. Let $\psi : L \to N$ be a homomorphism. Since $\psi e \in S$, $\mathrm{t}_M(\psi e)$ is a direct summand of M. We assert that

$$\mathrm{t}_L(\psi) = \mathrm{t}_M(\psi e) \cap eM.$$

If $m \in \mathrm{t}_L(\psi)$, then $\psi(m) \in Z_2(N) \subseteq Z_2(M)$, and so $\psi(m) \in Z_2(M)$. Since $m \in L = eM$, $m = em$. Hence $\psi(m) = \psi(em) \in Z_2(M)$. Therefore $m \in \mathrm{t}_M(\psi e) \cap eM$. For the reverse inclusion, let $m \in \mathrm{t}_M(\psi e) \cap eM$. Then $m \in eM$ and $\psi e(m) = \psi(m) \in Z_2(M) \cap N = Z_2(M)$. Hence $m \in \mathrm{t}_L(\psi)$, proving the assertion.

Since M is T-Rickart, $\mathrm{t}_M(\psi e) = e'M$ for some $e'^2 = e' \in S$. We will show that $\mathrm{t}_L(\psi) = e'M \cap eM$ is a direct summand of L. Since $Z_2(M) \subseteq \mathrm{t}_M(\psi e) = e'M$, we have $\mathrm{t}_M((1 - e')) = e'M$. As M is T-Rickart, $\mathrm{t}_M((1 - e')e)$ is a direct summand of M. We claim that

$$\mathrm{t}_M((1 - e')e) = eM \cap e'M \oplus (1 - e)M.$$

If $m \in \mathrm{t}_M((1 - e')e)$, then $((1 - e')e)(m) \in Z_2(M)$. Consequently, $m = em + (1 - e)m \in eM \cap e'M \oplus (1 - e)M$ because $em \in \mathrm{t}_M((1 - e')) = e'M$. Hence $\mathrm{t}_M((1 - e')e) \subseteq eM \cap e'M \oplus (1 - e)M$. The other inclusion is clear. Since $\mathrm{t}_M((1 - e')e)$ is direct summand of M, the modular law shows that $eM \cap e'M$ is a direct summand of L.

(2)⇒(3). The statement is clear by taking N a direct summand of M.

(3)⇒(1). Take $L = N = M$. ■

In view of Proposition 3.7, it can be seen that t-Baer modules have SSIP for direct summands which contain the second singular submodule. In the following proposition we prove that T-Rickart modules have SIP for direct summands that contain the second singular submodule.

Proposition 3.12. *Let M be a T-Rickart module.*

(1) *If L and N are direct summands of M with $Z_2(M) \subseteq L$, then $L \cap N$ is a direct summand of M.*
(2) *M has SIP for direct summands that contain $Z_2(M)$.*

Proof. (1) Let $L = eM$ and $N = e'M$, where e and e' are idempotent elements of S. Consider the projection $1 - e : M \to (1 - e)M$. By Proposition 3.11, $\mathrm{t}_N((1-e)|_N) = \mathrm{t}_N((1-e)e')$ is a direct summand of N. It can be seen that $\mathrm{t}_N((1 - e)e') = \mathrm{t}_M((1 - e)) \cap e'M$. Since $Z_2(M) \subseteq L = eM$, we have $\mathrm{t}_M(1 - e) = eM$. As $\mathrm{t}_N((1 - e)e') \leq^{\oplus} N$ and $N \leq^{\oplus} M$, $\mathrm{t}_N((1 - e)e')$ is a direct summand of M, as desired.

(2) Apply (1). ■

The next theorem gives a condition equivalent to being T-Rickart in terms of $\mathrm{t}_M(I)$, where I is a finitely generated left ideal of $S = \mathrm{End}(M)$.

Theorem 3.13. *An R-module M is T-Rickart if and only if* $\mathrm{t}_M(I) \leq^{\oplus} M$ *for every finitely generated left ideal I of S.*

Proof. Let M be a T-Rickart module and $I = S\phi_1 + \cdots + S\phi_n$ ($n \in \mathbb{N}$) be a finitely generated left ideal of S, where $\phi_i \in S$. An inspection shows that $\mathrm{t}_M(I) = \bigcap_{i=1}^{n} \mathrm{t}_M(\phi_i)$. Since M is T-Rickart, $\mathrm{t}_M(\phi_i) \leq^{\oplus} M$ for each $1 \leq i \leq n$. As $Z_2(M) \subseteq \mathrm{t}_M(\phi_i)$ for each $1 \leq i \leq n$, and M has SIP for direct summands which contain $Z_2(M)$ by Proposition 3.12(2), it follows that $\mathrm{t}_M(I) = \bigcap_{i=1}^{n} \mathrm{t}_M(\phi_i)$ is a direct summand of M. The converse implication is clear since $\mathrm{t}_M(S\phi) = \mathrm{t}_M(\phi) \leq^{\oplus} M$ for each $\phi \in S$. ∎

Now, we characterize right Σ-t-extending rings in terms of T-Rickart modules. Note the contrast with [14, Theorem 2.25] which shows that the rings R for which every R-module is Rickart are exactly the semisimple rings.

Theorem 3.14. *The following are equivalent for a ring R:*

(1) *every R-module is t-Baer;*
(2) *every R-module is T-Rickart;*
(3) *every R-module is t-extending;*
(4) *R is Σ-t-extending.*

Proof. (1)⇒(2) is clear.

(2)⇒(3). Let M be a T-Rickart R-module; we will show that M is t-extending. Let C be a t-closed submodule of M. Consider the R-module $M \oplus (M/C)$. Since each R-module is T-Rickart by (2), $M \oplus (M/C)$ is a T-Rickart R-module. By Proposition 3.11(2), M is (M/C)-T-Rickart. If $\pi : M \to M/C$ is the canonical epimorphism, then $\mathrm{t}_M(\pi) = \{m \in M \mid \pi(m) \in Z_2(M/C)\}$ is a direct summand of M. Since C is a t-closed submodule in M, by Proposition 2.2(b), M/C is nonsingular and so $Z_2(M/C) = 0$. Therefore $\mathrm{t}_M(\pi) = \mathrm{Ker}(\pi) = C$. Thus C is a direct summand of M and so M is t-extending.

(3)⇒(4)⇒(1) follows from Proposition 2.2(c). ∎

In the next theorem, we characterize the rings R for which every free R-module is T-Rickart.

Theorem 3.15. *Let R be a ring. The following are equivalent;*

(1) *every free R-module is T-Rickart;*
(2) *every projective R-module is T-Rickart;*
(3) *$R = Z_2(R) \oplus R'$, where R' is a hereditary R-module.*

Proof. (1)⇒(2). Let M be a projective R-module. Thus $M \leq^{\oplus} F$ for some free R-module F. By (1), F is T-Rickart, and Theorem 3.3 implies that M is T-Rickart.

(2)⇒(3). Since R_R is T-Rickart, we have $Z_2(R) \leq^{\oplus} R$ by Theorem 3.4. Let $R = Z_2(R) \oplus R'$. We will show that R' is hereditary. Let $I \leq R'$. There exists a free R-module F such that I is a homomorphic image of F, say under $\phi : F \to I$. Then we can take ϕ as an endomorphism of F. As R' is nonsingular, I is nonsingular. We claim that $\mathrm{Ker}(\phi) = \mathrm{t}_F(\phi)$. Indeed, if $m \in \mathrm{t}_F(\phi)$, then $\phi(m) \in Z_2(F) \cap I = 0$; hence $\phi(m) = 0$. Since every projective R-module is T-Rickart, F is T-Rickart, and so $\mathrm{Ker}(\phi)$ is a direct summand of F. Thus $I = \mathrm{Im}(\phi)$ is projective and hence R' is hereditary.

(3)⇒(1). Let $F = R^{(\Lambda)}$ be a free R-module and ϕ be an endomorphism of F. By (3), we have $F = Z_2(R)^{(\Lambda)} \oplus R'^{(\Lambda)}$. Set $F' = R'^{(\Lambda)}$. It is clear that $Z_2(F) = Z_2(R)^{(\Lambda)}$. Thus $F = Z_2(F) \oplus F'$. Since $Z_2(F) \subseteq \mathrm{t}_F(\phi)$, we have $\mathrm{t}_F(\phi) = Z_2(F) \oplus F' \cap \mathrm{t}_F(\phi)$. Let $F' = eF$ where $e^2 = e \in \mathrm{End}(F)$. Clearly $e\phi e \in \mathrm{End}(F')$ and $\mathrm{Ker}(e\phi e) = F' \cap \mathrm{t}_F(\phi)$. Since R' is hereditary, $R'^{(\Lambda)}$ is hereditary. So $F'/\mathrm{Ker}(e\phi e) \cong \mathrm{Im}(e\phi e) \leq F'$ is projective. Thus $F' \cap \mathrm{t}_F(\phi)$ is a direct summand of F'. Therefore $\mathrm{t}_F(\phi)$ is a direct summand of F, and hence F' is T-Rickart. ∎

4. Strongly T-Rickart modules. In this section we introduce the notion of strongly T-Rickart R-modules. Also, we collect some basic properties of such modules.

DEFINITION 4.1. An R-module M is called *strongly T-Rickart* if $\mathrm{t}_M(\phi)$ is a fully invariant direct summand of M for each $\phi \in \mathrm{End}(M)$.

It is clear that each Z_2-torsion module is strongly T-Rickart, and the notion of strongly T-Rickart and strongly Rickart are equivalent for nonsingular modules.

THEOREM 4.2. *The following statements are equivalent for an R-module M:*

(1) *M is strongly T-Rickart;*
(2) *M is T-Rickart and each direct summand of M which contains $Z_2(M)$ is fully invariant;*
(3) *$M = Z_2(M) \oplus M'$ where M' is strongly Rickart;*
(4) *$M = Z_2(M) \oplus M'$ and for each $\phi \in \mathrm{End}(M)$, $\mathrm{t}_M(\phi) \cap M'$ is a fully invariant direct summand of M';*
(5) *for each $\phi \in \mathrm{End}(M)$, $\phi^{-1}(Z_2(M))$ is a fully invariant direct summand of M'.*

Proof. (1)⇒(2). Let M be a strongly T-Rickart. It is clear that M is T-Rickart. Let N be a direct summand of M which contains $Z_2(M)$, hence there exists $e^2 = e \in \mathrm{End}(M)$ such that $N = eM$. Since $Z_2(M) \subseteq eM$, we have $\mathrm{t}_M((1 - e)) = eM$, and so N is fully invariant.

(2)⇒(1). Let $\phi \in \mathrm{End}(M)$. Since M is T-Rickart, we obtain $\mathrm{t}_M(\phi) \leq^{\oplus} M$. As $Z_2(M) \subseteq \mathrm{t}_M(\phi)$, $\mathrm{t}_M(\phi)$ is fully invariant direct summand. Hence M is strongly T-Rickart.

(1)⇒(3). Since M is strongly T-Rickart, $\mathrm{t}_M(1_S) = Z_2(M)$ is a direct summand of M. Let $M = Z_2(M) \oplus M'$. We show that M' is strongly Rickart. If $\phi \in \mathrm{End}(M')$, then $1_{Z_2(M)} \oplus \phi \in \mathrm{End}(M)$. Since M is strongly T-Rickart, $\mathrm{t}_M(1_{Z_2(M)} \oplus \phi) = Z_2(M) \oplus \mathrm{r}_{M'}(\phi)$ is a fully invariant direct summand of M. Since $\mathrm{t}_M(1_{Z_2(M)} \oplus \phi)$ is a direct summand of M, we obtain $\mathrm{r}_{M'}(\phi) \leq^{\oplus} M'$.

Now we show $\mathrm{r}_{M'}(\phi)$ is fully invariant in M'. Let $f \in \mathrm{End}(M')$ and $m \in \mathrm{r}_{M'}(\phi)$. Thus $1_{Z_2(M)} \oplus f \in \mathrm{End}(M)$. Since $\mathrm{t}_M(1_{Z_2(M)} \oplus \phi)$ is fully invariant, $(1 \oplus f)(m) = f(m) \in \mathrm{r}_{M'}(\phi)$. Thus $\mathrm{r}_{M'}(\phi)$ is fully invariant, as desired.

(3)⇒(4). Let $\phi \in \mathrm{End}(M)$. If $M' = eM$, where $e^2 = e \in \mathrm{End}(M)$, then $\mathrm{t}_M(\phi) \cap M' = \mathrm{r}_{M'}(e\phi e)$ where $e\phi e \in \mathrm{End}(eM) = eSe$. Since M' is strongly Rickart, $\mathrm{t}_M(\phi) \cap M'$ is a fully invariant direct summand of M'.

(4)⇒(1). Let $\phi \in$ End(M). As $Z_2(M) \subseteq \mathrm{t}_M(\phi)$, we have $\mathrm{t}_M(\phi) = Z_2(M) \oplus \mathrm{t}_M(\phi) \cap M'$. Since $\mathrm{t}_M(\phi) \cap M'$ is a direct summand of M', $\mathrm{t}_M(\phi)$ is direct summand of M. We show that $\mathrm{t}_M(\phi)$ is fully invariant in M. If f is canonical projection $f : M \to M'$, then $1 - f : M \to Z_2(M)$. Since $\mathrm{t}_M(\phi) \cap M' \leq M'$, we have $f(\mathrm{t}_M(\phi) \cap M') = \mathrm{t}_M(\phi) \cap M'$. Let $g \in S$. Then $g = (1 - f)g + fg$. So we have $g(\mathrm{t}_M(\phi)) = ((1 - f)g)(\mathrm{t}_M(\phi)) + fg(\mathrm{t}_M(\phi))$. It is clear that $((1 - f)g)(\mathrm{t}_M(\phi)) \subseteq Z_2(M)$ and $fg(\mathrm{t}_M(\phi)) = fg(Z_2(M)) + fg(\mathrm{t}_M(\phi) \cap M')$. Since $g(Z_2(M)) \subseteq Z_2(M)$, we have $fg(Z_2(M)) = 0$. As $\mathrm{t}_M(\phi) \cap M'$ is a fully invariant submodule of M',

$$fg(\mathrm{t}_M(\phi) \cap M') = fgf(\mathrm{t}_M(\phi) \cap M') \subseteq \mathrm{t}_M(\phi) \cap M'.$$

Thus $g(\mathrm{t}_M(\phi)) \subseteq \mathrm{t}_M(\phi)$, and so $\mathrm{t}_M(\phi)$ is a fully invariant direct summand. Hence M is strongly T-Rickart.

(1)⇔(5) is clear as $\mathrm{t}_M(\phi) = \phi^{-1}(Z_2(M))$. ∎

It is clear that strongly T-Rickart modules are T-Rickart, but the following example shows that the converse is not true.

Example 4.3. Let F be a field and $R = \left(\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right)$, and let M be an R-module. Then $R \oplus Z_2(M)$ is a T-Rickart R-module. Since R is not strongly Rickart, by Theorem 4.2, $R \oplus Z_2(M)$ is not strongly T-Rickart.

Theorem 4.4. *If M is strongly T-Rickart, then so is every direct summand of M.*

Proof. Let N be a direct summand of M and $M = N \oplus K$ for some $K \leq M$. Since M is strongly T-Rickart, it is T-Rickart. By Theorem 3.3, N and K are T-Rickart. Since K is T-Rickart, Theorem 3.4 implies $K = Z_2(K) \oplus K'$ for some $K' \leq K$. Let N_1 be a direct summand of N, say $N = N_1 \oplus N_2$ and $Z_2(N) \subseteq N_1$. The module $Z_2(K) \oplus N_1$ satisfies

$Z_2(K) \oplus N_1 \leq^{\oplus} M$ and is fully invariant in M since $Z_2(M) \subseteq Z_2(K) \oplus N_1$ (as M is strongly T-Rickart by Theorem 4.2, every direct summand which contains $Z_2(M)$ is fully invariant).

Further, if $f \in \mathrm{End}(N)$, then $1_{\mathrm{End}(K)} \oplus f \in \mathrm{End}(M)$. As $1_{\mathrm{End}(K)} \oplus f(Z_2(K) \oplus N_1) \subseteq Z_2(K) \oplus N_1$, therefore $f(N_1) \subseteq N_1$. Thus N_1 is fully invariant.

Since N is T-Rickart and each direct summand of N that contains $Z_2(N)$ is fully invariant in N, Theorem 4.2 shows that N is strongly T-Rickart. ∎

THEOREM 4.5. *Let M_1 and M_2 be two modules. The following are equivalent:*

(1) *$M = M_1 \oplus M_2$ is strongly T-Rickart;*

(2) (i) *M_1 is strongly T-Rickart and $M_1 = Z_2(M_1) \oplus M_1'$ for some strongly Rickart module M_1';*

(ii) *M_2 is strongly T-Rickart and $M_2 = Z_2(M_2) \oplus M_2'$ for some strongly Rickart module M_2';*

(iii) $\mathrm{Hom}(M_1', M_2') = 0$ *and* $\mathrm{Hom}(M_2', M_1') = 0$.

Proof. (1)⇒(2). (i) Theorem 4.4 implies that M_1 is strongly T-Rickart, so $M_1 = Z_2(M_1) \oplus M_1'$ for some strongly Rickart module M_1' by Theorem 4.2. (ii) is similar to (i).

(iii) Since M is strongly T-Rickart and

$$M = Z_2(M_1) \oplus Z_2(M_2) \oplus M_1' \oplus M_2' = Z_2(M) \oplus M_1' \oplus M_2',$$

by Theorem 4.4, $M_1' \oplus M_2'$ is strongly Rickart, and so each direct summand of $M_1' \oplus M_2'$ is fully invariant. Hence M_1' and M_2' are fully invariant in $M_1' \oplus M_2'$. We know

$$\mathrm{End}(M_1' \oplus M_2') = \begin{pmatrix} \mathrm{End}(M_1') & \mathrm{Hom}(M_2', M_1') \\ \mathrm{Hom}(M_1', M_2') & \mathrm{End}(M_2') \end{pmatrix}.$$

As

$$M_1' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}(M_1' \oplus M_2') \quad \text{and} \quad M_2' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}(M_1' \oplus M_2')$$

and M_1' and M_2' are fully invariant in $M_1' \oplus M_2'$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in S_l(\mathrm{End}(M_1' \oplus M_2')) \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in S_l(\mathrm{End}(M_1' \oplus M_2')).$$

An inspection shows that $\mathrm{Hom}(M_1', M_2') = 0$ and $\mathrm{Hom}(M_2', M_1') = 0$.

(2)⇒(1). By (i) and (ii), $M = Z_2(M) \oplus M_1' \oplus M_2'$. We will show that $M_1' \oplus M_2'$ is strongly Rickart. Since $\mathrm{Hom}(M_1', M_2') = 0$ and $\mathrm{Hom}(M_2', M_1') = 0$, we have

$$\mathrm{End}(M_1' \oplus M_2') = \begin{pmatrix} \mathrm{End}(M_1') & 0 \\ 0 & \mathrm{End}(M_2') \end{pmatrix}.$$

Let $f = f_1 \oplus f_2 \in \mathrm{End}(M_1' \oplus M_2')$, where $f_1 \in \mathrm{End}(M_1')$ and $f_2 \in \mathrm{End}(M_2')$. Since M_1' and M_2' are strongly Rickart, we have $\mathrm{r}_{M_1'}(f_1) = e_1 M_1'$ for some $e_1 \in S_l(\mathrm{End}(M_1'))$, and $\mathrm{r}_{M_2'}(f_2) = e_2 M_2'$ for some $e_2 \in S_l(\mathrm{End}(M_2'))$. Therefore

$$\mathrm{r}_{M_1' \oplus M_2'}(f_1 \oplus f_2) = \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} (M_1' \oplus M_2').$$

Since $e_1 \in S_l(\mathrm{End}(M_1'))$ and $e_2 \in S_l(\mathrm{End}(M_2'))$,

$$\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in S_l(\mathrm{End}(M_1' \oplus M_2')).$$

Thus $\mathrm{r}_{M_1' \oplus M_2'}(f_1 \oplus f_2)$ is a fully invariant direct summand of $M_1' \oplus M_2'$. Hence $M_1' \oplus M_2'$ is strongly Rickart, and so by Theorem 4.2, M is strongly T-Rickart. ∎

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